LECTURE 14

Synchrotron Radiation

Radiation of particles (e.g. electrons) gyrating around a magnetic field is called cyclotron radiation when the electrons are non-relativistic and synchrotron radiation when the electrons are relativistic. The spectrum of cyclotron emission frequency is a peak at the frequency of gyration. Synchrotron frequency picks up higher harmonics and is not as simple.

Total emitted synchrotron power:

Motion of a particle in a magnetic field is governed by

$$\mathbf{a}^{\mu} = \frac{e}{mc} F^{\mu}_{\nu} u^{\nu} \tag{417}$$

with components

$$d_t(\gamma \mathbf{v}) = \frac{q\mathbf{v} \times \mathbf{B}}{mc} \tag{418}$$

and

$$d_t \gamma = \frac{q\mathbf{v} \cdot \mathbf{E}}{mc^2} = 0. \tag{419}$$

Thus γ and $|\mathbf{v}|^2$ are constants so

$$\gamma d_t \mathbf{v} = \frac{q\mathbf{v} \times \mathbf{B}}{mc} \tag{420}$$

and

$$\gamma d_t \mathbf{v}_{||} = 0 \tag{421}$$

and

$$d_t \mathbf{v}_{\perp} = \frac{q}{\gamma mc} \mathbf{v}_{\perp} \times \mathbf{B}. \tag{422}$$

fig 6.1

The magnitude of $\mathbf{v}$ is a constant, and the parallel component is unchanged, the motion represents helical motion along a magnetic field. This is circular motion in the frame where

$\mathbf{v}_{||} = 0$. The gyration frequency is $\omega_B = qB/\gamma mc$. The acceleration ($a_\perp = \omega_B v_\perp$) is perpendicular to the velocity and so the power emitted is

$$P = \frac{2q^2}{3c^3}\mathbf{a}'^2 = \frac{2q^2}{3c^3}\gamma^4(a_\perp^2 + \gamma^2 a_{||}^2) = \frac{2q^2}{3c^3}\gamma^4 a_\perp^2 = \frac{2}{3}r_0^2 c\beta_\perp \gamma^2 B^2, \tag{423}$$

where we are now assuming that the electrons are raditating so $r_0 = e^2/m_e c^2$ is the classical elecron radius as defined earlier.

If we average over pitch angles $0 \leq \psi \leq \pi$ then

$$\langle \beta_\perp^2 \rangle = \frac{\beta^2}{4\pi}\int sin^2\psi d\Omega = \frac{2\beta^2}{3}. \tag{424}$$

Then

$$P = \frac{4}{9}cr_0^2\beta^2\ \gamma^2 B^2 = \frac{4}{3}\sigma_T c\beta^2\gamma^2 U_m \simeq \frac{4}{3}\sigma_T c\gamma^2 U_m. \tag{425}$$

where $\sigma_T = 8\pi r_0^2/3$.

Spectrum of Synchrotron Radiation

fig 6.2

To estimate this, note that observer sees emission during the time the particle moves from point 1 to 2 below. Consider the extremely relativistic regime $\gamma >> 1$.

The emission from synchrotron radiation will be spread over a broader range of frequency than just the gyro-frequency because the time interval over which observer sees emission is much smaller than the gyro-frequency. The uncertainty relation then implies the broader spectrum.

Radius of curvature is $a = \Delta s/\Delta\theta$. We see that $\Delta\theta = 2/\gamma$ so $\Delta s = \frac{2a}{\gamma}$. The radius of curvature is also given by $asin\alpha = v/\omega_B$, which follows from

$$\gamma m\frac{\Delta v}{\Delta t} = (q/c)\mathbf{v}\times\mathbf{B} = \gamma m v^2\frac{\Delta\theta}{\Delta s}, \tag{426}$$

where $|\Delta v| = v\Delta\theta$ and $\Delta s = v\Delta t$.

Thus

$$a = \Delta s/\Delta\theta = \gamma mcv/qBsin\alpha = v/\omega_B sin\alpha, \tag{427}$$

where α is the pitch angle. Therefore

$$\Delta s = 2v/\gamma\omega_B sin\alpha. \tag{428}$$

The particle passes between points 1 and 2 in a time $t_2 - t_1$ so that

$$(t_2 - t_1) \sim \Delta s/v = \frac{2}{\gamma\omega_B sin\alpha}. \tag{429}$$

Defining t_{A1} and t_{A2} as the arrival times of radiation at the observation point from points 1 and 2, the difference $t_{A2} - t_{A1}$ is less than $t_2 - t_1$ by an amount $\Delta s/c$, which is the time for radiation to move Δs. Thus

$$(t_{A2} - t_{A1}) \sim \frac{2}{\gamma\omega_B sin\alpha}(1 - v/c). \tag{430}$$

For $v \sim c$ this becomes

$$(t_{A2} - t_{A1}) \sim \frac{1}{\gamma^3\omega_B sin\alpha}. \tag{431}$$

Thus for a finite pitch angle, observed pulse duration is γ^3 times smaller than gyroperiod. Pulse frequency cutoff will be something like

$$\omega_c \sim 1/\Delta t_A = \gamma^3\omega_B sin\alpha. \tag{432}$$

fig 6.3

Now since the electric field, as reflected in (389) and (390) is only a function of the product $\theta\gamma$, we can write $E(t) = F(\gamma\theta)$, with t in the lab frame. Let $s = t = 0$ when pulse is along the axis of the observer. Then

$$t = \Delta t_A = (s/v)(1 - v/c). \tag{433}$$

and

$$\theta \sim (s/a), \tag{434}$$

so

$$\theta\gamma = \frac{vt}{(1-v/c)a} = 2\gamma^3 t\omega_B sin\alpha \sim \omega_c t. \tag{435}$$

Thus

$$E(t) \propto g(\omega_c t). \tag{436}$$

The Fourier transform is then

$$\tilde{E}(\omega) \propto \int_{-\infty}^{\infty} g(\zeta) e^{i\omega\zeta/\omega_c} d\zeta, \tag{437}$$

where $\zeta = \omega_c t$.

The spectrum $dW/d\omega d\Omega \propto \tilde{E}^2$. Integrating over solid angle and dividing by the orbital period gives the time averaged power per frequency

$$dW/dtd\omega = (\omega_B/2\pi)dW/d\omega = P(\omega) = KF(\omega/\omega_c), \tag{438}$$

where K is a constant which can be fixed by integrating and comparing to the total power derived earlier. Since the total power emitted for $\beta \sim 1$ is given by

$$P_{tot} \simeq \frac{2q^4 B^2 \gamma^2 sin^2\alpha}{3m^2c^3} = K\int F(\omega/\omega_c)d\omega = \omega_c K \int F(\xi)d\xi, \tag{439}$$

where $\xi \equiv \omega/\omega_c$. Then using

$$\omega_c = \frac{\gamma^2 qBsin\alpha}{mc} \tag{440}$$

$$P(\omega) = KF(\omega/\omega_c)d\omega = \omega_c KF(\xi)d\xi = \frac{3\sqrt{3}}{4\pi mc^2} q^3 Bsin\alpha F(\omega/\omega_c), \tag{441}$$

where the numerical integral factor is arbitrary, and comes from choice of normalization of the dimensionless integral.

Spectral Index for power law electron distribution

Often the emission spectrum can be described by a power law in frequency for some range. That is,

$$P(\omega) \propto \omega^{-s} \tag{442}$$

where s is the spectral index. Electrons also often follow a power law distribution when they have been accelerated. For relativistic electrons, consider an energy distribution

$$N(\gamma)d\gamma = C\gamma^{-p}d\gamma. \tag{443}$$

Then the total power radiated is given by

$$P_{tot}(\omega) = C\int_{\gamma_1}^{\gamma_2} P(\omega)\gamma^{-p}d\gamma \propto \int_{\gamma_1}^{\gamma_2} F(\omega/\omega_c)\gamma^{-p}d\gamma. \tag{444}$$

If we change variables of integration to $x \equiv \omega/\omega_c$ and use $\omega_c \propto \gamma^3 \omega_B \propto \gamma^2$ we get

$$P_{tot}(\omega) \propto \omega^{\frac{1-p}{2}} \int_{x_1}^{x_2} F(x) x^{\frac{p-3}{2}} dx. \tag{445}$$

For wide enough frequency limits, take $x_1 \sim 0$ and $x_2 \sim \infty$. Then we have

$$P_{tot}(\omega) \propto \omega^{\frac{1-p}{2}} \int_{0}^{\infty} F(x) x^{\frac{p-3}{2}} dx. \tag{446}$$

The spectral index is then related to the particle distribution index:

$$s = \frac{p-1}{2}. \tag{447}$$

For shocks, p is often $2-4$ so spectral index is $0.5-1.5$. Relevant for radio jets in AGN (e.g. Blackman 1996).

Note that $F(x)$ is something like a Gaunt factor, but it is narrowly peaked helping to justify the assumption of wide limits of integration. fig 6.6

Summary of Results: (1) emission into $1/\gamma$ half angle, (2) emission up to critical frequency ω_c and dependence only on ω/ω_c, and (3) spectral index for power law electron distribution is $s = (p-1)/2$.