LECTURE 6 Radiation Spectrum

The radiation spectrum (histogram of energy as a function of frequency) cannot be measured at an instant of time because of the "uncertainty" relation $\delta\omega\delta t \gtrsim 1$. True for waves in general. Instead we will eventually consider radiation received from a fixed location averaged over a "long" time.

Consider finite radiation pulse and consider the scalar $E(t) = \hat{\mathbf{a}}_1 \cdot \mathbf{E}$ pulling out one component of $\mathbf{E}$. We then have, using Fourier transforms,

$$\tilde{E}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} E(t) e^{i\omega t} dt, \tag{163}$$

so that

$$E(t) = \int_{-\infty}^{\infty} \tilde{E}(\omega) e^{-i\omega t} d\omega, \tag{164}$$

also note

$$\tilde{E}(-\omega) = \tilde{E}^*(\omega). \tag{165}$$

The $\tilde{E}$ has the power spectrum info. Write the energy/(time· Area) as

$$\frac{dW}{dt dA} = cE(t)^2/4\pi \tag{166}$$

from the result above for the Poynting flux. For the total energy per area we have, integrating

$$\frac{dW}{dA} = c\int_{-\infty}^{\infty} E^2(t) dt/4\pi. \tag{167}$$

But

$$= c\int_{-\infty}^{\infty} E(t)^2 dt/4\pi = (c/2)\int_{-\infty}^{\infty} |\tilde{E}(\omega)|^2 d\omega \tag{168}$$

by Parsevals theorem relating the Fourier transform of a product to the product of the Fourier transforms. Further, since $|\tilde{E}(\omega)|^2 = |\tilde{E}(-\omega)|^2$, we take the integral from 0 and multiply by 2 to obtain:

$$dW/dA = c\int_0^{\infty} |E(w)|^2 d\omega, \tag{169}$$

and

$$\frac{dW}{dA d\omega} = c|E(w)|^2. \tag{170}$$

But this is the total energy per frequency per area for the entire pulse. If pulse repeats over the time T or if it is more or less stationary for long times, we can determine the spectrum for some finite part of an "infinitely" long pulse, based on the results for finite pulse. (The finite pulse issue enters to ensure that the integrals above are finite). Spectrum is not useful if the characteristic time changes $T \leq 1/\delta\omega$, where $\delta\omega$ is the characteristic frequency resolution. We can write

$$\frac{dW}{dAd\omega T} \simeq \frac{dW}{dAd\omega dt} = c|E(w)|^2/T, \tag{171}$$

which is a power spectrum per unit area for the duration T, assuming that the signal has approximately same properties over that duration. (fig 7)

Elliptical Polarization, Stokes Parameters for Monochromatic Waves

Plane waves are linearly polarized. Waves of arbitrary polarization can be generated from a linear combination of two orthogonal linearly polarized waves. Consider the electric field

$$\mathbf{E} = (E_1\hat{\mathbf{x}} + E_2\hat{\mathbf{y}})e^{-iwt}. \tag{172}$$

We can write the complex amplitudes

$$E_1 = \varepsilon_1 e^{i\phi_1} \tag{173}$$

and

$$E_2 = \varepsilon_2 e^{i(\pi/2+\phi_2)} \tag{174}$$

so

$$\mathbf{E} = \varepsilon_1\hat{\mathbf{x}}e^{i(\phi-wt)} + \varepsilon_2\hat{\mathbf{y}}e^{i(\pi/2+\phi-wt)}. \tag{175}$$

The real part is thus

$$E_x = \varepsilon_1 cos(\omega t - \phi_1) \tag{176}$$

$$E_y = \varepsilon_2 sin(\omega t - \phi_2). \tag{177}$$

(fig 8)

These equations show that the electric field traces out an ellipse, as we will now see.

The general equation for an ellipse can be written as

$$(X/a)^2 + (Y/b)^2 = 1. \tag{178}$$

If we write

$$E_{x'} = \varepsilon_0 cos\beta coswt \tag{179}$$

and

$$E_{y'} = -\varepsilon_0 sin\beta coswt, \tag{180}$$

for $-\pi/2 < \beta < \pi/2$. Then these satisfy (178) using $X = E_{x'}$, $Y = E_{y'}$, $a = \varepsilon_0 cos\beta$ and $b = \varepsilon_0 sin\beta$. When the wave is propagating toward us, the ellipse is traced out clockwise for $0 < \beta < \pi/2$ and counterclockwise for $-\pi/2 < \beta < 0$. These represent right handed or left handed polarization respectively (negative and positive helicity). When $\beta = \pm\pi/4$ the ellipse becomes a circle and we have circular polarization. For $\beta = 0, \pm\pi/2$ we have linear polarization.

To show that (176) and (177) also form an ellipse, perform a rotation on the x', y' components (179) and (180) (multiply by the rotation matrix) so that we have

$$E_x = \varepsilon_0(cos\beta cos\chi cos\omega t + sin\beta sin\chi sin\omega t) \tag{181}$$

and

$$E_y = \varepsilon_0(-cos\beta sin\chi cos\omega t + sin\beta sin\chi sin\omega t). \tag{182}$$

Then these match (176) and (177) if we set

$$\varepsilon_1 cos\phi_1 = \varepsilon_0 cos\beta cos\chi \tag{183}$$

$$\varepsilon_1 sin\phi_1 = \varepsilon_0 sin\beta sin\chi \tag{184}$$

$$\varepsilon_1 cos\phi_2 = -\varepsilon_0 cos\beta sin\chi \tag{185}$$

$$\varepsilon_2 sin\phi_2 = \varepsilon_0 sin\beta cos\chi. \tag{186}$$

We can solve for $\varepsilon_0, \beta, \chi$ given ε_1, ϕ_1, ε_2, ϕ_2. This is normally done with the monochromatic Stokes Parameters:

$$I = \varepsilon_0^2 = \varepsilon_1^2 + \varepsilon_2^2 \tag{187}$$

$$Q = \varepsilon_0^2 cos2\beta cos2\chi = \varepsilon_1^2 - \varepsilon_2^2 \tag{188}$$

$$U = \varepsilon_0^2 cos2\beta sin2\chi = 2\varepsilon_1\varepsilon_2 cos(\phi_1 - \phi_2). \tag{189}$$

$$V = \varepsilon_0^2 sin2\beta = 2\varepsilon_1\varepsilon_2 sin(\phi_1 - \phi_2). \tag{190}$$

Pure elliptical polarization (polarization of monochromatic waves) is controlled by the 3 parameters $(\varepsilon_0, \chi, \beta)$ but there are 4 Stokes parameters. This is because they are related by

$$I^2 = Q^2 + U^2 + V^2. \tag{191}$$

Further algebra gives

$$\varepsilon_0^2 = I, \tag{192}$$

highlighting that I is related to the intensity of the wave. (Recall our calculation of the energy density of a plane wave above). We have

$$sin 2\beta = V/I \tag{193}$$

and thus V measures the circularity of the wave, related to the ratio of the power of the linearly independent components. If $\beta = 0$, then the wave is linearly polarized and $V = 0$. If $\beta = \pi/4$, $V/I = 1$ implying circular polarization. Finally,

$$\tan 2\chi = U/Q, \tag{194}$$

so that U or Q relate to the orientation of the ellipse that the electric field vector traces when propagation to a given set of axes. However for circular polarization $Q = U = 0$ and $I = V$ from (193), or (191).

Stokes Parameters for Quasi Monochromatic Waves

In general, the observed emission is a superposition of wave components with different polarizations and different frequencies. The waves are generally not monochromatic. The amplitude and phase for a given frequency can change with time. We have

$$E_1(t) = \varepsilon_1(t)e^{i\phi_1(t)} \tag{195}$$

and

$$E_2(t) = \varepsilon_2(t)e^{i(\pi/2-\phi_2(t))}. \tag{196}$$

Assume that the time variation is relatively slow, such that over a time period of $1/\omega$ the wave looks fully polarized, but such that for $\delta t >> 1/\omega$ the amplitude and phases can vary. Then $\delta\omega\delta t \sim 1$ implies $\delta\omega << \omega$ and so we call this limit a quasi-monochromatic wave of bandwidth $\delta\omega$ and coherence time δt.

Most waves are measured in the time averaged sense by a detector. In addition, measuring devices often split waves into a linear combinations of two independent electric field polarization components. Assuming any time for the wave to pass through measurement apparatus is short compared to the coherence time δt, then the splitting of waves into the two components is a linear transformation of the form

$$E_{x'} \equiv E_1' = \lambda_{11}E_1 + \lambda_{12}E_2 \tag{197}$$

$$E_{y'} \equiv E_2' = \lambda_{21}E_1 + \lambda_{22}E_2 \tag{198}$$

Where the λs represent the effect of the measuring apparatus. The time average sum of the squares of the x' and y' components are measured. Using the identity (160) we have for the x' component

$$\langle Re(\mathbf{E}^*_{x'}\mathbf{E}_{x'})\rangle = 2\langle [ReE_1'e^{-iwt}]^2\rangle = |\lambda_{11}^2|\langle E_1E_1^*\rangle + |\lambda_{12}^2|\langle E_2E_2^*\rangle + \lambda_{11}\lambda_{12}^*|\langle E_1E_2^*\rangle + \lambda_{12}\lambda_{11}^*|\langle E_2E_1^*\rangle, \tag{199}$$

where the terms with the residual e^{-iwt} have been dropped assuming the quasi-monochromatic approximation, and we only consider averages over longer times.

The λs are known features of the apparatus. The four different terms which represent averages of the radiation field can be written in terms of Stokes parameters

$$I = \langle E_1E_1^*\rangle + \langle E_2E_2^*\rangle = \langle \varepsilon_1^2 + \varepsilon_2^2\rangle \tag{200}$$

$$Q = \langle E_1E_1^*\rangle - \langle E_2E_2^*\rangle = \langle \varepsilon_1^2 - \varepsilon_2^2\rangle \tag{201}$$

$$U = \langle E_1E_2^*\rangle + \langle E_2^*E_1^*\rangle = \langle 2\varepsilon_1\varepsilon_2 cos(\phi_1 - \phi_2)\rangle \tag{202}$$

$$V = -i(\langle E_1^*E_2^*\rangle - \langle E_2^*E_1^*\rangle) = \langle 2\varepsilon_1\varepsilon_2 sin(\phi_1 - \phi_2)\rangle. \tag{203}$$

These are the generalized Stokes parameters which reduce to the monochromatic ones when the amplitudes and phases are time independent. We cannot distinguish waves with the same Stokes parameters. But here,

$$I^2 \geq Q^2 + U^2 + V^2. \tag{204}$$

Physically this makes sense when the wave is not completely elliptically polarized, and if there were part of the radiation which is not polarized then this would be an underestimate. The result follows from the Schwartz inequality

$$\langle E_1E_2^*\rangle\langle E_2E_1^*\rangle \leq \langle E_1E_1^*\rangle\langle E_2E_2^*\rangle, \tag{205}$$

where the equality holds when the ratio of E_1/E_2 is independent of time. (I leave as an exercise to show this from (195) and (196)). A completely unpolarized wave has no preferred orientation so $\langle \varepsilon_1^2 \rangle = \langle \varepsilon_2^2 \rangle$ and $Q = U = V = 0$.