LECTURE 3

Einstein Coefficients

Kirchoff's law relating emission to absorption for a thermal emitter must involve microscopic physics. Consider system with two energy states with statistical weights g_1 and g_2 respectively. Transition from 2 to 1 is by emission and from 1-2 by absorption. State 1 has energy E and state 2 has energy $E + h\nu$.

1. Define the Einstein A_{21} coefficient as the probability per unit time for spontaneous emission.

2. The absorption of a photon $\propto$ density of photons or the mean intensity at frequency ν_0. Energy difference between two levels is not sharp, but broadened, and so we also need to consider the line profile function ϕ obeying $\int_0^\infty \phi(\nu)d\nu = 1$. This is usually quite narrow. The transition probability per unit time for absorption is

$$B_{12}\bar{J} = B_{12}\int_0^\infty \bar{I}_\nu \phi(\nu)d\nu, \tag{63}$$

where B_{12} is the Einstein B coefficient and $J_\nu = \bar{I}_\nu$, the intensity integrated over solid angle. Here $\bar{J} \sim J_{\nu_0}$ for narrow line profiles.

Derivation of Planck's law led Einstein to include stimulated emission. It can be thought of as negative absorption and has a coefficient B_{21} such that $B_{21}\bar{J}$ is the transition probability per time for stimulated emission, also proportional to the intensity.

Relation between Coefficients

Micro-physical relations that are independent of temperature will hold regardless of whether processes are in thermodynamic equilibrium or not, but we can use case of thermodynamic equilibrium to get at them when possible.

In thermodynamic equilibrium the number of transitions per unit time per volume into state 1 are equal to the transitions out. If n_1 and n_2 are number densities of atoms in states 1 and 2 we have

$$n_1 B_{12}\bar{J} = n_2 A_{21} + n_2 B_{21}\bar{J}. \tag{64}$$

This gives (wrong in text)

$$\bar{J} = \frac{A_{21}/B_{21}}{(n_1/n_2)(B_{12}/B_{21}) - 1}, \tag{65}$$

where

$$n_1/n_2 = (g_1/g_2)e^{h\nu/kT} \tag{66}$$

so

$$\bar{J} = \frac{A_{21}/B_{21}}{(g_1/g_2)(B_{12}/B_{21})e^{h\nu/kT} - 1}. \tag{67}$$

But $J_\nu \sim B_\nu$ for thermodynamic equilibrium so we have the Einstein detailed balance relations which relate atomic properties

$$g_1 B_{12} = g_2 B_{21} \tag{68}$$

and

$$A_{21} = 2(h\nu^3/c^2)B_{21}. \tag{69}$$

These do not depend on temperature and must hold independently of thermodynamic equilibrium. If we can determine any one of the Einstein coefficients we get the other two.

Wien's law follows if you don't include stimulated emission. Recall that was the $h\nu >> kT$ regime, so $n_2 << n_1$ and the stimulated emission term is small.

Absorption and Emission Coefficients in Terms of Einstein Coefficients

Assume that line profile during spontaneous emission is same as $\phi(\nu)$ line profile. Then

$$j_\nu dV d\Omega d\nu dt = (h\nu/4\pi)\phi(\nu) n_2 A_{21} dV d\Omega d\nu dt, \tag{70}$$

so the emission coefficient equals

$$j_\nu = (h\nu/4\pi) n_2 A_{21} \phi(\nu). \tag{71}$$

For the absorption coefficient the total energy absorbed in dV and in dt is

$$h\nu n_1 B_{12}(1/4\pi) dV dt \int I_\nu \phi(\nu) d\nu d\Omega \tag{72}$$

but take $dV = dsdA$ (the cylinder). Then using $dI_\nu = \alpha_\nu I_\nu ds$, we have

$$\alpha_{\nu,abs} = (h\nu/4\pi) n_1 B_{12} \phi(\nu). \tag{73}$$

Stimulated emission can be thought of as inverse absorption so

$$\alpha_\nu = (h\nu/4\pi)(n_1 B_{12} - n_2 B_{21})\phi(\nu). \tag{74}$$

This is the full absorption coefficient.

Transfer Equation in terms of the Einstein Coefficients

$$dI_\nu/ds = -(h\nu/4\pi)(n_1 B_{12} - n_2 B_{21})\phi(\nu) I_\nu - (h\nu/4\pi) n_2 A_{21} \phi(\nu). \tag{75}$$

The source function is j_ν/α_ν and so

$$S_\nu = n_2 A_{21}/(n_2 B_{12} - n_2 B_{21}). \tag{76}$$

Using the Einstein relations above gives

$$\alpha_\nu = (h\nu/4\pi) n_1 B_{12} (1 - g_1 n_2/g_2 n_1)\phi(\nu) \tag{77}$$

and

$$S_\nu = (2h\nu^3/c^2)(g_2 n_1/g_1 n_2 - 1)^{-1}. \tag{78}$$

Masers/Lasers:

For a system in thermal equilibrium we have

$$n_1/n_2 = (g_1/g_2)e^{h\nu/kT} \tag{79}$$

which implies

$$n_1/g_1 > n_2/g_2 \tag{80}$$

But it is possible to pump atoms into the upper state to reverse this relation. Then (74) is negative, implying negative absorption. The intensity actually increases along the ray path exponentially with the optical depth.

Many interesting maser sources in astrophysics. NGC4258 is a galaxy for which $H_2 0$ masers (22.235, 321..GHz) in the central region trace a Keplerian warped accretion disk, which has provided some of the best evidence for black holes in nature.

Also, OH/IR sources: large cool giant star or supergiant star losing mass rapidly in winds and detected only in IR or masers in OH (1.665 Ghz). Can use OH masers in these sources to estimate their distances. The maser emission line from the expanding wind shell has a red and blue component because the near side part of the wind is moving at us, and the far side is moving away (centered at the star). But variability in the stellar wind will show up with a delay between the red and blue parts of the line due to the cross time across the source. Since radio telescopes can resolve the angular diameter, we can get the source diameter and thus the distance. More explicitly, $d = r/\theta = ct/\theta$, where d is distance, r is the "wind span" t is the measured time delay and θ is the measured angular diameter.

Masers would be an interesting paper topic.

Scattering

So far we have ignored scattering. Scattering can be considered an emission process that depends on the amount of radiation incident upon the medium doing the scattering. (Contrast: Thermal radiation does not depend on incident radiation.)

Consider electron scattering (scattering of photons by electrons). Here we assume isotropic, coherent (elastic) scattering. More elaborate coverage of these assumptions later.

The scattering emission coefficient is found by equating the power "absorbed"(think of scattering as absorption and immediate re-emission) per unit volume and frequency to the power emitted.

$$j_\nu = \sigma_\nu J_\nu. \tag{81}$$

Where σ_ν is the scattering coefficient (not to be confused with the cross section or Stefan Boltzmann constant!) This gives $S_\nu = J_\nu$ and

$$dI_\nu/ds = -\sigma_\nu(I_\nu - J_\nu). \tag{82}$$

The solution cannot be easily extracted from (45). Consider instead thinking of the scattering, absorption and emission processes in probabilistic terms for a single photon.

Consider a photon in an infinite homogeneous scattering region. Displacement of photon after N free paths is

$$\mathbf{R} = \sum_{a=1}^{N} \mathbf{r}_{(a)}. \tag{83}$$

where the sum is over free displacements, not vector indices. Mean square photon displacement is given by

$$l_*^2 = \langle \mathbf{R}^2 \rangle = \langle \sum_{a=1}^{N} \mathbf{r}_{(a)} \cdot \mathbf{r}_{(a)} \rangle, \tag{84}$$

as the cross terms vanish for isotropic scattering when there are a large number of scatterings. (Book does not mention this requirement.) Each average in this case is a mean free path so we have

$$l_*^2 = Nl^2, \tag{85}$$

which indicates the mean square displacement of the photon. If the medium is finite, then we can determine N. For large optical depths, N is found by setting $l^* = L$, the typical size of the medium. so that $L/l = \tau$, the optical depth to scattering and $N \simeq \tau^2$. This is a large N result only, because in deriving it we assumed that each contribution to the sum on the right of (84) contributes the same amount. This is true only within an error of $\pm 1/N^{1/2}$, so the error is small for large N and large for small N.

For small optical depths the probability for scattering is $1 - e^{-\tau} \sim \tau$, which is equal to the expected number of scatterings in transversing the medium. The reason is that $e^{-\tau}$ is probablity for a photon NOT to scatter in trasversing an optical depth τ. Thus $1 - e^{-\tau}$ is the probability to scatter in traversing and optical depth τ. Thus per optical depth, this gives the number of scatterings.