LECTURE 20

Plasma Effects

Debeye Length:

Consider plasma composed of charge Ze and electrons of charge e. For charge neutrality we have

$$n_e Z_i = n_i, \tag{591}$$

summation implied. The ions attract the electrons. Focus on a single ion and let us see the influence of the ambient plasma on shielding the potential of this ion. The Coulomb law takes the form

$$\nabla^2 \phi = -4\pi \rho_c \tag{592}$$

where

$$\rho_c = Z_i e \delta(\mathbf{r}) - e n_e e^{e\phi/kT} + e n_i Z_i e^{-eZ_i\phi/kT}, \tag{593}$$

assuming $T_i = T_e$ and n_i and n_e are the average mean densities, and assuming Boltzmann spatial distributions. Let us assume for the moment that the exponentials are small (justified later). The first terms in the expansion cancel under the assumption of lowest order charge neutrality. Then we have

$$\nabla^2 \phi = -4\pi Z_i e \delta(\mathbf{r}) + 4\pi \frac{e^2}{kT}(n_e + n_i Z_i^2)\phi = -4\pi Z_i e \delta(\mathbf{r}) + 4\pi\phi/l_d^2 =, \tag{594}$$

where $l_d = (kT/e^2(n_e + n_i Z_i^2))^{1/2}$ is the Debeye length and is $\sim (kT/2e^2 n_e)^{1/2}$ for $Z_i = 1$. Spherical symmetry means that the only derivative comonents are r components, thus the laplacian becomes

$$\frac{1}{r^2}\frac{d}{dr} r^2 \frac{d\phi}{dr} = -4\pi Z_i e \delta(\mathbf{r}) + 4\pi\phi/l_d^2. \tag{595}$$

The appropriate solution which satisfies $\phi = eZ_i/r$ at $r = 0$ and 0 at $r = \infty$ is

$$\phi = \frac{Z_i e}{r} e^{-r/l_D}. \tag{596}$$

What this means is that the Coulomb potential is reduced at large r. The statistical attraction of electrons and repulsion of ions shield the primary charge.

Ions and electrons contribute equally to the shielding, because it depends on the ratio of particle speed to mass through the temperature, and not solely on the particle mass. The message is that for $r > l_d$, the free charge potential much be modified.

Conditions for the above analysis require that there be enough particles inside a volume $\sim l_d^3$ for a statistical treatment, that is $n_d l_d^3 >> 1$. (The inverse of this quantity is called the plasma parameter.) Note that the mean distance between electrons is $n_e^{-1/3}$. Thus the mean electrostatic energy$/kT$

$$e^2 n_e^{1/3}/kT \sim n_e^{-2/3} l_d^{-2} << 1, \tag{597}$$

in the statistical regime, thus justifying the earlier expansion.

Plasma Frequency

The electron contribution to the inverse Debeye length is

$$l_{d,e}^{-2} \sim 4\pi n_e e^2/kT = 4\pi n_e e^2/m_e v_e^2 = \omega_{p,e}^2/v_e^2, \tag{598}$$

where v_e is the electron thermal velocity and $\omega_{p,e} = 4\pi n_e^2/m_e$ is the electron plasma frequency.

The meaning is as follows. Imagine displacing the electrons of a plasma relative to the ions by an amount x. The resulting electric field is like that of a two plate capacitor with surface charge density $\pm e n_e x$, so that

$$E = 4\pi e n_e x. \tag{599}$$

This comes from integrating $\nabla \cdot \mathbf{E} = 4\pi\rho$. The equation of motion for the electron between the plates is then

$$\ddot{x} = -eE/m_e = -e^2 x 4\pi n_e/m_e = -\omega_{pe}^2 x/m_e. \tag{600}$$

Thus the electron plasma frequency is the natural oscillator frequency for an electron from the charge separation. Ions oscillate at a much slower rate and so one can consider them to be fixed. Numerically $\omega_{pe} = 5.6 \times 10^4 n_e^{1/2}$. Larger frequency forcing or disturbances will set up faster oscillations. Otherwise fast oscillations are set up to counteract the disturbances. No charge separation occurs for very disturbances on scales larger than the Debeye length. For such disturbances one is in the neutral fluid MHD regime.

Now lets come at this from a slightly different angle. Consider the linearized equation of motion of an electron fluid (take, $n, \mathbf{E}, \mathbf{B}, \mathbf{v}$ equal to zeroth order part + perturbation and consider only linear surviving terms). We get

$$m_e n_0 \partial_t \mathbf{v}_1 = -e n_0 \mathbf{E}_1 \tag{601}$$

From perturbed Maxwells equations we get

$$\nabla \times \mathbf{B}_1 = -(4\pi/c) n_o e \mathbf{v}_1 + (1/c)\partial_t \mathbf{E}_1 \tag{602}$$

and

$$\nabla \times \mathbf{E}_1 = -(1/c)\partial_t \mathbf{B}_1. \tag{603}$$

Note that no n_1 appears in these equations. Lets assume $e^{i\omega t}$ solutions for the time dependence. Then for (601) we have

$$\mathbf{v}_1 = -(ie/\omega m_e)\mathbf{E}_1. \tag{604}$$

Plugging this into (602) gives

$$\nabla \times \mathbf{B}_1 = -(i/c)\epsilon\omega\mathbf{E}_1, \tag{605}$$

where $\epsilon = 1 - \omega_p^2/\omega^2$. Note that in standard E+M, this represents the dielectric constant since by replacing of the $-iw$ in (606) with ∂_t we would recover

$$\nabla \times \mathbf{B}_1 = \partial_t \epsilon \mathbf{E}_1. \tag{606}$$

More on this in next section.

No E-M wave propagation at $\omega < \omega_{p,e}$

Assuming wave propagation solutions to Maxwell's equations of the form $e^{i\mathbf{k}\cdot\mathbf{x}-\omega t}$. Maxwell's equations for the plasma become

$$i\mathbf{k}\cdot\mathbf{E}= 4\pi\rho_c; \quad i\mathbf{k}\cdot\mathbf{B} = 0. \tag{607}$$

and

$$i\mathbf{k}\times\mathbf{E} = i\omega/c\mathbf{B}; \quad i\mathbf{k}\times\mathbf{B} = 4\pi\mathbf{j}/c - i\mathbf{E}\omega/c. \tag{608}$$

We again ignore the magnetic force (non-relativistic) and then for the equation of motion we have

$$\mathbf{v} = e\mathbf{E}/i\omega m_e. \tag{609}$$

The current density then satisfies

$$\mathbf{j} = \frac{ine^2}{\omega m_e}\mathbf{E} = \sigma\mathbf{E}. \tag{610}$$

Charge conservation gives

$$-i\omega\rho_c + i\mathbf{k}\cdot\mathbf{j} = 0, \tag{611}$$

so that

$$\rho_c = \mathbf{k}\cdot\mathbf{j}/\omega = \sigma\mathbf{k}\cdot\mathbf{E}/\omega. \tag{612}$$

If we again define the dielectric constant

$$\epsilon = 1 - 4\pi\sigma/i\omega = 1 - \omega_{p,e}^2/\omega^2, \tag{613}$$

we can then get "modified" vacuum equations:

$$i\mathbf{k}\cdot\epsilon\mathbf{E} = 0; \quad i\mathbf{k}\cdot\mathbf{B} = 0 \tag{614}$$

and

$$i\mathbf{k} \times \mathbf{E} = i\omega\mathbf{B}/c; \quad i\mathbf{k} \times \mathbf{B} = -i\omega\epsilon\mathbf{E}/c. \tag{615}$$

These are the source free Maxwell's equations. We solved them earlier, now we have a revised relation:

$$c^2k^2 = \epsilon\omega^2 = \omega^2[1 - (\omega_{p,e}/\omega)^2]. \tag{616}$$

Thus we see that

$$kc = (\omega^2 - \omega_{p,e}^2)^{1/2}. \tag{617}$$

Thus for $\omega < \omega_{p,e}$ the wave number is imaginary. This means that such waves do not propagate. Plugging back into $e^{i\mathbf{k}\cdot\mathbf{x}-\omega t}$ means that a wave propagating in the direction $\mathbf{x}$ will decay on length scale $\sim 2\pi c/\omega_p$, and instead reflects! This allows probing the density of the ionosphere (90-150km above surface).

Send a pulse of radiation upward. Where n is larger than critical, $\omega_p > \omega$ and the radiation is reflected. Electron density can be determined as a function of height. (MHz range) when one uses also the time measured for the signal to go up and come back down.

Dispersion Measure

For $\omega > \omega_p$, the E-M radiation propagates with phase velocity

$$v_{ph} = \omega/k = c/n_r \geq c \tag{618}$$

where $n_r = \epsilon^{1/2} = (1 - \omega_p^2/\omega^2)^{1/2}$ is the index of refraction. The group velocity travels at $v \leq c$

$$v_g = \partial\omega/\partial k = c(1 - \omega_p^2/\omega^2)^{1/2}. \tag{619}$$

When electron density varies, and thus the index of refraction, radiation travels along curved path. From Snell's law, the curved trajectories satisfy

$$d(n_r\hat{\mathbf{k}})/dl = \nabla n_r, \tag{620}$$

where l is the trajectory path length and $\hat{\mathbf{k}}$ is the ray direction.

One application of the spread in group velocities for waves of different frequencies is for pulsars. Pulsars have pulses with a range of frequencies per pulse. Interaction with ISM means that different frequency ranges will travel at different group velocities and thus reach observer at different times. If pulsar is a distance d away, then the time elapsed before the pulse reaches earth is

$$t_p = \int_0^d ds/v_g, \tag{621}$$

where ds is line of sight distance differential. Interstellar space has kHz plasma frequencies (Radio waves are in GHz). Thus assume $\omega >> \omega_p$. Then we have

$$v_g^{-1} = \frac{1}{c}\left(1 - \frac{\omega_p^2}{\omega^2}\right)^{-1/2} \sim \frac{1}{c}\left(1 + \frac{1}{2}\frac{\omega_p^2}{\omega^2}\right). \tag{622}$$

Thus

$$t_p \sim \frac{d}{c} + \frac{1}{2c\omega}\int_0^d \omega_p^2 ds. \tag{623}$$

The second term is the plasma correction. To get a convenient measured quantity take the frequency derivative to obtain

$$dt_p/d\omega = -\frac{4\pi e^2}{cm\omega^3}D, \tag{624}$$

where

$$D = \int_0^d n_e ds \tag{625}$$

is the dispersion measure. If we assume that the density is constant across interstellar space between the observer and the pulsar, we get the pulsar distance.