LECTURE 8

Velocity and Radiation Components of the Electromagnetic Fields

To get $\mathbf{E}$ and $\mathbf{B}$ at $\mathbf{r}$ and t we first get the retarded position and time r_{ret} and t_{ret}. The particle has $\mathbf{u} = \dot{\mathbf{r}}_0(t_{ret})$ and acceleration $\dot{\mathbf{u}} = d^2_{t_{ret}}\mathbf{r}_0(t_{ret})$. There is some messy algebra to obtain, from the potentials presented at the end of last lecture, the fields (see Jackson)

$$\mathbf{E}(\mathbf{r},t) = q\left[\frac{(\hat{\mathbf{n}} - \vec{\beta})(1-\beta^2)}{\kappa^3 R^2}\right] + \frac{q}{c}\left[\frac{\hat{\mathbf{n}} \times ((\hat{\mathbf{n}} - \vec{\beta}) \times \dot{\vec{\beta}})}{\kappa^3 R}\right] \tag{241}$$

and

$$\mathbf{B}(\mathbf{r},t) = [\hat{\mathbf{n}} \times \mathbf{E}], \tag{242}$$

where $\vec{\beta} = \mathbf{u}/c$ and $\kappa = (1 - \hat{\mathbf{n}} \cdot \vec{\beta})$. Where the brackets again indicate retarded time, and so only the conditions at t_{ret} matter even though particle is currently at $\mathbf{r}, t$.(fig3.1).

The first term in the electric field is just the relativistic generalization of Coulomb's law for moving particles. It falls off as $1/R^2$. This is the "velocity field." It points in the line toward the current position of the particle. Fig (9).

The second term is the radiation field. It falls off as $1/R$. Alas we see why the Poynting flux associated with radiation from an accelerating particle can be finite at large distances.

The effect can be seen from the Figure below. (fig 3.2 of R&L).

Suppose the particle is originally moving at constant speed and then it stops. This information is not immediately communicated to the exterior. Only for distances less than $c(t_1 - t_0)$ is the field informed that the particle decelerated, but outside this field is that of the particle had it not decelerated (that is, as if it were at x=1). Thus there is the kink in the field if the field is to match together for the two regions.

The thickness of this the interface region is constant and equal to the light travel distance during the period of deceleration or acceleration. Now take a ring whose central axis is the line of particle motion and the plane of the ring lies perpendicular to the field lines from the interface region, such that they penetrate the ring. Since $\nabla \cdot \mathbf{B} = \nabla \cdot \mathbf{E} = 0$, we have

$$\int \nabla \cdot \mathbf{E} d^3\mathbf{r}' = \int \mathbf{E} \cdot d\mathbf{S} = constant, \tag{243}$$

(same for **B** field). Thus the flux is conserved. But with time, the ring, area, while maintaining the same thickness, increases as R if it is to keep up with the interface region field lines. This means that to conserve flux the fields must fall of as $1/R$. That is where the $1/R$ radiation field comes from!

Energy/(Freq· Solid angle) for single particle

From Lecture 6 we have that

$$\frac{dW}{dAdt} = \frac{c}{4\pi} E^2(t). \tag{244}$$

Then using $dA = R^2 d\Omega$ we have

$$\frac{dW}{d\Omega dt} = \frac{c}{4\pi}(R(t)E(t))^2 = \frac{c}{4\pi}(Q(t))^2 = \tag{245}$$

where $\mathbf{Q}(t) = R(t)\mathbf{E}(t)$, so that

$$\frac{dW}{d\Omega d\omega} = c\tilde{Q}(\omega)^2 = \frac{c}{4\pi^2}\left|\int_{-\infty}^{\infty}[R\mathbf{E}(t)]e^{iwt}dt\right|^2 = \frac{q^2}{4\pi^2 c}\left|\int_{-\infty}^{\infty}[\hat{\mathbf{n}} \times ((\hat{\mathbf{n}} - \vec{\beta}) \times \dot{\vec{\beta}})\kappa^{-3}]e^{iwt}dt\right|^2. \tag{246}$$

The brackets again mean evaluated at retarded time $t' = t - R(t')/c$. Changing time variables using $dt = \kappa dt'$ and using the Taylor expansion $R(t') \sim |\mathbf{r}| - \hat{\mathbf{n}} \cdot \mathbf{r}_0$ for $\mathbf{r_0} << \mathbf{r}$ (expand around $\mathbf{r}_0 \sim 0$) gives

$$\frac{dW}{d\Omega d\omega} = \frac{q^2}{4\pi^2 c}\left|\int_{-\infty}^{\infty} \hat{\mathbf{n}} \times ((\hat{\mathbf{n}} - \vec{\beta}) \times \dot{\vec{\beta}})\kappa^{-2} e^{iw(t' - \hat{\mathbf{n}} \cdot \mathbf{r}_0(t')/c)} dt'\right|^2. \tag{247}$$

Then using the identity

$$\hat{\mathbf{n}} \times ((\hat{\mathbf{n}} - \vec{\beta}) \times \dot{\vec{\beta}})\kappa^{-2} = d_{t'}(\kappa^{-1}\hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \vec{\beta}) \tag{248}$$

we can integrate (247) by parts to obtain

$$\frac{dW}{d\Omega d\omega} = \frac{q^2}{4\pi^2 c}\left|\int_{-\infty}^{\infty} \hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \vec{\beta}) e^{iw(t' - \hat{\mathbf{n}} \cdot \mathbf{r}_0(t')/c)} dt'\right|^2. \tag{249}$$

The integration by parts (really just the chain rule for differentiation) led to two terms, one of which was of the form $\int d_{t'}(UV)dt'$, which vanished under the assumption that $\mathbf{E}$ falls off rapidly at $t \rightarrow \pm\infty$.

Radiation from Non-relativistic systems

In the non-relativistic limit $\beta << 1$. Also the ratio of the second term on the right of (241) to the first term on the right is $E_{rad}/E_{vel} = R\dot{u}/c$. For a particular frequency of oscillation,

$$E_{rad}/E_{vel} = uR\nu/c^2 = (u/c)(R/\lambda). \tag{250}$$

For the near zone, $R \leq \lambda$, the velocity field dominates, whilst for the far zone $R \geq \lambda c/u$, the radiation field dominates.

When $\beta << 1$,

$$\mathbf{E}_{rad} = (q/c)[\hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \dot{\vec{\beta}})/R] \tag{251}$$

and

$$\mathbf{B}_{rad} = [\hat{\mathbf{n}} \times \mathbf{E}_{rad}]. \tag{252}$$

Then

$$E_{rad} = B_{rad} = \frac{q\dot{\beta}}{Rc} sin\Theta, \tag{253}$$

where Θ is the angle between $\vec{\beta}$ and $\hat{\mathbf{n}}$. The Poynting flux is given by

$$\frac{dW}{dAdt} = S = \frac{c}{4\pi} E_{rad}^2 = \frac{c}{4\pi} \frac{q^2 \dot{\beta}^2}{R^2 c^2} sin^2\Theta. \tag{254}$$

This is directed along $\hat{\mathbf{n}}$. Using $dA = R^2 d\Omega$ we have

$$\frac{dW}{dtd\Omega} = \frac{q^2 \dot{\beta}^2}{4\pi c} sin^2\Theta. \tag{255}$$

The total power emitted is then

$$dW/dt = \frac{q^2 \dot{\beta}^2}{2c} \int_{-1}^{1} (1 - cos^2\Theta d(cos\Theta)) = \frac{2q^2 \dot{\beta}^2}{3c}, \tag{256}$$

which is Larmor's formula for emission from a single particle. (fig3.3)

Note that as a function of angle, there is a dipole pattern: no radiation emitted along the direction of acceleration. The maximum is emitted perpendicular to the acceleration. Also, the instantaneous direction of $\mathbf{E}_{rad}$ is determined by $\dot{\vec{\beta}}$ and $\mathbf{n}$ and 1-D acceleration means linear polarization in the $\dot{\vec{\beta}}$ and $\mathbf{n}$ plane.

Dipole Approximation

For a system of particles, differences in the retarded times can be ignored if the time scales for a system to change are $\tau >> L/c$, the light crossing time. Since this also characterizes a frequency of radiation, the equivalent condition for ignoring retarded time differences is if $L << c/\omega = \lambda$ Since the time for variation also characterizes the time for change in velocity of a particle, a long time for change in velocity over a particle orbit of radius l implies $\tau \sim l/u >> L/c$. But $l \leq L$, so $u << c$ is also an expression of the same limit.

Thus we can use the non-relativistic radiation fields in these regimes.

$$\mathbf{E}_{rad} = \sum_a \frac{q_a}{c}[\hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \dot{\vec{\beta}})/R_a]. \tag{257}$$

For large distances from the source, $R_a \sim R_0$, where R_0 can be defined as the distance from the geometric centroid of the particle system to the observer. Then

$$\mathbf{E}_{rad} = [\frac{1}{c^2 R_0}\hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \ddot{\mathbf{d}})], \tag{258}$$

where $\mathbf{d} = \sum_a q_a \mathbf{r}_a \simeq \int \rho_c \mathbf{r} d^3\mathbf{r}$, where the latter similarity follows for a nearly continuous charge distribution. We have

$$\frac{dP}{d\Omega} = \frac{\ddot{\mathbf{d}}^2}{4\pi c^3} sin^2\Theta \tag{259}$$

and

$$P = \frac{2\ddot{\mathbf{d}}^2}{3c^3} \tag{260}$$

For a non-relativistic dipole moment, the energy emitted per unit frequency is (see text).

$$dW/d\omega = 8\pi\omega^4 |\hat{d}(\omega)|^2 sin^2\Theta. \tag{261}$$

It can be shown by that the dipole term represents the first term in the small expansion parameter L/λ. (or $L\omega/c \sim (L/l)(u/c) << 1$), where l is the particle orbit scale and u is the particle velocity. As the system gets more and more relativistic, the lowest order expansion is not sufficient. The frequencies which show up in the radiation spectrum are not only that associated with the dipole oscillation of the particles, but higher harmonics, where the dipole produces the primary frequency ω_0 but and the quadrupole etc. produce the higher integer harmonics.