LECTURE 5

The first four lectures discussed general properties of radiation transfer and the some aspects of the specific case of a blackbody. The rest of the course builds up to trying to understand the physics behind the emission, absorption, and scattering coefficients of real systems, and how this relates to the material constituents.

Basic E&M

The electromagnetic force on a particle moving with velocity $\mathbf{v}$ and charge q is

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}/c), \tag{126}$$

where $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic fields. The rate of work done is

$$\mathbf{v} \cdot \mathbf{F} = d_t(mv^2/2) = q\mathbf{v} \cdot \mathbf{E}. \tag{127}$$

For many particles the force density is

$$\mathbf{f} = \rho_c \mathbf{E} + \mathbf{j} \times \mathbf{B}/c, \tag{128}$$

where $\rho_c = \sum_a q_a n_a$ is the total charge density (sum of charge times number density over particle type) and $\mathbf{j} = \sum_a q_a n_a \mathbf{v}_a$. Rate of work per unit volume is then

$$\sum_a \mathbf{f}_a \cdot \mathbf{v}_a = \sum_a q_a n_a \mathbf{v}_a \cdot \mathbf{E} = \mathbf{j} \cdot \mathbf{E}. \tag{129}$$

The rate of work done per unit volume is the rate of change of mechanical energy density:

$$d_t u_{mech} = \mathbf{j} \cdot \mathbf{E}. \tag{130}$$

Maxwell's Equations

$$\nabla \cdot \mathbf{D} = 4\pi \rho_c \tag{131}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{132}$$

$$\nabla \times \mathbf{E} + c^{-1} \partial \mathbf{B}/\partial t = 0 \tag{133}$$

$$\nabla \times \mathbf{H} - c^{-1} \partial \mathbf{D}/\partial t = 4\pi \mathbf{j}/c. \tag{134}$$

Note that $\mathbf{D} = \epsilon\mathbf{E}$. This comes from

$$\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P} \tag{135}$$

where $\mathbf{P} = \sum_a n_a \langle \mathbf{p}_a \rangle$ is the electric polarization= dipole moment density averaged over spatial region. (Note dipole moment is $\int \rho_c \mathbf{x}' d^3\mathbf{x}'$). For linear response, we take $\mathbf{P} \propto \mathbf{E}$ or $\mathbf{P} = \chi\mathbf{E}$ so

$$\mathbf{D} = (1 + 4\pi\chi)\mathbf{E} \equiv \epsilon\mathbf{E}, \tag{136}$$

where ϵ is the permittivity=dielectric constant.

Similarly for $\mathbf{H}$, we have the magnetization $\mathbf{M} = \sigma_a n_a \langle \mathbf{m_a} \rangle$, where the dipole moment is $\mathbf{m} = \int \mathbf{x}' \times \mathbf{j} d^3x'$. This leads to $\mathbf{B} = \mu\mathbf{H}$, where μ is the magnetic permeability.

Conservation laws of E& M

The law for charge conservation is

$$\partial_t \rho + \nabla \cdot \mathbf{j} = 0 \tag{137}$$

.

Let us now derive the Poynting theorem of electromagnetic energy conservation. Use Maxwell's equation (134) and dot with $\mathbf{E}$. This gives

$$\mathbf{j} \cdot \mathbf{E} = (1/4\pi)[c(\nabla \times \mathbf{H}) \cdot \mathbf{E} - \mathbf{E} \cdot \partial_t \mathbf{D}]. \tag{138}$$

Now use a vector identity on the first term on the right:

$$\mathbf{E} \cdot \nabla \times \mathbf{H} = \mathbf{H} \cdot \nabla \times \mathbf{E} - \nabla \cdot (\mathbf{E} \times \mathbf{H}). \tag{139}$$

(prove using index tensor notation and ϵ_{ijk} tensor). Use Maxwell's, equation (133), and assume the dielectric and magnetic permeability are independent of time to get

$$(1/4\pi)\partial_t(\epsilon E^2 + B^2/\mu) = -\mathbf{j} \cdot \mathbf{E} - (c/4\pi)\nabla \cdot (\mathbf{E} \times \mathbf{H}). \tag{140}$$

This is Poyntings theorem. If we integrate we can see its true meaning:

$$(1/4\pi)\partial_t \int (\epsilon E^2 + B^2/\mu) d^3x = -\int \mathbf{j} \cdot \mathbf{E} d^3x - (c/4\pi) \int (\mathbf{E} \times \mathbf{H}) \cdot d\mathbf{A}, \tag{141}$$

where we use the divergence theorem on the last term. This shows that the change in electromagnetic energy equals that lost to work plus that flowing out the boundary. $\mathbf{S} \equiv c\mathbf{E} \times \mathbf{H}/4\pi$ is the Poynting flux. It is the energy flux vector (energy per unit time per area). We could have instead put the material currents into $\mathbf{j}$ so that the constants do not appear in the time derivative term, and instead redefine $\mathbf{j}$.

Definition of Radiation Field

For electro/magnetostatic fields $\mathbf{E}$ and $\mathbf{B}$ fall as $1/r^2$. Thus at large distances $\int \mathbf{S} \cdot \mathbf{dA}$ goes as $1/r^4$.

However for time varying fields, we will find that there are components of E, B that are $\propto 1/r$, and so the Poynting flux from these terms is finite are large distances. This is the definition of the radiation field.

Plane Waves

Maxwell's equation in vacuum

$$\nabla \cdot \mathbf{E} = 0 \tag{142}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{143}$$

$$\nabla \times \mathbf{E} + c^{-1}\partial_t \mathbf{B} = 0 \tag{144}$$

$$\nabla \times \mathbf{B} - c^{-1}\partial_t \mathbf{E} = 0. \tag{145}$$

Then taking curl of (144), using (145) and the vector identity $\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E}$ we get

$$\nabla^2 \mathbf{E} - (1/c^2)\partial^2 \mathbf{E}/\partial t^2 = 0. \tag{146}$$

Assume

$$\mathbf{E} = \hat{\mathbf{a}}_1 E_0 e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)} \tag{147}$$

and

$$\mathbf{B} = \hat{\mathbf{a}}_2 B_0 e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)}, \tag{148}$$

where $\hat{\mathbf{a}}_1$ and $\hat{\mathbf{a}}_2$ are unit vectors and $\mathbf{k}$ is the wave vector. These are traveling wave solutions because the argument of the exponential is constant for surfaces which propagate in time in the $\mathbf{n}$ direction.

Plugging back into Maxwell's equations we have

$$i\mathbf{k} \cdot \hat{\mathbf{a}}_1 E_0 = 0, \tag{149}$$

$$i\mathbf{k} \cdot \hat{\mathbf{a}}_2 B_0 = 0 \tag{150}$$

and

$$i\mathbf{k} \times \hat{\mathbf{a}}_1 E_0 = i\omega \hat{\mathbf{a}}_2 B_0/c, \tag{151}$$

$$i\mathbf{k} \times \hat{\mathbf{a}}_2 B_0 = -i\omega \hat{\mathbf{a}}_1 E_0/c. \tag{152}$$

These imply that $\mathbf{k} \perp \hat{\mathbf{a}}_1, \hat{\mathbf{a}}_2$ and $\hat{\mathbf{a}}_2 \perp \hat{\mathbf{a}}_1$. They also imply

$$E_0 = B_0\omega/kc, \quad B_0 = E_0\omega/kc, \tag{153}$$

so

$$\omega^2 = c^2k^2 \tag{154}$$

and $E_0 = B_0$. The phase velocity $v_{ph} \equiv w/k = c$. Group velocity is also c because there is no medium and the group velocity depends on the index of refraction n:

$$v_g = \frac{c}{n(\omega) + \omega(\partial n/\partial\omega)}, \tag{155}$$

but for the vacuum $n = 1$ and is independent of ω so $v_g = v_{ph} = c$.

Wave Energy Flux and Density

First note that **E** and **B** vary sinusoidally in time, but in astro, we measure time averages for measurements of interest. (Measure time is longer than wave period).

First we need math result. We are going to calculate products of complex quantities (the electromagnetic fields) but we measure only the real part. So we are interested in the products of the real parts. Take two complex quantities $Q_1 = E_0e^{iwt}$ and $Q_2 = B_0e^{iwt}$. Then

$$Re(Q_1) = \frac{1}{2}(Q_1 + Q_1^*) = \frac{1}{2}(E_0e^{i\omega t} + E_0^*e^{-i\omega t}), \tag{156}$$

where $*$ indicates complex conjugate. Similarly,

$$Re(Q_2) = \frac{1}{2}(B_0e^{i\omega t} + B_0^*e^{-i\omega t}). \tag{157}$$

(These follow from recognizing any complex number can be written as $a + bi$ with a and b real.) Then

$$Re(Q_1)Re(Q_2) = (1/4)(E_0B_0e^{2i\omega}t + E_0B_0^* + E_0^*B_0 + E_0^*B_0^*e^{-2i\omega t}) \tag{158}$$

Take time average, then the last terms with exponentials vanish. Thus we have

$$Re(Q_1)Re(Q_2) = \frac{1}{4}(E_0B_0^* + E_0^*B_0). \tag{159}$$

But only the real part of each term contributes. Thus

$$Re(Q_1)Re(Q_2) = \frac{1}{4}Re(E_0B_0^* + E_0^*B_0) = \frac{1}{2}Re(E_0^*B_0) = \frac{1}{2}Re(E_0B_0^*). \tag{160}$$

Applying this to the time averaged Poynting flux for waves gives

$$\langle \mathbf{S} \rangle = \frac{c}{4\pi}\langle Re(\mathbf{E})\times Re(\mathbf{B})\rangle = \frac{c}{4\pi}\hat{\mathbf{k}}\langle |Re(\mathbf{E})||Re(\mathbf{B})|\rangle = \frac{c}{8\pi}\hat{\mathbf{k}}Re(E_0 B_0^*) = \frac{c}{8\pi}|E_0|^2 = \frac{c}{8\pi}\hat{\mathbf{k}}|B_0|^2 \tag{161}$$

For the electromagnetic energy density, we find

$$= \frac{1}{16\pi}Re(E_0 E_0^* + B_0 B_0^*) = \frac{1}{16\pi}|E_0|^2 + \frac{1}{16\pi}|B_0|^2 = \frac{1}{8\pi}|B_0|^2, \tag{162}$$

so $\langle u_{mag}\rangle\hat{\mathbf{k}} = \langle \mathbf{S}\rangle/c$, implying energy flows at speed c for vacuum plane waves (not surprising).