LECTURE 12

Angular Distribution of Emitted and Received Power

fig 4.9

Now consider dW' emitted into angle $d\Omega' = sin\theta' d\theta' d\phi'$, where θ' is to the x-axis. Let $\mu = cos\theta$ and $\mu' = cos\theta'$. We do not assume forward backward symmetry here so the momentum is not zero. Then

$$dW = \gamma(dW' + vdP'_x) = \gamma(1 + \beta\mu')dW'. \tag{371}$$

From (313) we have

$$\mu = \frac{\mu' + \beta}{1 + \beta\mu'} \tag{372}$$

so that differentiating and using $d\phi = d\phi'$ we get

$$d\Omega = \frac{d\Omega'}{\gamma^2(1 + \beta\mu')^2}. \tag{373}$$

Thus

$$dW/d\Omega = \gamma^3(1 + \beta\mu')^3(dW'/d\Omega') \tag{374}$$

To get the differential emitted power in the lab frame we divide by the time as determined from time dilation $dt = \gamma dt'$.

To get an observed differential power we must include the Doppler effect so that $dt_{obs} = \gamma(1 - \beta\mu)dt'$ from (316).

We then have

$$dP_e/d\Omega = \gamma^2(1 + \beta\mu')^3 dP'/d\Omega' = \gamma^{-4}(1 - \beta\mu)^{-3} dP'/d\Omega' \tag{375}$$

and

$$dP_{obs}/d\Omega = \gamma^4(1 + \beta\mu')^4 dP'/d\Omega' = \gamma^{-4}(1 - \beta\mu)^{-4} dP'/d\Omega' \tag{376}$$

where we have used the reciprocal of (318). The factor $\gamma^{-4}(1 - \beta\mu)^{-4}$ is peaked near $\theta = 0$ for small θ which is highlights the beaming effect described earlier.

(fig 4.10)

Apply to a moving particle. Recall that in the non-relativistic regime (and thus the proper frame here)

$$dP'/d\Omega' = \frac{q^2 a'^2}{4\pi c^3} sin^2\Theta' \tag{377}$$

Thus

$$dP/d\Omega = \frac{q^2}{4\pi c^3} \frac{(\gamma^2 a_{||}^2 + a_{\perp}^2)}{(1-\beta\mu)^4} sin^2\Theta', \tag{378}$$

where we have used (368) and (369).

acceleration || velocity: Then $\Theta' = \theta'$ so

$$sin^2\Theta' = sin^2\theta' = \frac{sin^2\theta}{\gamma^2(1-\beta\mu)^2}, \tag{379}$$

using the inverse of (313). Then

$$dP_{||}/d\Omega = \frac{q^2}{4\pi c^3} a_{||}^2 \frac{sin^2\theta}{(1-\beta\mu)^6}. \tag{380}$$

for acceleration ⊥ velocity: Then $Cos\Theta' = cos\phi' sin\theta'$ so

$$sin^2\Theta' = 1 - \frac{sin^2\theta cos^2\phi}{\gamma^2(1-\beta\mu)^2}. \tag{381}$$

Then

$$dP_{\perp}/d\Omega = \frac{q^2}{4\pi c^3} a_{\perp}^2 \frac{1}{(1-\beta\mu)^4} \left(1 - \frac{sin^2\theta cos^2\phi}{\gamma^2(1-\beta\mu)^2}\right). \tag{382}$$

extreme relativistic limit

Since

$$\beta = (1 - 1/\gamma^2)^{1/2} \sim 1 - \frac{1}{2\gamma^2} \tag{383}$$

and

$$\mu = cos\theta = 1 - \theta^2/2 \tag{384}$$

we have

$$(1 - \beta\mu) \simeq \frac{1 + \gamma^2\theta^2}{2\gamma^2}. \tag{385}$$

and (381) becomes

$$dP_{||}/d\Omega = \frac{16q^2a_{||}^2}{\pi c^3}\frac{\gamma^{12}\theta^2}{(1+\gamma^2\theta^2)^6} \tag{386}$$

while (382) becomes

$$dP_{\perp}/d\Omega = \frac{4q^2a_{\perp}^2}{\pi c^3}\gamma^8\frac{(1 - 2\gamma^2\theta^2 cos2\phi + \gamma^4\theta^4)}{(1+\gamma^2\theta^2)^6}. \tag{387}$$

(fig 4.11)

The beaming as a result of the $\theta\gamma$ factors is highlighted in the figures. Notice that the maximum powers occur for $\theta \sim 1/\gamma$. Take the limits for small and larger θ and compare to the figure, remembering that we are already in the small theta regime.

Specific Intensity and Invariants

Consider group of particles moving along x frame in comoving phase space $d^3\mathbf{x}'d^3\mathbf{p}'$. There is slight spread in momentum but not in energy. Because $E = mc^2 + mv^2/2$ in this frame and E is thus quadratic in v. Phase space is invariant because $d^3\mathbf{x} \rightarrow d^3\mathbf{x}/\gamma$ But $dp_x = \gamma(dp'_x + \beta dE/c)$, but dE is constant.

Thus $d^3\mathbf{x}d^3\mathbf{p}$ is an invariant.

Thus $dN/d^3\mathbf{x}d^3\mathbf{p}$ is also an invariant.

But we related the latter to the intensity in equation (18)

$$I_\nu = \sum_{\alpha=1}^{2}(h^4\nu^3/c^3)f_\alpha(\mathbf{x},\mathbf{p},t). \tag{388}$$

so I_ν/ν^3 is an invariant, as is S_ν/ν^3.

For the absorption coefficient α_ν. Consider thickness l of material of material moving along x with β. The optical depth as seen by an observer at angle θ to the vertical in the is invariant since $e^{-\tau}$ measures the fraction of photons passing through material.

fig 4.12

Then

$$\tau = \alpha_\nu (l/sin\theta) = \nu\alpha_\nu (l/\nu sin\theta) \tag{389}$$

is invariant, but $\nu sin\theta$ is the vertical momentum component and so it is not affected by the motion along the $x - axis$. Thus $\nu\alpha_\nu$ is an invariant.

Since $j_\nu = S_\nu/\nu$, j_ν/ν^2 is also an invariant.

LECTURE 13

Bremsstrahlung-free free emission

Radiation from acceleration of charge in Coulomb field of another charge.

Quantum treatment required, as process can produce photons comparable in energy to that of emitting particle, but do classical treatment first.

Bremsstrahlung is zero in dipole approximation for same type particles.

Electrons due most of radiation since acceleration is inversely proportional to mass.

Emission from Single Speed Electrons

Consider electron moving in field of nearly stationary ion (at origin).

Consider small-angle scattering regime.

Consider electron of charge $-e$ moving past ion of charge Ze with impact parameter b.

fig 5.1

Dipole moment is $\mathbf{d} = -e\mathbf{R}$ and so

$$\ddot{\mathbf{d}} = -e\dot{\mathbf{v}} \tag{390}$$

so Fourier transform gives

$$-\omega^2\tilde{\mathbf{d}}(\omega) = -\frac{e}{2\pi}\int_{-\infty}^{\infty} \dot{\mathbf{v}} e^{iwt} dt. \tag{391}$$

Consider asymptotic limits of large and small frequencies. Since $\tau \sim b/v$ is the interaction time, then the integral mainly contributes over this time.

When $\omega\tau >> 1$ exponential oscillates rapidly and is small.

When $\omega\tau << 1$ exponential is unity so

$$\tilde{\mathbf{d}} \sim \frac{e}{2\pi\omega^2}\Delta\mathbf{v}, \tag{392}$$

with $\Delta\mathbf{v}$ the change in velocity over the interaction time.

Using

$$\frac{dW}{d\omega} = \frac{8\pi\omega^4}{3c^3}|\tilde{d}(\omega)|^2 \tag{393}$$

we have

$$\frac{dW}{d\omega} = \frac{2e^2}{3\pi c^3}|\Delta\mathbf{v}|^2. \tag{394}$$

The acceleration is mainly perpendicular to the electron path. It is found by integrating the perpendicular component of acceleration:

$$\Delta v = \int a_\perp dt = \int_{-\infty}^{\infty} b/(b^2+vt^2)^{3/2}dt = 2e^2Z/m_e bv, \tag{395}$$

where we are in the $\omega\tau << 1$ limit, and extend the integral to infinity.

For small angle scatterings then

$$\frac{dW}{d\omega} = \frac{8e^2Z^2}{3\pi c^3 m_e^2 v^2 b^2} \tag{396}$$

For $\omega\tau << 1$ and 0 otherwise.

To get the total energy per unit frequency for a number density of ions n_i and electrons n_e with velocity $\mathbf{v}$, we note that the flux of electrons per ion is $n_e v$. Then, the number of electrons per unit time per ion that interact is $n_e v 2\pi b db$. Then multiplying by the number density of ions and integrating, we have

$$\frac{dW}{d\omega dV dt} = 2\pi n_i n_e v \int_{b_{min}}^{\infty} (dW(b)/d\omega) b db. \tag{397}$$

The main contribution is for $\omega\tau \leq 1$ or $b << v/\omega$ and so substitute (396) for the integrand

$$\frac{dW}{d\omega dV dt} = \frac{16Z^2e^6}{3c^3m_e^2 v} n_i n_e \; ln(b_{max}/b_{min}) \tag{398}$$

$b_{max} = v/\omega$. Two possibilities for b_{min}. One is classical, when $\Delta v = v$ (i.e. when small angle approximation is invalid). From (395) this is

$$b_{min} = \frac{4Ze^2}{\pi m_e v^2}. \tag{399}$$

Second is the quantum limit

$$b_{min} = h/mv. \tag{400}$$

When the second limit is < first, then $mv^2 > Z^2 Ry \sim Z^2 13.6\text{eV}$. Then the interaction time is short, scale is small, need quantum treatment.

Convention is to write the uncertainty as the Gaunt factor $g_{ff}(v, \omega)$ so

$$\frac{dW}{d\omega dV dt} = \frac{16\pi Z^2 e^6}{3\sqrt{3} c^3 m_e^2 v} n_i n_e g_{ff}. \tag{401}$$

The Gaunt factor is of order 1 usually.

Thermal Bremsstrahlung

We can integrate single speed expression above over thermal distribution of electrons. Probability for electron to have velocity in range $d^3\mathbf{v}$ for an isotropic distribution of velocities

$$dP = e^{-E/kt} d^3\mathbf{v} = v^2 e^{-mv^2/2kT} dv \tag{402}$$

Only subtlety is that lower bound for radiation contributing velocity integral must be $2h\nu < mv^2$, so that photon can be produced. Then

$$\frac{dW(T,\omega)}{dV dt d\omega} = \frac{\int_{v_{min}}^{\infty} [dW(v,\omega)/dV d\omega dt] v^2 e^{-mv^2/2kT} dv}{\int_0^{\infty} v^2 e^{-mv^2/2kT} dv} \tag{403}$$

The emission coefficient is then given by:

$$4\pi j_\nu^{ff} = \epsilon_{ff,\nu} = \frac{dW(T,\omega)}{dV dt d\omega} = 6.8 \times 10^{-38} Z^2 n_e n_i T^{-1/2} e^{-h\nu/kT} \bar{g}_{ff}(T,\nu). \tag{404}$$

Note the dependence on the number density squared for $n_e = n_i$.

The Gaunt factors tend to be of order a few for a wide range of velocity. (problem in book fig 5.3 with the Gaunt factor X-axis graph. $u = h\nu/kT$)

Note that the exponential tail is due to the cutoff in the velocity. Also, the spectrum is more or less flat until $h\nu/kT \sim 1$.

To get the total power per volume, we integrate over frequency. This gives

$$\frac{dW}{dt dV} = \left(\frac{2\pi kT}{3}\right)^{1/2} \left(\frac{32\pi e^6}{3hm^{3/2}c^3}\right) Z^2 n_e n_i g_B(T) = 10^{-27} T^{1/2} n_e n_i Z^2 g_B(T). \tag{405}$$

where $g_B(T)$ is frequency averaged Gaunt factor.

Thermal Bremsstrahlung

Consider electric field absorbing radiation while moving in field of an ion. (Free-free absorption).

For thermal processes we have Kirchoff's law

$$j_\nu^{ff} = \alpha_\nu^{ff} B_\nu(T), \tag{406}$$

where α_ν^{ff} is the free-free abs. coefficient and j_ν^{ff} is the associated emission coefficient as above. Plugging in for the Planck function and using ϵ_ν^{ff} above we have

$$\alpha_\nu^{ff} = 3.7 \times 10^8 T^{-1/2} Z^2 n_i n_e \nu^{-3} (1 - e^{-h\nu/kT}) \bar{g}_{ff} \tag{407}$$

In the Rayleigh-Jeans regime $h\nu/kT << 1$, expand exponential and then:

$$\alpha_\nu^{ff} = 0.018 T^{-3/2} Z^2 n_e n_i \nu^{-2} \bar{g}_{ff} \tag{408}$$

In the limit $h\nu >> kT$ then

$$\alpha_\nu^{ff} = 3.7 \times 10^8 T^{-1/2} Z^2 n_i n_e \nu^{-3} \bar{g}_{ff}. \tag{409}$$

Recall that the Rosseland mean absorption coefficient was defined when we considered plane-parallel atmospheres in the limit of Kirchoffs law at large optical depths leading to

$$F(z) = -\frac{16\sigma T^3}{3\alpha_R} \frac{\partial T}{\partial z}. \tag{410}$$

For Bremsstrahlung,

$$\frac{1}{\alpha_R^{ff}} = \frac{\int_0^\infty \frac{1}{\alpha_\nu^{ff}} \frac{\partial B_\nu}{\partial T} d\nu}{\int_0^\infty \frac{\partial B_\nu}{\partial T} d\nu} = \frac{1}{1.7 \times 10^{-25} T^{-7/2} Z^2 n_e n_i \bar{g}_R}. \tag{411}$$

Relativistic Bremsstrahlung

One can use the method of virtual quanta to get a relativistic correction to Bremsstrahlung emission. (Electrons moving relativistically). Ultimately, we want a correction in the lab frame, in which the electron is moving. Here is the procedure

1) First move to rest frame of electron.

2) The electron sees the ions field as pulse of electromagnetic radiation emitted by the ion (the virtual quanta). This pulse is scattered by the electron.

3) Then transform back to the rest frame of the ion to get the emission.

Roughly, consider the pulse of virtual quanta in the electron frame $dW'/dA'd\omega'$. Then project out the scattered part by multiplying by the electron scattering cross section. (σ_T for low freq $h\nu << mc^2$). This gives

$$(dW'/d\omega')_{sc} = \sigma_T dW'/dA'd\omega'. \tag{412}$$

Then note that dW' and $d\omega'$ transform the same way under a Lorentz transformation. So that

$$(dW/d\omega)_{sc} = (dW'/d\omega')_{sc}. \tag{413}$$

Then appeal to forward backward symmetry of the scattering, and write primed frame quantities such as ω' and b' in terms of quantities in the unprimed frame by a Lorentz transformation.

For a thermal distribution, one must integrate over the thermal distribution function. Then time integrating, one obtains.

$$dW/dVdt = 4\pi \int j_\nu d\nu 1.4 \times 10^{-27} Z^2 \bar{g}_B T^{1/2} n_i n_e (1 + 4.4 \times 10^{-10} T). \tag{414}$$

The second term on the right is the relativistic correction. Important for quite large temperatures.