LECTURE 4

Scattering + Absorption

Consider thermal material with absorption coefficient α_ν describing thermal emission and a scattering coefficient σ_ν describing coherent isotropic scattering. The transfer equation is then

$$dI_\nu/ds = -\alpha_\nu(I_\nu - B_\nu) + \sigma_\nu(J_\nu - I_\nu). \tag{86}$$

The source function

$$S_\nu = (\alpha_\nu B_\nu + \sigma_\nu J_\nu)/(\alpha_\nu + \sigma_\nu), \tag{87}$$

which is a weighted average of the thermal and scattering source functions. Can define $d\tau_\nu = (\sigma_\nu + \alpha_\nu)ds$ as the differential optical depth to both scattering and absorption, with extinction coefficient $(\sigma_\nu + \alpha_\nu)$.

Note limits that $S_\nu = B_\nu$ for $J_\nu = B_\nu$ (e.g. at large optical depths) and $S_\nu < B_\nu$ for regions where $J_\nu = 0$.

Random Walk for Combined Scattering and Absorption

Mean free path to both scattering and absorption

$$l_\nu = 1/(\sigma_\nu + \alpha_\nu). \tag{88}$$

The probability that a free photon will be absorbed rather than scattered is

$$\epsilon_\nu = \alpha_\nu/(\alpha_\nu + \sigma_\nu), \tag{89}$$

and the probability for scatter is the single scattering albedo, and equals $1 - \epsilon_\nu$.

Consider again a homogeneous infinite medium: For large number scatterings, the mean number of free paths is the inverse of the probability of absorption so the RMS net displacement of the photon is

$$l_*^2 = N l_\nu^2 = l_\nu^2/\epsilon_\nu = 1/(\alpha_\nu(\alpha_\nu + \sigma_\nu)). \tag{90}$$

The l_* is the net displacement for a photon from creation to absorption. It is the "effective diffusion length."

The "effective optical depth" is

$$\tau_* = R/l^* = [\tau_a(\tau_a + \tau_s)]^{1/2}, \tag{91}$$

where R is scale of system. When $\tau_* << 1$ we have a transluscent medium. Why not transparent?: because there can still be a lot of scattering. A transparent medium would also be translucent by this definition, the former being a stronger constraint in that addition to not being absorbed, there would be little scattering. However, the analysis that led to (91) above really assumes multiple scatterings.

For $\tau_* << 1$ the monochromatic luminosity from a thermal source would just be determined by the emission coefficient:

$$\tilde{L}_\nu = 4\pi j_\nu V = 4\pi \alpha_\nu B_\nu V. \tag{92}$$

For $\tau_* >> 1$ thermal equilibrium between matter and radiation is reached, and in addition, the emission comes only from a thin layer of order l_*, on surface of area A. Thus

$$L_\nu \simeq 4\pi \alpha_s B_\nu l_* A = 4\pi \epsilon_\nu^{1/2} B_\nu A. \tag{93}$$

This does not quite reduce to a black-body for zero scattering (i.e. $\epsilon_\nu = 1$), so numerical factor is not quite accurate. One part of the reason is that $l_*^2 = l_x^2 + l_y^2 + l_z^2 \simeq 3l_x^2$, and since the radial direction in a star for example, corresponds to only 1 of the three directions, rather than use l_* in (93) we should use $l_*/3^{1/2}$.

Note that at $\tau_* > 1$ matter and radiation approach thermal equilibrium, thus l_* is also called the "effective thermalization length."

Radiative Diffusion Equation

Source function becomes black body at large effective optical depths for homogeneous systems. Actually homogeneity is not realistic though local homogeneity may be OK, as in the Sun or stars.

Consider "plane parallel" assumption: properties of an atmosphere depend only on depth. Assume planar geometry. Intensity then depends on angle from normal θ such that $ds = dz/cos\theta \equiv dz/\mu$ and

$$\mu \partial I_\nu(z,\mu)/\partial z = -(\alpha_\nu + \sigma_\nu)(I_\nu - S_\nu). \tag{94}$$

If we consider a region deep in the atmosphere, the intensity varies slowly on scale of mean free path. Intensity approximates a blackbody but with small variation. We describe this as follows: To zeroth order in $cos\theta$ we have

$$I_\nu^{(0)} = J_\nu^{(0)} = S_\nu^{(0)} = B_\nu(T). \tag{95}$$

Then we iterate and plug the next higher term back into (94), that is

$$I_\nu^{(1)} \simeq B_\nu(T) - \frac{\mu}{\alpha_\nu + \sigma_\nu}\frac{\partial B(T)}{\partial z}. \tag{96}$$

The flux is then

$$F_\nu(z) \simeq \int I_\nu^{(1)}(z,\mu) cos\theta d\Omega = 2\pi \int_{-1}^{1} I_\nu^{(1)}(z,\mu)\mu d\mu = -\frac{4\pi}{3(\alpha_\nu+\sigma_\nu)}\frac{\partial B_\nu(T)}{\partial T}\frac{\partial T}{\partial z}. \tag{97}$$

Integrating over all frequencies, we have

$$F(z) \simeq -\frac{4\pi}{3}\frac{\partial T}{\partial z}\int_0^\infty (\alpha_\nu+\sigma_\nu)^{-1}\frac{\partial B_\nu}{\partial T}d\nu \tag{98}$$

Note that

$$\int_0^\infty (\alpha_\nu+\sigma_\nu)^{-1}\frac{\partial B_\nu}{\partial T}d\nu = \frac{(\alpha_\nu+\sigma_\nu)^{-1}\frac{\partial B_\nu}{\partial T}d\nu}{\int_0^\infty \frac{\partial B_\nu}{\partial T}d\nu}\int_0^\infty \frac{\partial B_\nu}{\partial T}d\nu \equiv \frac{1}{\alpha_R}\int_0^\infty \frac{\partial B_\nu}{\partial T}d\nu, \tag{99}$$

where α_R is the Rosseland mean absorption coefficient α_R. But

$$\int_0^\infty \frac{\partial B_\nu}{\partial T}d\nu = \partial B(T)/\partial T = 4\sigma T^3/\pi, \tag{100}$$

using the Stefan-Boltzmann law. Thus

$$F(z) = \frac{-16\sigma T^3}{3\alpha_R}\frac{\partial T}{\partial z}. \tag{101}$$

Key properties: 1) Flux increases with decreasing temperature gradient. Makes sense since isotropy gets progressively violated at larger radii (nearer to the stellar surface). 2) Radiative energy transport is similar to heat conduction.

Absorption Lines vs. Emission lines, & Limb Darkening

From (87) and (89) above we have the source function

$$S_\nu = (1-\epsilon_\nu)J_\nu + \epsilon_\nu B_\nu. \tag{102}$$

Ignore scattering for the moment, and assume thermal emission. We are back to

$$S_\nu = B_\nu. \tag{103}$$

Recall solution of radiative transfer equation in this case: we have from (46) for $I_\nu(0) = 0$

$$I_\nu = B_\nu(1 - e_\nu^{-\tau_\nu}). \tag{104}$$

For small optical depths, this goes as

$$I_\nu \simeq \tau_\nu B_\nu, \tag{105}$$

so the intensity is larger in lines where the optical depth is larger.

But for a source with $\tau \geq 1$, we do not see the whole source, only the layers which are optically thin. The temperature decreases outward in a medium whose optical depth decreases with radius, so we then are looking at layers of lower temperature for lines, since the lines' optical depth is larger for a given distance of ray travel when compared to the continuum. Thus we would see absorption lines for large optical depths in a system with radially decreasing temperature.

For a point source behind a emitting and absorbing region in local thermal equilibrium we would have

$$I_\nu = I_\nu(0)e^{-\tau_\nu} + B_\nu(1 - e^{-\tau_\nu}). \tag{106}$$

When $I_\nu(0)$ is brighter than the foreground, $(I_\nu(0) - B_\nu(T))e^{-\tau_\nu}$ decreases with increasing optical depth. Since we see into the star less deeply for the line frequencies, less emission will come from the deeper regions, and more from the outer regions.

A related issue explains limb darkening. White light image of Sun shows edges darker than center because the edge view penetrates less deeply into Sun for given optical depth, and thus sees emission coming from regions of lower temperature than the face on view.

Eddington Approximation

Rosseland approximation employed the fact that system approached black body at large effective optical depths. Eddington only assumes isotropy, thus is good at smaller optical depths. We write a power series in $\mu = cos\theta$

$$I_\nu(\tau, \mu) = a_\nu(\tau) + b_\nu(\tau)\mu. \tag{107}$$

If we take moments of this equation we obtain

$$J_\nu = \frac{1}{2}\int_{-1}^{1} I_\nu d\mu = a \tag{108}$$

$$H_\nu = \frac{1}{2}\int_{-1}^{1} \mu I_\nu d\mu = b/3 \tag{109}$$

$$K_\nu = \frac{1}{2}\int_{-1}^{1} \mu^2 I_\nu d\mu = a/3. \tag{110}$$

The fact that $K_\nu = J_\nu/3$ is the Eddington approximation, and is equivalent to $p_\nu = u_\nu/3$ found in first lecture. The relation is found here for a more general system with slight anisotropy.

Writing $d\tau(z) = -(\alpha_\nu + \sigma_\nu)dz$ we then have

$$\mu dI_\nu/d\tau = I_\nu - S_\nu. \tag{111}$$

Then taking moments in μ and integrating from $-1 \le \mu \le 1$ we get

$$dH_\nu/d\tau = J_\nu - S_\nu. \tag{112}$$

and

$$dK_\nu/d\tau = H = \frac{1}{3}\partial J_\nu/\partial\tau. \tag{113}$$

Combining these last two we obtain

$$d^2 J_\nu/d\tau^2 = 3(J_\nu - S_\nu) = 3\epsilon_\nu(J_\nu - B_\nu), \tag{114}$$

where the latter eq. follows from (87). If ϵ_ν does not depend on z, we can write this

$$d^2 J_\nu/d\tau_*^2 = (J_\nu - B_\nu). \tag{115}$$

using the effective optical depth. These are forms of the radiative diffusion equation. If we have $B_\nu(\tau)$, we can solve for J_ν and get S_ν from (87). Then we have what we need to solve (111).

Text goes onto discuss two-stream approximation, for obtaining the two boundary conditions. Assume the radiation propagates at two angles $I_\nu^\pm = I_\nu(\tau, \mu = \pm 1/3^{1/2})$. Then plug into the moment equations (108) and (109) to relate $H, J, I^\pm$. The boundary conditions can then be found from

$$I_\nu^+ = J_\nu + \frac{1}{\sqrt{3}}\partial J_\nu/\partial\tau \tag{116}$$

and

$$I_\nu^- = J_\nu - \frac{1}{\sqrt{3}}\partial J_\nu/\partial\tau. \tag{117}$$

The reason for this odd choice of μ is that it reproduces exactly $K_\nu = J_\nu/3$, the Eddington approx.

Moments

Consider integrating the intensity over various vector multiples of $\mathbf{k}$. We have

$$cu_\nu = \int I_\nu d\Omega \tag{118}$$

and

$$\mathbf{F}_\nu = \int \mathbf{k} I_\nu d\Omega \tag{119}$$

$$c\mathbf{P}_\nu = \int \mathbf{k}\mathbf{k} I_\nu d\Omega. \tag{120}$$

From fluid mechanics or E&M, different components of stress tensor refer to rate of momentum transfer across surfaces of different orientations. Rate of momentum transfer per photon across surface of unit normal $\mathbf{k}$ is is $c\mathbf{kp}$. Thus the total rate of momentum transfer is given locally by the radiation stress tensor

$$\mathbf{P} = \sum_{\alpha=1}^{2} \int c\hat{\mathbf{k}}\mathbf{p} f_\alpha d^3p = \sum_{\alpha=1}^{2} \int c\hat{\mathbf{k}}\hat{\mathbf{k}}(h^4\nu^3/c^3) f_\alpha d\Omega d\nu = (1/c) \int \hat{\mathbf{k}}\hat{\mathbf{k}} I_\nu d\Omega d\nu \quad (121)$$

The total derivative of I_ν obeys

$$\partial_t I_\nu + (1/c)\mathbf{k} \cdot \nabla I_\nu = -\alpha_\nu I_\nu + j_\nu - \sigma_\nu I_\nu + \sigma_\nu \int \Phi(\hat{\mathbf{k}}, \hat{\mathbf{k}}') I_\nu(\mathbf{k}) d\Omega' \quad (122)$$

Compare this more general form to (86): In our previous discussions we have ignored the 1st term on the left and the last term on the right. If we integrate over all solid angles of photon paths, we have, for "front back symmetric scattering"

$$\partial_t u_\nu + \nabla \cdot \mathbf{F}_\nu = -c\alpha_\nu u_\nu + 4\pi j_\nu. \quad (123)$$

The next highest moment is obtained by multiplying (122) by $\mathbf{k}$ and integrating over solid angle, assuming (j_ν is independent of $\mathbf{k}$) to obtain

$$\partial_t \mathbf{F}_\nu + c\nabla \cdot \mathbf{P}_\nu = -(\alpha_\nu + \sigma_\nu)\mathbf{F}_\nu. \quad (124)$$

Note that right side represents a radiation force on matter. (Sink for radiation appears as source for matter and vice versa. Removal of radiation momentum means gain in momentum of matter.) Note that to fully solve the system of equations, one needs an infinite set of equation because each equation depends on a higher moment. This is the closure problem. (Also found in turbulence and MHD).

The Eddington approximation, combined with the approximation that the systems evolve on time scales slower than the travel time associated with a mean free path (e.g. ratio is 10^{-28}) for sun, allows ignoring of the time derivative terms.

Note the general form of conservation law:

$$\partial_t(quantity\ density + \nabla \cdot (quantity\ flux) = (quantity\ sources - quantity\ sinks) \quad (125)$$