LECTURE 18

Spectrum for isotropic photon distribution off of power law, isotopic distribution of electrons

The $q(\epsilon) = (4\pi/c)I_n(\epsilon) = F_n/c$. We will use this below. Note also that

$$j_n(\epsilon_f) = N\sigma_T \frac{1}{2}\int_{-1}^{1} I_n(\epsilon_f, \mu)d\mu = N\sigma_T I_n \tag{531}$$

for isotropic I_n.

We have for the total scattered power per volume per energy

$$\frac{dE}{dV d\epsilon_f dt} = 4\pi\epsilon_f j_n(\epsilon_f) \tag{532}$$

Then for $N(\gamma) = C\gamma^{-p}$, we have

$$\frac{dE}{dV d\epsilon_f dt} = \frac{1}{2}c\sigma_T C \int d\epsilon q(\epsilon)(\epsilon_f/\epsilon)\int d\gamma\gamma^{-(p+2)}(1 - \epsilon_f/4\gamma^2\epsilon + ...) \tag{533}$$

Change from γ to $x = \epsilon_f/4\gamma^2\epsilon$. We have

$$\frac{dE}{dV d\epsilon_f dt} \propto \epsilon_f^{\frac{1-p}{2}} \int d\epsilon\ \epsilon^{\frac{p-1}{2}} q(\epsilon)\int_{x_1}^{x_2}(1 - x + ..)dx. \tag{534}$$

with $x_1 = \epsilon_f/4\gamma_1^2\epsilon$ and $x_2 = \epsilon_f/4\gamma_2^2\epsilon$. If $\gamma_2 >> \gamma_1$ and $q(\epsilon)$ peaks strongly at $\bar{\epsilon}$ then we ignore second integral (like for synch.). We then have

$$\frac{dE}{dV d\epsilon_f dt} \propto \epsilon_f^{\frac{1-p}{2}} \int d\epsilon\ \epsilon^{\frac{p-1}{2}} q(\epsilon). \tag{535}$$

Note this is restricted to the regime $4\gamma_1^2\bar{\epsilon} << \epsilon_f << 4\gamma_2^2\bar{\epsilon}$.

Note that the power law is the same as for synchrotron!.

For an initial photon black body distribution, we have

$$q(\epsilon) = \frac{8\pi\epsilon^2}{h^3c^3}\frac{1}{e^{\epsilon/kT} - 1} \tag{536}$$

and

$$\frac{dE}{dV d\epsilon_f dt} \propto (kT)^{\frac{p+5}{2}}\epsilon_f^{\frac{1-p}{2}}. \tag{537}$$

Note the strong dependence on temperature. Also note that the temperature is that of the incident radiation.

Energy transfer for repeated scattering, Y parameter

When does scattering affect total photon energy? again assume Thomson limit in rest frame of electron.

Define "y" parameter such then when $y > 1$, scattering changes spectrum and energy wheras when $y < 1$, scattering does not affect energy but can still change spectrum. The appropriate definition is

$$y = \frac{\delta\epsilon}{\epsilon}(\text{mean number of scatterings}). \tag{538}$$

Consider a thermal electron distribution in the non-relativistic limit. Recall that

$$\epsilon'_f \sim \epsilon'\left(1 - \frac{\epsilon'}{mc^2}(1 - Cos\Theta)\right), \tag{539}$$

when $\epsilon' << mc^2$. If we average over Θ we get

$$\frac{\Delta\epsilon'}{\epsilon'} = -\frac{\epsilon'}{mc^2}, \tag{540}$$

where this is a small term. Expanding to lowest order in the two small parameters ϵ/mc^2 and kT/mc^2 we have

$$\frac{\Delta\epsilon}{\epsilon} = -\frac{\epsilon}{mc^2} + \alpha kT/mc^2. \tag{541}$$

Assume now that $e-$ and γ are in complete thermal equilibrium but interact only through scattering. Ignore stimulated emission. Photons thus have a Bose-Einstein distribution in which particle number cannot change. We are in the classical regime, where we ignore the stimulated emission. (This means that $e^{\epsilon-\mu} >> 1$, This gives

$$n_\epsilon = K\epsilon^2 e^{-\epsilon/kT}, \tag{542}$$

where K includes e^{μ} among the other constants. so that

$$\langle\epsilon\rangle = \frac{\int \epsilon \frac{dN}{d\epsilon} d\epsilon}{\int \frac{dN}{d\epsilon} d\epsilon} = 3kT \tag{543}$$

and

$$\langle\epsilon^2\rangle = 12(kT)^2. \tag{544}$$

In this limit, no net energy can be exchanged between photons and electrons so we have

$$\langle\Delta\epsilon\rangle = 0 = \frac{\alpha kT}{mc^2}\langle\epsilon\rangle - \frac{\langle\epsilon^2\rangle}{mc^2} = \frac{3kT}{mc^2}(\alpha - 4)kT, \tag{545}$$

thus $\alpha = 4$. Thus for non-relativistic electrons in thermal equilibrium,

$$(\Delta\epsilon)_{NR} = \frac{\epsilon}{mc^2}(4kT - \epsilon). \tag{546}$$

This shows that when $\epsilon > 4kT$, the energy is transferred from photons to electrons. When $\epsilon < 4kT$ energy is transferred from electrons to photons, inverse Compton scattering.

Now consider the limit $\gamma >> 1$. Then ignoring the energy transfer in the electron rest frame we recall that

$$\epsilon' = \epsilon\gamma(1 - \beta cos\theta) \tag{547}$$

and

$$\epsilon_f = \epsilon'_f\gamma(1 + \beta cos\theta'_f). \tag{548}$$

These give

$$\epsilon_f = \epsilon\gamma^2(1 - \beta cos\theta)(1 + \beta cos\theta'_f) \tag{549}$$

or averaging over both angles gives

$$\epsilon_f - \epsilon = \epsilon(\gamma^2(1 - \beta^2\langle(cos\theta)(cos\theta'_f)\rangle) - 1). \tag{550}$$

But since the average is over both angles, only when they are equal does the average contribute. The first integral then gives a delta function and the second then is an average over $(cos\theta)^2$. The average of which is 1/3. Thus we have, using $\gamma^2 - 1 = \gamma^2\beta^2$

$$\epsilon_f - \epsilon = \epsilon(\gamma^2(1 - \beta^2/3) - 1) = . \tag{551}$$

This can be written:

$$(\Delta\epsilon)_R \sim \frac{4}{3}\gamma^2\epsilon. \tag{552}$$

Then for thermal distribution of relativistic electrons $E \propto \gamma^2$ and thus

$$\langle\gamma^2\rangle = \frac{\langle E^2\rangle}{(mc^2)^2} = 12\left(\frac{kT}{mc^2}\right)^2. \tag{553}$$

Thus

$$(\Delta\epsilon)_R \sim \frac{4}{3}\gamma^2\epsilon = 16\epsilon\left(\frac{kT}{mc^2}\right)^2. \tag{554}$$

That is the first factor for the y parameter.

For the second factor (recall chap 1 of book) we recall that the mean number of scatterings is just the size of the medium over the mean free path all squared: $N_s = (R/l)^2 = \tau^2$ for $\tau >> 1$, and $N_s \sim 1 - e^{-\tau} \sim \tau$ for $\tau << 1$. Thus we take

$$N_s = Max(\tau_{es}^2, \tau_{es}) \tag{555}$$

and $\tau_{es} = \rho\kappa_{es}R$, where κ_{es} is the electron scattering opacity and is 0.4cm^2/g for ionized hydrogen.

Thus for the non-relativistic regime we have for the y parameter using (546) and (555) we have

$$y = \left(\frac{4kT}{mc^2} - \epsilon\right) Max(\tau_{es}^2, \tau_{es}) \sim \left(\frac{4kT}{mc^2}\right) Max(\tau_{es}^2, \tau_{es}), \tag{556}$$

where the latter similarity follows for $\epsilon << 4kT$.

For the relativistic regime we have using (554) and (555)

$$y = 16\left(\frac{kT}{mc^2}\right)^2 Max(\tau_{es}^2, \tau_{es}). \tag{557}$$

In the non-relativistic regime, it can be shown that the gain in energy of the photon as is scatters off non-relativistic electrons satisfies $\epsilon_f/\epsilon_i = e^y$. (pset 3).

In general, the Compton y parameter is frequency dependent. Then $\tau_{es} = \tau_{es}(\nu)$. Then since absorption takes away photons before they can scatter indefinitely, we determine τ_{es} as the value such that $\tau_*\nu = 1$, where the latter is the effective optical depth. Define $l_*(\nu)$ as the effective path the photon travels before absorption. When $l_* < R$ we have

$$\tau_{es}(\nu) = \rho\kappa_{es}l_*(\nu), \tag{558}$$

Since

$$l_* = lN^{1/2} \sim l\left(\frac{\alpha_\nu + \sigma_\nu}{\alpha_\nu}\right)^{1/2} \sim [\alpha_\nu(\alpha_\nu + \sigma_\nu)]^{-1/2} \tag{559}$$

since $l = (\alpha_\nu + \sigma_\nu)^{-1}$. If we then note that $\kappa_a(\nu) = \rho\alpha_\nu$ and $\kappa_{es}(\nu) = \rho\sigma_\nu$ we have

$$\tau_{es}(\nu) = \rho\kappa_{es}l_*(\nu) = \rho\kappa_{es}[\alpha_\nu(\nu)(\alpha_\nu + \sigma_\nu)]^{-1/2} = \left(\frac{\kappa_{es}^2}{\kappa_a^2 + \kappa_{es}\kappa_a}\right)^{1/2}. \tag{560}$$

Note that κ_a and κ_{es} are the absorption and scattering opacities.