LECTURE 17

Compton power for isotropic distributions of photons and relativistic electrons (single scattering)

Let $N(\gamma)$ be the number of electrons per volume between γ and $\gamma + d\gamma$. Then

$$P_{tot} = \int P_c N(\gamma) d\gamma \text{ erg s}^{-1}\text{cm}^{-3} \tag{512}$$

Choose a power law so that $N(\gamma) = C\gamma^{-p}$ for a range of γ. Then

$$P_{tot} = \frac{4}{3}\sigma_T c U_\gamma C \frac{\gamma_{max}^{3-p} - \gamma_{min}^{3-p}}{3-p} \text{ erg s}^{-1}\text{cm}^{-3} \tag{513}$$

for $\beta \sim 1$.

We can also compute the total power from a distribution of non-relativistic electrons. For $\gamma \sim 1$ we have $\langle \beta^2 \rangle = 3kT_e/mc^2$ so that

$$P_{tot} = \frac{4kT_e}{mc^2} c\sigma_T n_e U_\gamma \tag{514}$$

Inverse Compton Spectra (isotropic distribution of photons and electrons)

Spectrum depends on both incident distribution of photons and energy distribution of the electrons. But we first calculate for scattering of mono-energetic photons off of mono-energetic electrons assuming isotropy and single scattering. This means we just have to later average over energy distributions of photons and electrons to get overall spectrum spectrum (that averaging over distribution functions will be done in next section.).

Assume again the Thomson scattering regime $\epsilon' \sim \gamma\epsilon << mc^2$. Let us assume that the scattering in the rest frame is isotropic. We thus assume

$$\frac{d\sigma'}{d\Omega'} = \frac{\sigma_T}{4\pi} = \frac{2}{3}r_0^2. \tag{515}$$

Following the text use I_n here to mean photon number intensity rather than energy intensity. Then $I_n d\epsilon dA d\Omega dt$ is the number of photons passing through area dA in time dt within solid angle $d\Omega$ and energy $d\epsilon$. This intensity is just the specific intensity divided by the energy. Same for source and emission functions.

Consider now isotropic incident photon field as mono-energetic

$$I_n(\epsilon) = F_0 \delta(\epsilon - \epsilon_0), \tag{516}$$

where F_0 is the number of photons per unit area, per time, per steradian. (The flux per solid angle). Note that δ function always has units of 1 over the argument. We scatter these photons off of a beam of electrons of number density N and energy γmc^2 moving along the x-axis.

fig 7.2

The incident intensity in the rest frame, from Lorentz transformation properties of I_n, is

$$I'_n(\epsilon', \mu') = F_0 \left(\frac{\epsilon'}{\epsilon_0}\right)^2 \delta(\epsilon - \epsilon_0). \tag{517}$$

Then using the Doppler formula $\epsilon = \gamma\epsilon'(1 + \beta\mu')$ we have

$$I'_n(\epsilon', \mu') = \left(\frac{\epsilon'}{\epsilon_0}\right)^2 F_0 \delta\left(\gamma\epsilon'(1 + \beta\mu') - \epsilon_0\right) = \left(\frac{\epsilon'}{\epsilon_0}\right)^2 \frac{F_0}{\gamma\epsilon'\beta} \delta\left(\frac{\gamma\epsilon' - \epsilon_0}{\gamma\epsilon'\beta} + \mu'\right). \tag{518}$$

Here μ' is the cosine of the angle between the electron travel direction (x-axis) and the incident photon.

The emission function in the rest frame for isotropic distribution of photons is

$$j'_n(\epsilon'_f) = N'\sigma_T \frac{1}{2} \int_{-1}^{1} I'_n(\epsilon'_f, \mu') d\mu' \tag{519}$$

which here is the number of emitted photons per volume per steradian per time, and $N'\sigma_T$ is the scattering coefficient. Note $N = \gamma N'$. We replace ϵ' by ϵ'_f by using the rest frame Thomson scattering approximation. Then

$$j'_n(\epsilon'_f) = \frac{N'\sigma_T \epsilon'_f F_0}{2\epsilon_0^2 \gamma\beta} \tag{520}$$

for the range

$$\frac{\epsilon_0}{\gamma(1+\beta)} < \epsilon_f < \frac{\epsilon_0}{\gamma(1-\beta)}, \tag{521}$$

and zero otherwise. This range corresponds to the range for which $-1 < \mu' < 1$ which has to be the case since μ' is a cosine of the angle between incident photon and electron velocity.

We really want $j_n(\epsilon_f, \mu_f)$, the value in the lab frame. Recalling that j_ν/ν^2 is an invariant, we thus have j_n/ϵ is an invariant. Using this we have

$$j_n(\epsilon_f, \mu_f) = j'_n(\epsilon'_f)\frac{\epsilon_f}{\epsilon'_f} = \frac{N\sigma_T\epsilon_f F_0}{2\epsilon_0^2\gamma^2\beta} \tag{522}$$

when

$$\frac{\epsilon_0}{\gamma^2(1+\beta)(1-\beta\mu_f)} < \epsilon_f < \frac{\epsilon_0}{\gamma^2(1-\beta)(1-\beta\mu_f)}, \tag{523}$$

using the Doppler shift formula for ϵ'_f, and $N = \gamma N'$. Note also that for large β and $\mu_f = 1$ the inequality range can be written

$$\epsilon_0 < \epsilon_f < 4\epsilon_0\gamma^2, \tag{524}$$

while for large β and $\mu_f = -1$ the inequality range can be written

$$\frac{\epsilon_0}{4\gamma^2} < \epsilon_f < \epsilon_0. \tag{525}$$

For an isotropic distribution of electrons we average over the angle between the electron and emitted photon:

$$j_n(\epsilon_f) = \frac{1}{2}\int_{-1}^{1} j_n(\epsilon_f, \mu_f)d\mu_f. \tag{526}$$

Note that the limits above can be written in terms of μ_f, that is the non-zero region of j_n occurs when

$$\beta^{-1}\left(1 - \frac{\epsilon_0}{\epsilon_f}(1+\beta)\right) < \mu_f < \beta^{-1}\left(1 - \frac{\epsilon_0}{\epsilon_f}(1-\beta)\right). \tag{527}$$

For $\epsilon_f/\epsilon_0 < (1-\beta)/(1+\beta)$ or For $\epsilon_f/\epsilon_0 > (1+\beta)/(1-\beta)$, μ_f does not fall in range $-1 < \mu < 1$ so the integral must vanish. The finite contribution comes from within the integral limits given by

$$-1 < \mu_f < \frac{1}{\beta}\left(1 - \frac{\epsilon_0}{\epsilon_f}(1-\beta)\right) \tag{528}$$

for $(1-\beta)/(1+\beta) < \epsilon_f/\epsilon_0 < 1$ and

$$\frac{1}{\beta}\left(1 - \frac{\epsilon_0}{\epsilon_f}(1+\beta)\right) < \mu_f < 1 \tag{529}$$

for $1 < \epsilon_f/\epsilon_0 < (1+\beta)/(1-\beta)$. Since $j_n(\epsilon_f)$ is isotropic, the integrals just amount to multiplication of the integrand of (521) by each of the bounds and subtracting. When the dust clears, using (517), we have

$$j_n(\epsilon_f) = \frac{N\sigma_T F_0}{4\epsilon_0\gamma^2\beta^2}Q \tag{530}$$

where

$$Q = (\beta - 1) + (\beta + 1)\frac{\epsilon_f}{\epsilon_0} \tag{531}$$

for $(1-\beta)/(1+\beta) < \epsilon_f/\epsilon_0 < 1$ and

$$Q = \frac{\epsilon_f}{\epsilon_0}(\beta - 1) + (\beta + 1) \tag{532}$$

for $1 < \epsilon_f/\epsilon_0 < (1+\beta)/(1-\beta)$. (Which is 1 ¡ $\epsilon_f/\epsilon_0 < 4/\gamma^2$ for large γ).

The integrals satisfy:

$$\int_0^\infty j(\epsilon_f)d\epsilon_f = N\sigma_T F_0 \tag{533}$$

and

$$\int_0^\infty j(\epsilon_f)(\epsilon_f - \epsilon_0)d\epsilon_f = \frac{4}{3}\gamma^2\beta^2 Nc\sigma_T(\epsilon_0 F_0/c). \tag{534}$$

The first means that number of photons are conserved since it is the number of photons scattering per volume per solid angle per unit time. The second is just the average gain in energy per unit time measured as radiation per electron where $\epsilon_0 F_0/c$ is the average photon energy density. That is it is the POWER emitted per electron.

Notice that in contrast to Bremsstrahlung, this equation is only proportional to the number density of electrons rather than the number density squared.

The $\epsilon_0 j(\epsilon_f)/N\sigma_T F_0$ is plotted below. fig 7.3a

One can see the shift with larger β as the energy is gained because, highlighting how the average energy of the scattered photons are increased with larger γ. At very small γ there should be a peak around the initial photon energy with no gain or loss. As the γ is increased, there will be some loss as well as gain, because the photon distribution is isotropic and catch-up interactions deplete photon energy. But with large γ the beaming means electron sees most photons as head on, and the overall energy gain is produced.

When $\beta \sim 1$, we are in the second regime of Q above, which actually now corresponds to (524). Define $x = \epsilon_f/4\gamma^2\epsilon_0$, and then

$$j_n(\epsilon_f) = \frac{1}{2\gamma^2\epsilon_0}NF_0\sigma_T(1-x), \tag{535}$$

and applies only in range $0 < x < 1$ because of (524) and the fact that the limit below (527) can be written as $0 < x < 1$ for large γ. This follows directly from using (527) in (525) and using $\beta + 1 \sim 2$ and $\gamma^2 = 1/(1 - \beta^2)$, and some algebra.

We have assumed everything to be isotropic. When this emission function for the relativistic regime is compared to the exact expression, the result is below: fig 7.3b