LECTURE 2

Equation of Radiative Transfer

Condition that I_ν is constant along rays means that

$$dI_\nu/dt = 0 = \partial_t I_\nu + c\mathbf{k} \cdot \nabla I_\nu, \tag{29}$$

where $c\mathbf{k} \cdot \nabla = dI_\nu/ds$ is the ray-path derivative. This is equation is the statement that there are no sources or sinks. But real systems may have sources and sinks: emission (source) and absorption (sink).

If light travels past a system faster than the time scale under which the system evolves then $\partial_t I_\nu = 0$ and we have then

$$dI_\nu/dt = 0 = c\mathbf{k} \cdot \nabla I_\nu = dI_\nu/ds = \text{sources} \ - \ \text{sinks}. \tag{30}$$

Emission coefficient

Spontaneous emission coefficient j_ν is defined as the energy emitted per unit time per unit solid angle, per unit volume

$$dE = j_\nu dV d\Omega dt d\nu, \tag{31}$$

where j_ν has units $erg/cm^3 \cdot steradian \cdot sec \cdot hz$. The emissivity is defined as $\epsilon_\nu = 4\pi j_\nu/\rho$, where ρ is the mass density. Since $dV = dAds$, we thus have

$$dI_\nu = j_\nu ds. \tag{32}$$

Absorption coefficient

Consider the propagation of a beam through a cylinder (again) of cross section dA and length ds, populated with a number density n of absorbers (particles), with absorption cross section σ_ν. (fig 6)

The loss of intensity, or the energy absorbed out of the beam is given by

$$dE_\nu = (I_{i,\nu} - I_{f,\nu}) dA d\Omega dt d\nu = I_\nu (n\sigma_\nu dA ds) d\Omega dt d\nu. \tag{33}$$

Thus,

$$dI_\nu = -n\sigma_\nu I_\nu ds. \tag{34}$$

The absorption coefficient α_ν is defined such that

$$dI_\nu = -\alpha_\nu I_\nu ds, \tag{35}$$

so $\alpha_\nu = n\sigma_\nu$ and α_ν is units of cm^{-1}. Sometimes it's written $\alpha_\nu = \rho\kappa_\nu$, where ρ is the mass density and κ_ν is the mass absorption or "opacity" coefficient. Note that the emission and absorption coefficients have different units. Also, note that absorbing particles are assumed to be randomly distributed and much smaller than inter-particle spacing.

Note that absorption is used loosely: Stimulated emission is also included in the absorption coefficient because it is proportional to the incident intensity.

We have thus derived the radiative transfer equation which we will soon solve:

$$dI_\nu/ds = -\alpha_\nu I_\nu + j_\nu. \tag{36}$$

Solutions of Equation of Radiative Transfer

A primary goal is to see what the values of α_ν and j_ν are for different absorption and emission processes. Scattering complicates things since radiation from the initial solid angle is scattered into a different solid angle for which the scatter depends on the initial solid angle, so we need numerics. Consider some simple cases here.

Emission Only:

$$dI_\nu/ds = j_\nu, \tag{37}$$

where the solution is

$$I_\nu(s) = I_\nu(s_0) + \int_{s_0}^{s} j_\nu(s')ds'. \tag{38}$$

Intensity increase is equal to the integrated emission coefficient along line of sight.

Absorbtion Only: Then $j_\nu = 0$. In this case

$$I_\nu(s) = I_\nu(s_0) Exp[-\int_{s_0}^{s} \alpha(s')ds']. \tag{39}$$

The intensity decreases exponentially along the ray path.

Optical Depth

Define

$$d\tau_\nu = \alpha_\nu ds, \tag{40}$$

or

$$\tau_\nu(s) = \int_{s_0}^{s} \alpha(s')ds'. \tag{41}$$

The optical depth is measure of how transparent a medium is to radiation. For $\tau_\nu \geq 1$ the medium is optically thick, and for $\tau_\nu < 1$ the medium is optically thin. We rewrite the transfer equation (36) as

$$dI_\nu/d\tau_\nu = -I_\nu + S_\nu, \tag{42}$$

where we define the source function $S_\nu \equiv j_\nu/\alpha_\nu$ as the ratio of the emission to the absorption coefficient. Optical depth is a physically meaningful variable because it contains the information about transparency along the path.

Now solve (42) by integrating factor: Multiplying by e^{τ_ν}, we obtain:

$$d(I_\nu e^{\tau_\nu})/d\tau_\nu = e^{\tau_\nu} S_\nu, \tag{43}$$

so the solution is

$$I_\nu e^{\tau_\nu} = I_\nu(0) + \int_0^{\tau_\nu} S_\nu(\tau'_\nu)e^{\tau'_\nu}d\tau'_\nu \tag{44}$$

or

$$I_\nu = I_\nu(0)e^{-\tau_\nu} + \int_0^{\tau_\nu} S_\nu(\tau'_\nu)e^{-(\tau_\nu - \tau'_\nu)}d\tau'_\nu. \tag{45}$$

The right side is the sum of the initial intensity attenuated by absorption, and the second term is the emission source attenuated by absorption. For a constant source function we have

$$I_\nu = I_\nu(0)e^{-\tau_\nu} + S_\nu(1 - e^{-\tau_\nu}) = S_\nu(\tau_\nu) + e^{-\tau_\nu}(I_\nu(0) - S_\nu(\tau_\nu)). \tag{46}$$

At large optical depth, $I_\nu \simeq S_\nu$. Why? because the source function is the local input of radiation–little is retained from contributions to I_ν at distances far from the location of interest when $\tau_\nu >> 1$.

Note also that there is a relaxation process, where I_ν tends toward the source function: If $I_\nu > S_\nu$, then $dI_\nu/d\tau_\nu < 0$. If $I_\nu < S_\nu$, then $dI_\nu/d\tau_\nu > 0$.

Mean Free Path

Interpreting (39) as measuring probability of absorption for optical depth τ_ν, the mean optical depth for exponential attenuation is

$$\langle \tau_\nu \rangle = \int_0^{\infty} \tau_\nu e^{-\tau_\nu} d\tau_\nu = 1. \tag{47}$$

The associated mean free path (MFP) is

$$l_\nu = \langle\tau_\nu\rangle/\alpha_\nu = 1/n\sigma_\nu. \tag{48}$$

MFP is important concept for particles as well as radiation.

Thermal Radiation and Blackbody Radiation

Thermal radiation is radiation emitted from matter in thermal equilibrium.

A blackbody absorbs all incident radiation.

Blackbody radiation is radiation which itself is in thermal equilibrium.

Consider a thermally insulated box kept at temp T. $I_\nu = I_\nu(T) = B_\nu$, the Planck function for such a black body box. Dependence only on temperature can be seen by considering two connected boxes at same T. Energy cannot flow without violating second law. Thus I_ν is same in both boxes.

I_ν is isotropic because otherwise there would be a net energy flow from one part of the box to another which is not possible when entire system is at uniform T.

For a blackbody,

$$I_\nu = B_\nu(T) = \frac{2h\nu^3/c^2}{Exp[h\nu/kT] - 1}, \tag{49}$$

the Planck function, and is only a function of the temperature and frequency.

Kirchoff's Law

Put thermally emitting material at temperature T, and source function $S_\nu(T)$ in the box.

If $S_\nu > B_\nu$ then the intensity I_ν will tend to increase such that $I_\nu > B_\nu$ and if $S_\nu < B_\nu$ then the intensity $I_\nu < B_\nu$. But in equilibrium, the new box is still a blackbody, and so $S_\nu = B_\nu$ or $j_\nu = \alpha_\nu B_\nu$.

Now consider the rad transfer eqn again with $S_\nu = B_\nu$:

$$dI_\nu/ds = -\alpha_\nu I_\nu + \alpha_\nu B_\nu(T). \tag{50}$$

For a black body (or for free space) the left side is zero, and thus $I_\nu = B_\nu$. For thermal radiation in a not necessarily optically thick system, the left side need not be zero for example if the source function temperature changes fast with position and dI_ν/ds cannot keep up. Thermal radiation is $S_\nu = B_\nu$ whilst blackbody radiation is $I_\nu = B_\nu$.

Blackbody Thermodynamics

The text derives some results: The Stefan-Boltzmann law for the energy density:

$$u(T) = aT^4, \tag{51}$$

where a is a constant. Recall from above that $u = 4\pi/c \int B_\nu(T) d\nu$. The integral (57) over frequency is $B(T) = (ac/4\pi)T^4$, where $a = 8\pi^5 k^4/15c^2h^3$, obtained from (49).

Then since flux from isotropically emitting surface is πI_ν we have

$$F = \int F_\nu d\nu = \pi B(T), \tag{52}$$

so we also have that the the total flux for a black body

$$F = \sigma T^4, \tag{53}$$

where $\sigma = 0.25ac = 5.67 \times 10^{-5} erg\ cm^{-2}\ {}^oK^{-4} sec^{-1}$.

Properties of the Planck Law

1. Rayleigh-Jeans limit

Taking the $h\nu << kT$ limit of the Planck law of (49) we expand the exponential and obtain

$$B_\nu^{RJ}(T) \sim 2\nu^2 kT/c^2, \tag{54}$$

which is the Rayleigh-Jeans limit. This usually applies in the radio regime. It is the classical regime. It cannot apply to all frequencies, because integrating would imply total energy divergence.

2. Wien Limit

If we take the limit $h\nu >> kT$ we have

$$B_\nu^{RJ}(T) \sim 2(h\nu^3/c^2)e^{-h\nu/kT}, \tag{55}$$

which is the Wien regime.

3. Temperature derivative

A blackbody curve with the higher temperature is always completely above a curve with a lower temperature. This follows from the temperature derivative of $B_\nu(T)$.

4. Wien Displacement Law

Peak frequency can be found by maximizing $\partial B_\nu/\partial\nu$, which leads to

$$\nu_{max}/T = 2.82k/h = 5.9 \times 10^{10} Hz/{}^oK \tag{56}$$

Peak frequency shifts linearly with the temperature. This is the Wien Displacement Law. Same can be done for wavelength to find

$$\lambda_{max} T = 0.290 cm^o K. \tag{57}$$

Note that $\lambda_{max}\nu_{max} \neq c$. Can see that the Rayleigh-Jeans law is valid for $\nu << \nu_{max}$.

Other temperatures commonly used

1. Brightness Temperature

This is defined by the temperature of the blackbody having the same brightness (=intensity) at that frequency. We define it by

$$I_\nu = B_\nu(T_b). \tag{58}$$

(note paper/research topic: Brightness Temperature puzzle in some active galaxy radio jets. Coherent emission or relativistic effect?) In radio astronomy one often sees this formula in the Rayleigh-Jeans regime ($h\nu << kT$). One has there

$$T_b = (c^2/2\nu^2 k) I_\nu \tag{59}$$

and for thermal emission,

$$dT_b/d\tau_\nu = -T_b + T, \tag{60}$$

which is solved by

$$T_b = T_b(0)e^{-\tau_\nu} + T(1 - e^{-\tau_\nu}). \tag{61}$$

$T_b \sim T$ for $\tau_\nu >> 1$.

Only if source is a blackbody is the brightness temperature the same at all frequencies.

For optically thin thermal emission, brightness temperature gives lower temp than that of the actual source; this is because a blackbody gives maximum intensity of thermal emitter at temperature T. (to see this use (61) with $T_b(0) = 0$).

Why is this?: A body which is not optically thick can be thought of as a system that is not fully "black" and absorbs less than a blackbody. If this system were in thermal equilibrium, then its emission must equal its absorption. But if it absorbs less than a blackbody it must also emit less then a blackbody to remain in equilibrium.

2. Color Temperature

The color temperature is the temperature obtained by fitting a blackbody shape to an observed, approximate blackbody flux.

3. Effective Temperature

This is obtained by setting the observed flux to that of blackbody form from the Stefan-Boltzmann law. That is

$$T_{eff} = (F/\sigma)^{1/4}. \tag{62}$$