Handout 15: Virial Theorem E = P.E. + K.E = (1/2)P.E. = -K.E.

- The virial theorem is crucial for an overview of the stellar interior
  - See discussion Chap. 2 of C&O
- Holds when P.E. is from a force ~ $1/r^2$
  - i.e. gravity and electrostatic forces
- Consider orbit of satellite around Earth

$$\frac{m \cdot v^2}{r} = F_{gravity} = \frac{G \cdot m \cdot M}{r^2}$$

$$KE = \frac{1}{2} \cdot m \cdot v^2 = \frac{1}{2} \cdot \frac{G \cdot m \cdot M}{r} = \frac{-1}{2} \cdot PE.$$

Drag causes increase in speed

- Non-intuitive consequences
  - Atmospheric drag causes r to decrease
  - P.E. decreases
  - |P.E.| increases
  - K.E. **increases**
  - Satellite speeds up!
    - Gas drag causes speed to **increase**
    - Half the decrease in P.E. goes into **heating** up the atmosphere

Virial theorem in globular clusters, clusters of galaxies

- Assuming equilibrium has been reached
  - High K.E. ➔ high total mass
  - Missing mass discovered in galaxy clusters
    - Zwicky, 1930's
- For the stellar interior, a virial theorem results if
  - Hydrostatic equilibrium $dP/dr = -g\rho$
  - Pressure from ideal gas E.O.S $P = nkT$
    - And $\langle K.E. \rangle = (3/2)kT$
  - Massive particles, non-relativistic speeds

Derivation of Virial Theorem for stars

$$\frac{d}{dr}P = -\rho \cdot \frac{G \cdot M_r}{r^2}$$

multiply by $4\pi r^3$ and integrate from r = 0 to R

Right Hand Side

$$RHS = -\int_0^R \rho \cdot \frac{G \cdot M_r}{r} \cdot 4 \cdot \pi \cdot r^2 \, dr$$

$$4 \cdot \pi \cdot r^2 \cdot \rho \cdot dr = dm$$

$$-dm \cdot \frac{G \cdot M_r}{r} = dPE$$

$$RHS = Gravitational\_PE\_of\_star$$

The virial theorem for stars

$$LHS = \int_0^R \frac{d}{dr} P \cdot 4 \cdot \pi \cdot r^3 \, dr$$

integrate by parts $\quad \frac{d}{dr}\left(P \cdot 4 \cdot \pi \cdot r^3\right) = \frac{d}{dr} P \cdot 4 \cdot \pi \cdot r^3 + P \cdot 12 \cdot \pi \cdot r^2$

$P(r = R) = 0$ and $r(r = 0) = 0$ so

$$LHS = -3 \cdot \int_0^R P \cdot 4 \cdot \pi \cdot r^2 \, dr = -3 \cdot \int_0^R n \cdot k \cdot T \cdot 4 \cdot \pi \cdot r^2 \, dr = -3 \cdot \frac{2}{3} \cdot \int_0^R n \cdot \frac{3}{2} \cdot k \cdot T \cdot 4 \cdot \pi \cdot r^2 \, dr = -2 \cdot KE$$

$KE = \frac{-1}{2} \cdot PE$ the virial theorem

Consequences for stellar evolution

- Consider collapse of a star
  - As R decreases, PE decreases

$$PE = -\int_0^R \frac{G \cdot M_r}{r} \cdot \left(4 \cdot \pi \cdot r^2 \cdot \rho\right) dr$$

PE will be of order $\dfrac{-G \cdot M^2}{R}$

i.e. for constant density $PE = \dfrac{-3}{5} \cdot \dfrac{G \cdot M^2}{R}$

For a star, gas is compressible, therefore density increases as r decreases
Therefore more tightly bound than constant density
Therefore PE is more negative than this

Kelvin-Helmholtz contraction

- Contract from 100 to 1 R_{sun}
  - PE goes from near zero to $-GM_{sun}^2/R_{sun}$
    - KE increases by –(1/2)ΔPE
      - Star gets hotter inside
    - Where did the other half of the PE go?
      - It was radiated away as photon luminosity
  - Kelvin-Helmholtz timescale = -P.E./2*L
    - For the sun at R = R_{sun}, timescale = 10^7 yr
  - This is
    - Timescale for contraction to the Main Sequence
    - Age of the sun if only gravitational energy available

Nuclear energy sources

- Life on earth is at least 3 to 4 billion years old
  - Age of sun much longer than 10 million years
  - Nuclear energy sources crucial
    - **Thermal equilibrium**
      - Energy flux radiated from the surface exactly equal to the energy generated by nuclear fusion reactions in the interior
      - This condition defines the locus of the Main Sequence on the H-R diagram
      - $\varepsilon(\rho,T)$ is the rate of energy generation, **power/mass**

Thermal equilibrium

$$L_{radiant} = \int_0^R 4 \cdot \pi \cdot r^2 \cdot \rho \cdot \varepsilon \, dr$$

Temperature in the sun

- The virial theorem gives the average temperature of the sun

$$\frac{3}{2} \cdot k \cdot T \cdot N = \frac{-1}{2} \cdot PE = G \cdot \frac{M_{sun}^{2}}{R_{sun}}$$

$$N = \frac{M_{sun}}{\mu \cdot m_H} \qquad \mu = 0.6$$

$$T_{ave} = \frac{2}{3} \cdot \frac{\mu \cdot m_H \cdot G}{k} \cdot \left(\frac{M_{sun}}{R_{sun}} \right) = 10^{7} \cdot K$$

amazing

Just by looking at the sun for ~ 1 hour ➔ hydrostatic equilibrium
➔ Virial theorem ➔ interior is 10 million K!

Sun temperatures of 10 million K ➔ nuclear fusion reactions

- High velocities necessary to overcome coulomb barrier
- 4 H ➔ eventually, He + energy
  - And energy ~ 1% mc^2
    - Very significant, c.f. (1/2)PE
      - 1% $Mc^2 = 10^{52}$ ergs
      - $GM^2/2R = 10^{48}$ ergs
    - Rather than the K-H time of 10^7 yr, we have 10^{11} years for nuclear, Main Sequence lifetime
      - Stellar evolution ➔ only 10% of H burnt on M.S.