LECTURE 21

Faraday Rotation

Consider a fixed magnetic field. This means the plasma is anisotropic since the plasma properties will be different perpendicular to and parallel to the magnetic field. We also can expect to find the cyclotron frequency entering the problem for non-relativistic plasmas. Ignoring the particle pressure is called the cold plasma approximation, which we will do.

The dielectric constant is now a tensor, and also waves have different polarizations so in general the coefficient multiplying the $e^{(i\mathbf{k}\cdot\mathbf{x}-\omega t)}$ in the waves just discussed is not a constant.

Lets assume that the fixed field is stronger than the fields associated with the propagating wave. Then

$$m\frac{d\mathbf{v}}{dt} = -e\mathbf{E} - \frac{e}{c}\mathbf{v}\times\mathbf{B} \tag{615}$$

Consider a circularly polarized wave

$$\mathbf{E}(t) = Ee^{-i\omega t}(\hat{\epsilon}_1 \mp i\hat{\epsilon}_2). \tag{616}$$

Assume that the waves propagate along the uniform field, which we take to be in the z-direction, ϵ_3.

We then have

$$\mathbf{v}(t) = \frac{e}{im(\omega \pm \omega_B)}\mathbf{E}(t). \tag{617}$$

Comparing to (598) and (602), we have a new dielectric constant

$$\epsilon_{R,L} = 1 - \frac{\omega_p^2}{\omega(\omega \pm \omega_B)}, \tag{618}$$

where R, L corresponds to $-, +$. These travel with different velocities since the dielectric constant enters the wave velocity. Thus the plane of a linearly polarized wave, which is the superposition of two circularly polarized waves rotates as the wave propagates. That is Faraday rotation. fig 8.1

The phase angle of rotation of the electric field vector can be computed from

$$\phi_{R,L} = \int_0^d k_{R,L} ds, \tag{619}$$

where

$$k_{R,L} = \frac{\omega}{c}\sqrt{\epsilon}_{R,L}. \tag{620}$$

The plane polarized wave gets rotated by an angle $\Delta\theta = 0.5(\phi_R - \phi_L)$. If we assume that $\omega >> \omega_p$ and $\omega >> \omega_B$ we have

$$k_{r,L} = \frac{\omega}{c}\left[1 - \frac{\omega_p^2}{2\omega^2}(1 \mp \omega_B/\omega)\right]. \tag{621}$$

Thus

$$\Delta\theta = \frac{1}{2}\int_0^d (k_R - k_L) ds = \frac{1}{2}\int_0^d (c\omega^2)^{-1}\omega_B\omega_p^2 ds. \tag{622}$$

Then substituting for ω_p and ω_B we have

$$\Delta\theta = \frac{2\pi e^3}{m^2c^2\omega^2}\int_0^d nB_{||} ds = 1.2 \times 10^4 \lambda^2 \int_0^d nB_{||} ds. \tag{623}$$

By measuring the relative rotation at different frequencies we can get info about the magnetic field.

Combining the pulsar dispersion measure and the Faraday rotation, we can get an estimate of the mean magnetic field. Suppose for example that we have a signal from a pulsar measured to have a time delay of $dt_p/d\omega = 10^{-5}\mathrm{s}^2$ and a Faraday rotation that varies with frequency as $d\Delta\theta/d\omega = 2 \times 10^{-4}$sec. Let us assume the measurements are made around frequency of 100MHz and that we don't know the distance of the pulsar. From this we can estimate the mean magnetic field defined as

$$\langle B_{||}\rangle = \frac{\int nB_{||} ds}{\int n ds}. \tag{624}$$

We simply take the derivative of (623) and divide by (613) to get

$$\frac{d\Delta\theta/d\omega}{dt_p/d\omega} \sim 2 \times 10^7 B_{||}\mathrm{s}^{-1}. \tag{625}$$

Substituting $dt_p/d\omega = 10^{-5}\mathrm{s}^2$ and $d\Delta\theta/d\omega = 2 \times 10^{-4}$sec, we get $B_{||} \sim 10^{-6}$ G. Which is typical for the ISM. Note that this is independent of the frequency of emission.

Effects of plasma on high energy radiation

The radiation from all of the processes that we have discussed is subject to the influence of the plasma, e.g. little radiation will propagate below the plasma frequency. Faraday polarization will also set up a depolarization screen.

There are some effects on the high energy processes themselves which change the radiative process fundamentally. Need to then consider the induced motions and subsequent emission from particles in the ambient medium responding to the radiation. We consider a dielectric constant of the plasma which is independent of frequency, which is a cheat, but will capture basic ideas.

Maxwell's equations in the presence of a constant dielectric become

$$\nabla \cdot \mathbf{E} = 4\pi\rho/\epsilon;\ \nabla \cdot \mathbf{B} = 0 \tag{626}$$

$$\nabla \times \mathbf{E} = \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t};\ \nabla \times \mathbf{B} = \frac{4\pi\mathbf{j}}{c} + \frac{\epsilon}{c}\frac{\partial \mathbf{E}}{\partial t} \tag{627}$$

These equations result from Maxwell's equations with the substitution

$$\mathbf{E} \to \mathbf{E}\sqrt{\epsilon};\ \mathbf{B} \to \mathbf{B}, \tag{628}$$

$$\mathbf{A} \to \mathbf{A};\ \phi \to \sqrt{\epsilon}\phi \tag{629}$$

and

$$e \to \frac{e}{\sqrt{\epsilon}};\ c \to \frac{c}{\sqrt{\epsilon}}. \tag{630}$$

Recall that these are solved by the Liénard-Wiechart potentials

$$\phi\sqrt{\epsilon} = \frac{-e/\sqrt{\epsilon}}{\kappa(t_{ret})R(t_{ret})} \tag{631}$$

where

$$\tilde{\kappa} = 1 - \mathbf{n} \cdot \vec{\beta}\sqrt{\epsilon} \tag{632}$$

and

$$\mathbf{A} = -\frac{e\mathbf{u}(t_{ret})}{c\tilde{\kappa}(t_{ret})R(t_{ret})}, \tag{633}$$

so that

$$\sqrt{\epsilon}\mathbf{E}_{rad}(\mathbf{r}, t) = \frac{-e}{c}\left[\frac{\mathbf{n}}{\tilde{\kappa}^3 R} \times \{(\mathbf{n} - \tilde{\beta}) \times \dot{\tilde{\beta}}\}\right] \tag{634}$$

and

$$\mathbf{B}_{rad} = [\mathbf{n} \times \sqrt{\epsilon}\mathbf{E}_{rad}] \tag{635}$$

Cherenkov effect

Charge moving uniformly in a vacuum cannot radiate: No acceleration, no force, no energy change. Also true for a charge moving uniformly in dielectric medium if the phase velocity exceeds particle velocity. This is because the modified L-W potentials still then has only the term proportional to $1/R^2$.

However, if medium has index of refraction larger than 1, then the particle speed can be larger than the phase velocity. Then $\tilde{\kappa} = 1 - \beta n_r cos\theta$ can vanish when $cos\theta = 1/n_r\beta$. ($n_r \equiv \sqrt{\epsilon}$). The potentials can become quite large at places where this holds, and now there can be effective radiation from the $1/R^2$ term (consider Poynting flux argument, not necessarily zero at large distances in this case).

Note also that for $v > c/n_r$, the potentials can be determined by two retarded positions. fig 8.2

Note the cone, for this regime. The resulting radiation is confined within the cone of angle θ. The faster the particle the narrower the cone. Radiation propagates at c/n_r at the angle θ. Note that

$$c/n_r < v < c \tag{636}$$

for Cherenkov.

Note that the Cherenkov effect operates for weakly ionized plasmas, because for highly ionized plasmas, the conductivity is large and the conductivity dominates in determining the index of refraction. Sometimes when people refer to a "plasma" they mean a "fully ionized plasma" so be careful. Actually, for that case, we have also been ignoring the temperature contribution to the conductivity and dynamics (=cold plasma approximation).