Handout 13: Curve of Growth

- The strength of a line depends on the column density N above the $\tau = 1$ level

$$N = \int_{\tau_{cont}=1}^{0} n \, dz \qquad \tau_{line} = \sigma_{line} \cdot N$$

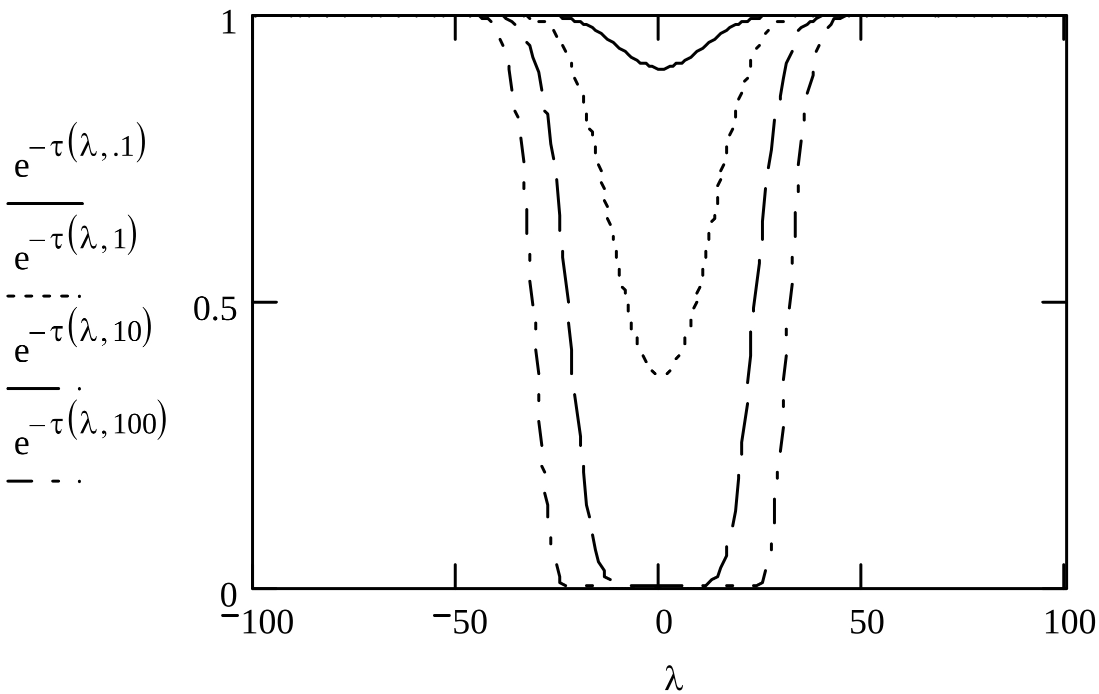
- For $\tau_{line} \ll 1$, W ~ τ_{line} ~ N
  - Use “reversing layer” model for atmosphere

$$\tau_{line} < \blacksquare < 1 \qquad F_{\lambda} = e^{-\tau_{\lambda}} \cdot F_{cont} = \left(1 - \tau_{\lambda}\right) \cdot F_{cont}$$

$$\frac{F_{cont} - F_{\lambda}}{F_{cont}} = \tau_{\lambda}$$

Line saturates in core, growth in W due to line wings

$w := 10$ $\qquad$ $\tau(\lambda, \tau_0) := \tau_0 \cdot e^{\frac{-\lambda^2}{2 \cdot w^2}}$ $\qquad$ $\lambda := -100..\,100$



Max τ depends on atomic physics and line width as well as N (#/cm^2)

Atomic absorption cross section: $$\sigma(\lambda) = \frac{2\cdot\pi\cdot e^2}{m\cdot c}\cdot f\cdot\phi(\lambda)$$

$$\text{atomic\_factor} = \frac{2\cdot\pi\cdot e^2}{m\cdot c} = cm^2\cdot\frac{\text{radians}}{\text{sec}}$$

$$\text{QM\_transition\_strength} = f = \text{of\_order\_1\_for\_electric\_dipole}$$

$$\text{line\_shape\_function} = \phi \qquad \int \phi\,d\omega = \int \phi\cdot 2\cdot\pi\,d\nu = \frac{2\cdot\pi\cdot c}{\lambda_0^{\,2}}\cdot\int \phi\,d\lambda = 1$$

For optically thin line

$$W = \int \tau_{\lambda}\, d\lambda = \left(N \cdot \frac{2 \cdot \pi \cdot e^2}{m \cdot c} \cdot f\right) \cdot \int \phi\, d\lambda = N \cdot \frac{2 \cdot \pi \cdot e^2}{m \cdot c} \cdot f \cdot \frac{\lambda_0{}^2}{2 \cdot \pi \cdot c}$$

or $$\frac{W}{\lambda_0} = N \cdot \frac{2 \cdot \pi \cdot e^2}{m \cdot c} \cdot f \cdot \frac{\lambda_0}{2 \cdot \pi \cdot c}$$ reason for axes on fig. 9.22

- For optically thick line
  - Line totally dark in center
  - Starts to brighten when $\tau = 1$
    - See graph of optically thick lines, note where $\tau = 1$
  - $W \cong \lambda_2 - \lambda_1$
    - **i.e. width of line at $\tau = 1$ points**
    - **Solve λ for $N\sigma_{\lambda} = 1$**

Doppler broadening, gaussian line shape, W grows like sqroot[ln($N\sigma_0$)]

solve $$\tau(\lambda) = \tau_0 \cdot e^{\frac{-\lambda^2}{2 \cdot w^2}} = 1$$

$$|\lambda| = \sqrt{2 \cdot w^2 \cdot \ln(\tau_0)} = \sqrt{2 \cdot w^2 \cdot \ln(N \cdot \sigma_0)}$$

$$W = 2 \cdot \sqrt{2 \cdot w^2 \cdot \ln(N \cdot \sigma_0)}$$

- Very slow growth of W with N
  - Small error in W implies a large error in N, the abundance
- Since gaussian ➔ zero very quickly
  - “damping wings” dominate W for very large τ_0
  - W ~ sqroot($N\sigma_0$)

Very optically thick, damping wings, W grows like sqroot[Nσ_0]

solve: $$\tau(\lambda) = \tau_0 \cdot \frac{\Gamma^2}{\lambda^2 + \Gamma^2} = 1$$ Lorentzian, damping line shape

we're far from line center, so $\lambda >> \Gamma$, giving

$$|\lambda| = \Gamma \cdot \sqrt{\tau_0} = \Gamma \cdot \sqrt{N \cdot \sigma_0}$$

$$W = 2 \cdot \Gamma \cdot \sqrt{N \cdot \sigma_0}$$

This explains the shape of the curve-of-growth given in Figure 9.22. Once the curve of growth is established, the strength of absorption lines (i.e. W) can be used to measure abundances.

As an example, consider the Na D line. W = 0.73 Angstrom and f = 0.65

Na abundance in the solar atmosphere

- First note the optical depth at line center is >> 1
  - Doppler width only 0.1 Angstrom
    - Optical depth at line center > 100
  - From Fig. 9.22
    - $\text{Log}(W/\lambda) = -3.9$
    - $\text{Log}(Nf\lambda/5000\ \text{Angstrom}) = 14.8$
    - Therefore $N = 10^{15}$ (NaI atoms)/cm^2
  - Consider Boltzmann factor
    - ~ 99% of NaI atoms are in ground state

Na abundance *cf*. H is 3.5 10^{-6}

- Saha ionization
  - NaII/NaI = 2430
  - NaIII/NaII very small
  - $N_{Na} = N_{NaI+NaII} = 2431 \times 10^{15}$ (Na atoms)/cm^2
- Repeating for Hα line

$$N_H = \frac{1.1 \cdot \dfrac{gm}{cm^2}}{1.67 \cdot 10^{-24} \cdot gm} = 0.7 \cdot 10^{24} \cdot cm^{-2}$$

$$\frac{N_{Na}}{N_H} = \frac{2.4 \cdot 10^{18}}{0.7 \cdot 10^{24}} = 3.5 \cdot 10^{-6}$$