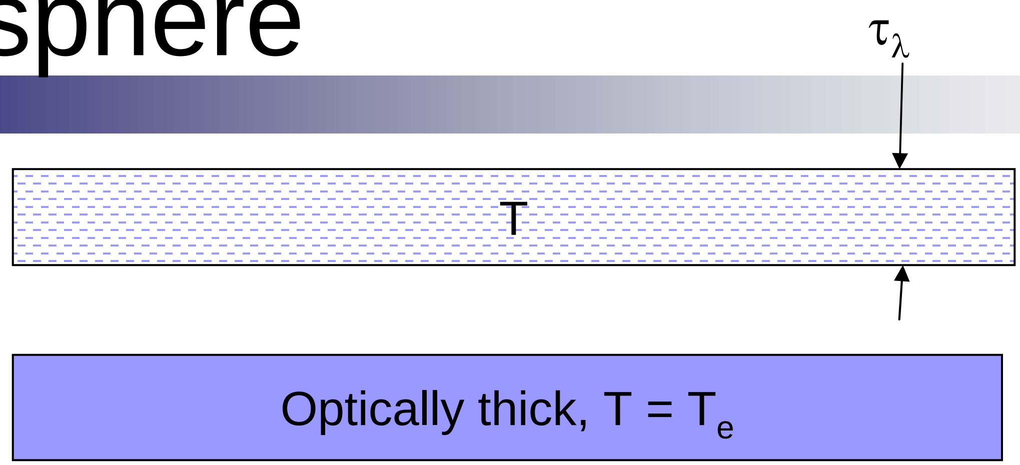
Handout 11: Solutions to equation of Radiative Transfer

- Consider the case where the source function S is constant in space
  - i.e. T.E. and T = const.
    - First approximation
- Solution $$I_\lambda(\tau_\lambda) = I_\lambda(0)\cdot e^{-\tau_\lambda} + S_\lambda\cdot\left(1 - e^{-\tau_\lambda}\right)$$
    - If S = 0,
      - merely extinction, Intensity attenuated by exp(-τ_λ)
    - If I(0) = 0, i.e. emission from the box
      - I = S(1 - exp(-τ_λ))
      - Take case $\tau \rightarrow 0$

Special solutions

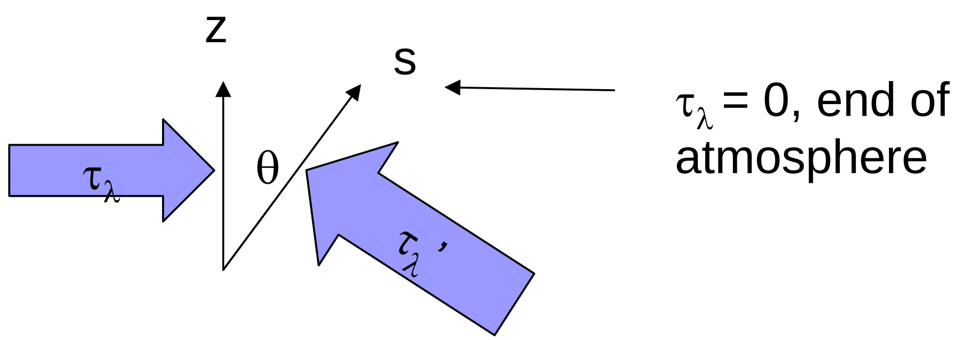
- $\tau \rightarrow 0$, $I_\lambda = \tau_\lambda S_\lambda$
  - Linearly proportional to τ_λ
  - Optically thin hot gas gives **emission lines**
    - Kirchoff law #1
- as $\tau \rightarrow$ infinity
  - $I_\lambda = S_\lambda$
    - Filament of incadescent lamp will give a Planck, black-body spectrum
      - Kirchoff law #2
    - In T.E., nothing radiates better than a blackbody
      - Important!

“Reversing layer” model for stellar atmosphere



- $T << T_e$, $I_\lambda = B_\lambda(T_e)\exp(-\tau_\lambda)$
  - Absorption lines in spectrum
- $T >> T_e$
    - i.e. corona, T ~ 1,000,000 K
  - Where optical depth small, $I_\lambda \rightarrow B_\lambda(T_e)$
  - Optical depth large, $I_\lambda \rightarrow S_\lambda = B_\lambda(T)$
    - Emission lines when T is increasing outward

Definition of optical depth into a plane-parallel atmosphere



- We use optical depth as a surrogate for physical depth in the atmosphere
  - $d\tau_\lambda' = d\tau_\lambda/\cos(\theta)$
  - $d\tau_\lambda = -\kappa_\lambda \rho dz$
    - Optical depth increases **into** the atmosphere

Xfer eq'n becomes $\frac{dI_\lambda}{d\tau_{\lambda'}} = \cos(\theta)\cdot\frac{dI_\lambda}{d\tau_\lambda} = I_\lambda - S_\lambda$

Assume radiation flow through atmosphere is represented by a mean opacity, i.e. the Rosseland Mean. This is exact if the opacity is independent of wavelength, which is called a **grey atmosphere**. This is a fair approximation for the sun, because of the broad nature of the H- opacity that dominates the continuous opacity. It is really accurate for hot stars, where electron scattering is the dominant opacity, and does not depend on wavelength.

First, **integrate** equation of transfer **over 4π ster**, giving **moments.**

$d\tau = \kappa\cdot\rho\cdot dz$ and $d\tau' = \frac{d\tau}{\cos(\theta)}$

$$I = \int_0^\infty I_\lambda\, d\lambda \qquad S = \int_0^\infty S_\lambda\, d\lambda = \frac{\sigma}{\pi}\cdot T^4$$

in Thermal Equilibrium

$\frac{d}{d\tau}\cos(\theta) = 0$ so $\frac{d}{d\tau}\int_{4\cdot\pi} I\cdot\cos(\theta)\, d\Omega = \int_{4\cdot\pi} I\, d\Omega - S\cdot\int_{4\cdot\pi} 1\, d\Omega$

1st moment

0th moment

4π

Relationship between 1^{st} and 0^{th} moments: $d(F_{rad})/d\tau = 4\pi(\langle I \rangle - S)$

- Very important!
  - Since there are no sources or sinks of energy in the atmosphere in the steady state
    - $F_{rad} = \text{constant} = \sigma T_e^4$ as a fct. of τ
    - $\langle I \rangle = S$, the **Source function** equals the average intensity
      - Assuming L.T.E., $S = B(T)$, so $\langle I \rangle = (\sigma/\pi)T^4$
        - Relates T to $\langle I \rangle$
    - Solution of $\langle I \rangle$ vs. τ gives **temperature structure** in the atmosphere

2^{nd} moment

- Multiply by $\cos(\theta)$, integrate over 4π ster

$$\frac{d}{d\tau}\int_{4\cdot\pi} I\cdot\cos(\theta)^2\,d\Omega = \int_{4\cdot\pi} I\cdot\cos(\theta)\,d\Omega - S\cdot\int_{4\cdot\pi}\cos(\theta)\,d\Omega$$

cP_{rad} **F_{rad}** **zero**

so

$$\frac{d}{d\tau}P_{rad} = \frac{1}{c}\cdot F_{rad} = \frac{1}{c}\cdot\sigma\cdot T_e^{\,4}$$

constant with depth

or

$$P_{rad}(\tau) = \frac{1}{c}\cdot F_{rad}\cdot\tau + C$$

where C = const.

Solution of const. C requires an approximation – Eddington 2-stream

- Intensity has 2 values
  - I_{out} is const., $\theta = 0 \rightarrow \pi/2$
  - I_{in} ($< I_{out}$) is const., $\theta = \pi/2 \rightarrow \pi$
- At top of atmosphere ($\tau = 0$)
  - $I_{in} = 0$
  - $\pi I_{out} = F_{rad}$
- Inside the atmosphere ($\tau > 0$)
  - $F_{rad} = \pi I_{out} - \pi I_{in}$ = const. with τ
    - A **constant** difference

Eddington (2-stream) approximation

- Calculate the average intensity, and the radiation pressure
  - $<I> = (1/2)(I_{out} + I_{in})$

$$P_{rad} = \frac{1}{c} \cdot \int_{4\cdot\pi} I \cdot \cos(\theta)^2 \, d\Omega = \frac{4\cdot\pi}{3\cdot c} \cdot \frac{1}{2} \cdot (I_{out} + I_{in})$$

$\tau = 0$

$$P_{rad} = \frac{2\cdot\pi}{3\cdot c} \cdot I_{out}$$

$$P_{rad}(\tau = 0) = C$$

so

$$C = \frac{2}{3\cdot c} \cdot \pi \cdot I_{out} = \frac{2}{3\cdot c} \cdot F_{rad}$$

and

$$P_{rad}(\tau) = \frac{F_{rad}}{c} \cdot \left(\tau + \frac{2}{3} \right)$$

Temperature structure in atmosphere

- Use $F_{rad} = \sigma T_e^4$ and $P_{rad} = (4\pi/3c)\langle I \rangle$
- $\langle I \rangle = (3\sigma/4\pi)(\tau + 2/3)\ T_e^4$
  - But source fct. $S = \langle I \rangle$
  - And in L.T.E., $S = B = (\sigma/\pi)T^4$
- So $T^4(\tau) = (3/4)\ T_e^4(\tau + 2/3)$
  - **Eddington approx**. to T vs. depth in atmos.
- At $\tau=0$, $T^4(0) = (1/2)\ T_e^4$, not zero
  - i.e. dust grain at blackbody aperture gets **half** the heating