PHY411 Lecture notes on Constraints

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Contents

1 Mechanical constraints

1.1 Lagrangian multipliers

The problem is to maximize a function $f(x, y)$ subject to a constraint $g(x, y) = 0$. Define

$$L(x, y, \lambda) = f(x, y) - \lambda g(x, y).$$

This Lagrangian is a function of coordinates only. Solve

$$\frac{\partial L}{\partial x} = \frac{\partial f}{\partial x} - \lambda \frac{\partial g}{\partial x} = 0 \quad (1)$$

$$\frac{\partial L}{\partial y} = \frac{\partial f}{\partial y} - \lambda \frac{\partial g}{\partial y} = 0 \quad (2)$$

$$\frac{\partial L}{\partial \lambda} = g(x, y) = 0. \quad (3)$$

The first two of these equations implies that the xy gradients $\boldsymbol{\nabla} f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$ is in the same direction as $\boldsymbol{\nabla} g$ (see Figure 1).

A solution to these equations can give a *local* maximum or minimum. The result may not be a *global* maximum or minimum.

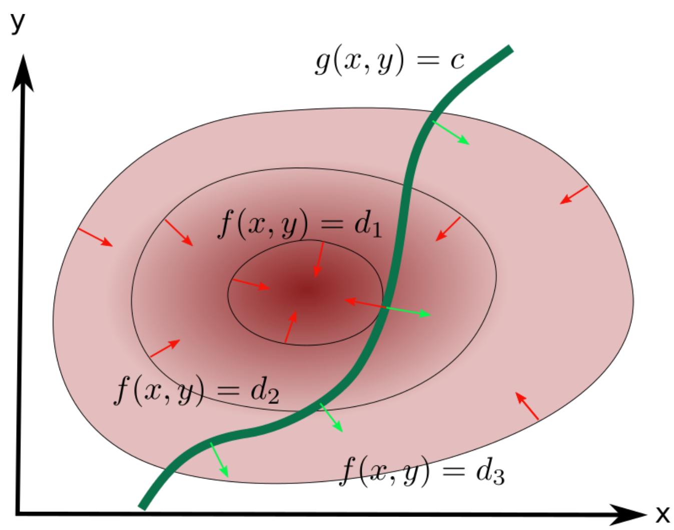

Figure 1: Maximization of $f(x, y)$ with the constraint $g(x, y) = c$ occurs where $\nabla f \propto \nabla g$.

1.2 Holonomic constraints

The adjective *holonomic* is from Greek, meaning 'whole'. For constraints of a mechanical system holonomic means expressible as a function of the coordinates and time. In differential geometry holonomy has to do with how a quantity is changed after it is transported about a loop. In physics non-holonomic is used to describe a system with path dependent dynamics or state.

A set of **holonomic** constraints for a classical system with equations of motion generated by a Lagrangian are a set of functions

$$f^k(\mathbf{x}, t) = 0. \tag{4}$$

Here k is an index for each constraint function. The constraints only depend on coordinates, not velocities.

An example is motion of a particle in $\mathbb{R}^3$ that is constrained to lie on a sphere. The constraint is a single function,

$$f(x, y, z) = x^2 + y^2 + z^2 - R^2 = 0$$

where R is the radius of the sphere. A dynamical system with holonomic constraints is called *holonomic.*

We consider a trajectory $\mathbf{x}(t)$, and take the time derivative of $f(\mathbf{x}) = 0$ (independent of time)

$$\frac{df}{dt} = \boldsymbol{\nabla} f \cdot \mathbf{v} = 0. \tag{5}$$

A time independent constraint automatically gives velocity perpendicular to $\boldsymbol{\nabla} f$.

The Newtonian equation of motion is

$$m\ddot{\mathbf{x}} = \mathbf{F} + \mathbf{C} \tag{6}$$

where $\mathbf{F}$ is an external force and $\mathbf{C}(\mathbf{x}, t)$ is a force that is caused by the constraint. A single constraint in three dimensions describes a two-dimensional surface. With a single constraint in three dimensions, there are three unknown components of constraint force $\mathbf{C}$ and this does not give enough information to determine all three of them. We can specify that the constraint force is normal to the surface

$$\mathbf{C} = \lambda \boldsymbol{\nabla} f(\mathbf{x}). \tag{7}$$

With the motion restricted to $f(\mathbf{x}, t) = 0$, the motion $\mathbf{v} = \dot{\mathbf{x}}$ must be perpendicular to $\boldsymbol{\nabla} f(\mathbf{x})$. This means that

$$\mathbf{C} \cdot \mathbf{v} = 0.$$

The constraint force is always perpendicular to the direction of motion, and the constraint does no work on the system.

With a Lagrangian, K constraints and a Newtonian system with potential U we can modify the Lagrangian in the absence of constraints to include the constraints;

$$L(\dot{\mathbf{x}}, \mathbf{x}, t) = \frac{\dot{\mathbf{x}}^2}{2} - U(\mathbf{x}) + \sum_k \lambda_k f^k(\mathbf{x}).$$

Lagrange's equations are consistent with the Newtonian equation of motion (equation 6).

This can be solved similar to the Lagrangian multiplier problem described above by minimizing the action

$$S(\gamma) = \int \tilde{L}(\dot{\mathbf{x}}, \mathbf{x}, t, \boldsymbol{\lambda}) dt \tag{8}$$

with Lagrangian

$$\tilde{L}(\dot{\mathbf{x}}, \mathbf{x}, t, \boldsymbol{\lambda}) = L(\dot{\mathbf{x}}, \mathbf{x}, t) + \sum_k \lambda_k f^k(\mathbf{x}) \tag{9}$$

and treating the $\boldsymbol{\lambda}$ as Lagrange multipliers. The equations of motion (derived from $\tilde{L}$) are

$$\frac{d}{dt}\frac{\partial L(\dot{\mathbf{x}}, \mathbf{x}, t)}{\partial \dot{\mathbf{x}}} = \frac{\partial L(\dot{\mathbf{x}}, \mathbf{x}, t)}{\partial \mathbf{x}} + \sum_k \lambda_k \frac{\partial f^k}{\partial \mathbf{x}} \tag{10}$$

$$f^k(\mathbf{x}) = 0 \tag{11}$$

This is true for a general, not just Newtonian, Lagrangian L with holonomic constraints. We interpret

$$\mathbf{F}^k = \lambda_k \frac{\partial f^k}{\partial \mathbf{x}}$$

as the constraint forces. As we will see below, these could include non-conservative forces.

1.3 Externally applied forces

Lagrange's equations are modified with

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{q}}} - \frac{\partial L}{\partial \mathbf{q}} = \mathbf{F}_{ext} \tag{12}$$

where $\mathbf{F}_{ext}$ is a sum of externally applied forces. A Lagrangian system can be modified to include external forces by adding them directly to Lagrange's equations.

1.4 Example of holonomic constraints: a disk on an inclined plane

A cylinder of radius a rolls without slipping down a plane inclined at an angle θ to the horizontal. The distance x represents the displacement of the center of mass of the cylinder parallel to the surface of the plane, and ϕ represent the angle of rotation of the cylinder about its symmetry axis. Rolling without slipping implies that x and ϕ obey the constraint

$$f(x, \phi) = x - a\phi = 0.$$

The Lagrangian without constraints is

$$L(x, \phi, \dot{x}, \dot{\phi}, t) = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I\dot{\phi}^2 + mgx\sin\theta \tag{13}$$

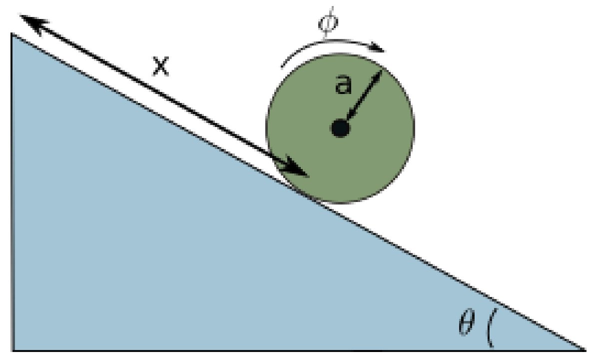

Figure 2: A disk rolling without slipping down an inclined plane.

where m, I, g are the mass of the disk, its moment of inertia and the acceleration due to gravity. With the constraint we add

$$L(x, \phi, \dot{x}, \dot{\phi}, t, \lambda) = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I\dot{\phi}^2 + mgx\sin\theta + \lambda(x - a\phi). \tag{14}$$

Lagrange's equation with the Lagrange multiplier λ give

$$\begin{aligned}
\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} &= \frac{\partial L}{\partial x} + \lambda\frac{\partial f}{\partial x} \\
m\ddot{x} &= mg\sin\theta + \lambda \\
\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} &= \frac{\partial L}{\partial \phi} + \lambda\frac{\partial f}{\partial \phi} \\
I\ddot{\phi} &= -\lambda a \\
\frac{\partial L}{\partial \lambda} &= x - a\phi = 0.
\end{aligned}$$

Note that we solve without assuming that λ is constant in time. The solution satisfies

$$\begin{aligned}
\ddot{x} &= \frac{g\sin\theta}{1 + I/(ma^2)} = a\ddot{\phi} \\
\lambda &= -\frac{mg\sin\theta}{1 + (ma^2)/I}.
\end{aligned} \tag{15}$$

As $\ddot{x}, \ddot{\phi}$ are constants, their initial values and velocities determine the solutions at later times. It happens that λ is time independent! The constraint forces are

$$\begin{aligned} F_x &= \lambda \frac{\partial f}{\partial x} = \frac{mg \sin\theta}{1 + (ma^2)/I} \\ F_\phi &= \lambda \frac{\partial f}{\partial \phi} = -\frac{mga \sin\theta}{1 + (ma^2)/I}. \end{aligned} \tag{16}$$

The force F_x arises from friction from the surface. The force F_ϕ arises from the torque on the cylinder, again due to friction on the surface, keeping the disk rolling without slipping.

Is there a Hamiltonian view for this problem? In the Lagrangian view point, the Lagrange multiplier λ can be considered an extra coordinate. The Lagrangian does not depend on $\dot{\lambda}$. Because the Lagrangian does not depend on $\dot{\lambda}$ there cannot be a momentum associated with $p_\lambda = \frac{\partial L}{\partial \dot{\lambda}}$. However this Hamiltonian

$$H(p_x, p_\phi; x, \phi; \lambda) = \frac{p_x^2}{2m} + \frac{p_\phi^2}{2I} - mgx \sin\theta - \lambda(x - a\phi) \tag{17}$$

gives

$$\begin{aligned} \frac{\partial H}{\partial p_x} &= \dot{x} = \frac{p_x}{m} \\ \frac{\partial H}{\partial p_\phi} &= \dot{\phi} = \frac{p_\phi}{I} \\ -\frac{\partial H}{\partial x} &= \dot{p}_x = mg\sin\theta + \lambda = m\ddot{x} \\ -\frac{\partial H}{\partial \phi} &= \dot{p}_\phi = -a\lambda = I\ddot{\phi}. \end{aligned} \tag{18}$$

These are the same equations of motion as derived with the Lagrangian viewpoint, but in order to find $\lambda(t)$ we must include the constraint

$$\frac{\partial H}{\partial \lambda} = x - a\phi = 0. \tag{19}$$

This restricts solutions to a submanifold. The constraint is a conserved quantity.

1.5 Non-holonomic constraints

Holonomic constraints can be expressed using a differential form. With a constraint $f(\mathbf{q}, t) = 0$,

$$df = \sum_i \frac{\partial f}{\partial q_i} dq^i + \frac{\partial f}{\partial t} dt.$$

Consider k constraints that depend on the velocities,

$$h^k(\mathbf{q}, \dot{\mathbf{q}}) = 0.$$

If these are linear in the velocities then we can write the constraints as

$$\sum_i g_i^k(\mathbf{q})\dot{q}^i = \mathbf{g}^k(\mathbf{q}) \cdot \dot{\mathbf{q}} = 0 \tag{20}$$

If we can find functions $f_j^k(\mathbf{q})$ such that

$$\frac{\partial f_i^k(\mathbf{q})}{\partial q_i} = g_i^k(\mathbf{q}) \tag{21}$$

then equation 20 can be written as

$$df^k(\mathbf{q}) = 0 \tag{22}$$

or

$$f^k(\mathbf{q}) - \text{constant} = 0 \tag{23}$$

Constraints in the form of equation 20 that can be written like equation 23 are *holonomic*, otherwise they are *non-holonomic*. Not all constraints that are linear in velocities are holonomic.

1.5.1 Lagrange-d'Alembert Principle

Suppose we have a set of k non-holonomic linear and time independent constraints

$$a_j^k(\mathbf{q})\dot{q}_j = \mathbf{a}^k(\mathbf{q}) \cdot \dot{\mathbf{q}} = 0. \tag{24}$$

The set of forces that are exerted due to the constraints are represented by $\mathbf{a}_k$. The above constraint equation is equivalent to assuming that constraint forces do no work. Note that the functions $a_k^j(\mathbf{q})$ do not depend on velocities. The dynamic non-holonomic or the Lagrange-d'Alembert equations of motion are

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{q}}} - \frac{\partial L}{\partial \mathbf{q}} = \sum_k \lambda_k \mathbf{a}_k. \tag{25}$$

We identify $\mathbf{F^k} = \lambda_k \mathbf{a}^k$ with the k constraint forces.

Energy is conserved for time independent non-holonomic systems. The energy is

$$E(\mathbf{q}, \dot{\mathbf{q}}) = \frac{\partial L(\mathbf{q}, \dot{\mathbf{q}})}{\partial \dot{\mathbf{q}}}\dot{\mathbf{q}} - L(\mathbf{q}, \dot{\mathbf{q}})$$

If we take the time derivative of this we find that

$$\begin{aligned}\dot{E} &= \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{q}}}\right)\dot{\mathbf{q}} + \frac{\partial L}{\partial \dot{\mathbf{q}}}\ddot{\mathbf{q}} - \frac{\partial L}{\partial \mathbf{q}}\dot{\mathbf{q}} - \frac{\partial L}{\partial \dot{\mathbf{q}}}\ddot{\mathbf{q}} \\ &= \lambda_k \mathbf{a}^k \cdot \dot{\mathbf{q}} \end{aligned} \tag{26}$$

and on the second line I have used equation 25. This vanishes according to our definition for the constraints (equation 24) so $\dot{E} = 0$.

The system can be described in Hamiltonian viewpoint with Hamiltonian and constraints in phase space

$$H(\mathbf{p}, \mathbf{q}; \lambda) = \mathbf{p}\dot{\mathbf{q}} - L(\mathbf{q}, \dot{\mathbf{q}}) + \lambda_j \mathbf{a}^k(\mathbf{q}) \cdot \dot{\mathbf{q}} \tag{27}$$

$$\mathbf{a}^k(\mathbf{q}) \cdot \dot{\mathbf{q}} = 0. \tag{28}$$

where $\dot{\mathbf{q}}(\mathbf{p}, \mathbf{q})$ follows from the Legendre transformation used to construct the Hamiltonian from a Lagrangian.

1.5.2 Some subtleties

The Lagrange-d'Alembert Principle is consistent with Hamilton's variational principle if path displacements are assumed to satisfy the constraints or

$$\sum_j a_j^k \delta q_j = 0$$

on the path. Only paths that satisfy the constraints are allowed when minimizing the path to derive the equations of motion.

Instead one could use Lagrange multipliers with the Lagrangian, namely

$$\tilde{L}(\mathbf{q}, \dot{\mathbf{q}}, \boldsymbol{\lambda}) = L(\mathbf{q}, \dot{\mathbf{q}}) + \sum_k \sum_j \lambda_k a_j^k \dot{q}_j.$$

Then the Euler-Lagrange equations are computed from this modified Lagrangian. Note the multipliers λ_k are time dependent. When this is done, you get what is called *variational non-holonomic* equations or *vakonomic* equations. Vakonomic is short for *variational axiomatic kind* as coined by Kozlov.

The two procedures give different equations of motion. With the dynamic Lagrange-d'Alembert equations, constraints are imposed on the variations, whereas in the variational problem, the constraints are imposed on the velocity vectors of the class of allowable curves. Why should the force vectors exactly arrange themselves to annihilate virtual displacements so that only paths that satisfy the constraints are possible?

Which viewpoint is correct? There is a consensus in the mechanics community (studying things like interconnected mechanical bodies) that Lagrange-d'Alembert equations, derived from considering forces, is correct. However, the variational viewpoint is appropriate to study *optimal control problems.*

1.5.3 Example of a system with non-holonomic constraints, the Rolling Disk

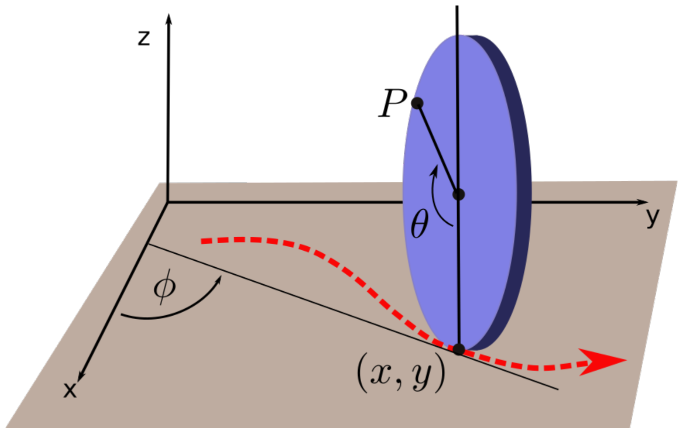

Figure 3: Geometry of a rolling disk. The disk rolls without slipping. This is an example of a dynamical system with non-holonomic constraints.

An example of a dissipation-free system is a vertically oriented disk that roles without slipping. The coordinate space is four dimensional $\mathbf{q} = (x, y, \phi, \theta)$. Here x, y is the position on the horizontal xy plane where the disk touches the plane; see Figure 3. The angle ϕ gives the orientation of the disk and determines which way the disk rolls. The angle θ gives the rotation angle of the disk. The constraints (rolling without slipping) can be written as

$$\begin{aligned} \dot{x} - R\cos\phi\ \dot{\theta} &= 0 \\ \dot{y} - R\sin\phi\ \dot{\theta} &= 0. \end{aligned} \tag{29}$$

We can write our two constraints (equations 29) as

$$\mathbf{a}_1 \cdot (\dot{x}, \dot{y}, \dot{\phi}, \dot{\theta}) = 0 \qquad \mathbf{a}_1 = (1, 0, 0, -R\cos\phi) \tag{30}$$

$$\mathbf{a}_2 \cdot (\dot{x}, \dot{y}, \dot{\phi}, \dot{\theta}) = 0 \qquad \mathbf{a}_2 = (0, 1, 0, -R\sin\phi). \tag{31}$$

The constraints are linear in velocities. The constraint equations cannot be written as a one form so they are nonholonomic.

The Lagrangian

$$L(x, y, \phi, \theta, \dot{x}, \dot{y}, \dot{\phi}, \dot{\theta}) = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}J\dot{\phi}^2 \tag{32}$$

where I is the disk's moment of inertia about the disk's minor axis, and J is the moment of inertia about an axis in the plane of the disk.

In addition to constraints we consider two controls. One control allows us to steer the disk (change ϕ), the other that affects the roll angle (θ). We can treat these two controls as external forces. The equations of motion with $q = (x, y, \phi, \theta)$, these two controls and the two constraints are

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{q}}} = u_\phi \mathbf{f}^\phi + u_\theta \mathbf{f}^\theta + \lambda_1 \mathbf{a}_1 + \lambda_2 \mathbf{a}_2 \tag{33}$$

$$\mathbf{f}^\phi = (0, 0, 1, 0), \qquad \mathbf{f}^\theta = (0, 0, 0, 1) \tag{34}$$

$$\frac{\partial L}{\partial \dot{\mathbf{q}}} = (m\dot{x}, m\dot{y}, J\dot{\phi}, I\dot{\theta}) \tag{35}$$

where λ_1, λ_2 are Lagrange multipliers from each constraint and $\mathbf{f}^\theta, \mathbf{f}^\phi$ are the directions of the external forces from our controls. The functions u_ϕ, u_θ specify how strongly we apply the two control forces. The resulting equations of motion, including the two constraints and the two controls, are

$$\begin{aligned} J\ddot{\phi} &= u_\phi \\ (I + mR^2)\ddot{\theta} &= u_\theta \\ \dot{x} &= R\cos\phi\ \dot{\theta} \\ \dot{y} &= R\sin\phi\ \dot{\theta}. \end{aligned} \tag{36}$$

The first two equations are independent of x, y. We can solve for ϕ, θ, from their initial conditions and from the controls. Then afterwards the variables x, y can be updated from the third and fourth equations.

Without the control forces, $\dot{\phi}$, $\dot{\theta}$ are constant frequencies. With $u_\phi = u_\theta = 0$ the solution is

$$\begin{aligned} \phi &= \omega_0 t + \phi_0 \\ \theta &= \Omega_0 t + \theta_0 \end{aligned} \tag{37}$$

where $\omega_0, \Omega_0, \phi_0, \theta_0$ are set from initial conditions and the solutions for x, y follow from equations 36,

$$\begin{aligned} x &= \frac{\Omega_0}{\omega_0} R\sin(\omega_0 t + \phi_0) + x_0 \\ y &= -\frac{\Omega_0}{\omega_0} R\cos(\omega_0 t + \phi_0) + y_0. \end{aligned} \tag{38}$$

1.6 The Knife Edge

The Lagrangian for a knife edge skating down an inclined plane is

$$L(x, y, \phi, \dot{x}, \dot{y}, \dot{\phi}) = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}J\dot{\phi}^2 + mgx\sin\alpha \tag{39}$$

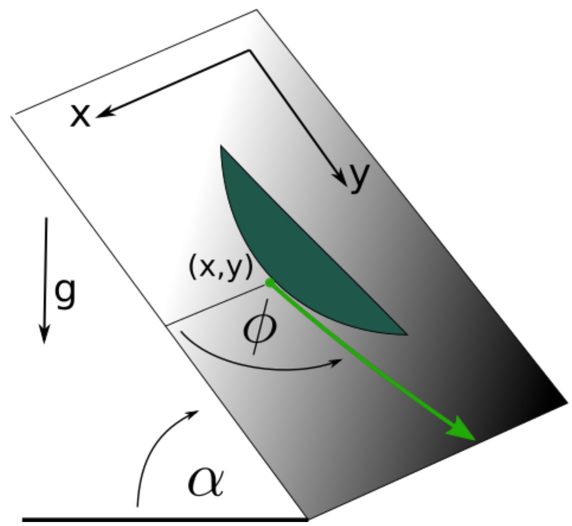

Figure 4: Motion of a knife on an inclined plane. The plane is inclined at angle α. The knife moves in the direction set by the orientation of its edge. This direction is set by angle ϕ. The knife touches the plane at position x, y.

with holonomic constraint

$$\dot{x} \sin \phi - \dot{y} \cos \phi. \tag{40}$$

Here g is gravity and J is the moment if inertia about a vertical axis through the contact point (x, y). The angle α denotes the inclination of the plane and ϕ the orientation of the knife edge on the plane.

The system cannot be solved explicitly in general. But it can be solved with initial conditions $x = \dot{x} = y = \dot{y} = 0$ and $\dot{\phi} = \omega$. The initial condition is rotating but not sliding. The non-holonomic case gives a cycloid motion but no sliding whereas the variational case gives sliding and oscillating solution.

1.7 Hamiltonian Formalism with Constraints

Suppose we have a Hamiltonian system with $2N$ dimensions $H(\mathbf{p}, \mathbf{q})$. The equations of motion in phase space are given by Hamilton's equations. We can restrict the phase space to a sub-manifold. What are the equations of motion? We define the sub-manifold with k constraint functions of phase space

$$\phi_k(\mathbf{p}, \mathbf{q}) = 0 \tag{41}$$

We can think of the problem as adding Lagrange multipliers to an action in phase space, on a path γ that gives $\mathbf{p}(t), \mathbf{q}(t)$

$$S(\gamma) = \int (\mathbf{p}\dot{\mathbf{q}} - H(\mathbf{p},\mathbf{q}) - \sum_k \lambda_k \phi_k(\mathbf{p},\mathbf{q})\ dt.$$

Equations of motion minimize the action S and at the same time obey the constraints. The constraints essentially modify the Hamiltonian

$$\tilde{H}(\mathbf{p},\mathbf{q}) = H(\mathbf{p},\mathbf{q}) + \sum_k \lambda_k \phi_k(\mathbf{p},\mathbf{q}) \tag{42}$$

giving equations of motion

$$\begin{aligned} \dot{q} &= \frac{\partial \tilde{H}}{\partial p} = \frac{\partial H}{\partial p} + \sum_k \lambda_k \frac{\partial \phi_k}{\partial p} \\ \dot{p} &= -\frac{\tilde{\partial H}}{\partial q} = -\frac{\partial H}{\partial q} - \sum_k \lambda_k \frac{\partial \phi_k}{\partial q}. \end{aligned} \tag{43}$$

To maintain a trajectory on the sub-manifold we require that

$$\dot{\phi}_k = \frac{\partial \phi_k}{\partial q}\dot{q} + \frac{\partial \phi_k}{\partial p}\dot{p} = 0 \tag{44}$$

Inserting equation 43 for $\dot{p}$ and $\dot{q}$ we find

$$\dot{\phi}_k = \frac{\partial \phi_k}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial \phi_k}{\partial p}\frac{\partial H}{\partial q} + \frac{\partial \phi_k}{\partial q}\sum_j \lambda_j \frac{\partial \phi_j}{\partial p} - \frac{\partial \phi_k}{\partial p}\sum_j \lambda_j \frac{\partial \phi_j}{\partial q} = 0 \tag{45}$$

$$= \{\phi_k, H\} + \sum_j \lambda_j \{\phi_k, \phi_j\} = 0. \tag{46}$$

If the matrix of Poisson brackets $\{\phi_k, \phi_j\}$ is non singular then equation 46 can be inverted to uniquely give $\boldsymbol{\lambda}$ as a function of $\mathbf{p}, \mathbf{q}$.

We take c_{ij} to be the inverse of $\{\phi_k, \phi_j\}$ so that $c_{ij}\{\phi_j, \phi_k\} = \delta_{ik}$ gives the identity matrix. It is possible to define a new Poisson bracket in terms of the old one

$$\{F_1, F_2\}' = \{F_1, F_2\} + \sum_{ij} \{\phi_i, F_1\} c_{ij} \{\phi_j, F_2\} \tag{47}$$

This gives a new Poisson bracket that is satisfied in the sub-manifold. We think about the constraint as giving a *dimensional reduction*. Complications arise if the number of constraints is odd as the sub-manifold must be even dimension for it to have symplectic dynamics. The dimension of the sub-manifold depends on the number of independent constraints.

We can generate the equations of motion with a new Poisson bracket, so we don't even need the original Hamiltonian. The Poisson bracket can replace the Hamiltonian, though the original Hamiltonian, if time independent, is conserved and it may be useful to refer to it as conserved quantity.

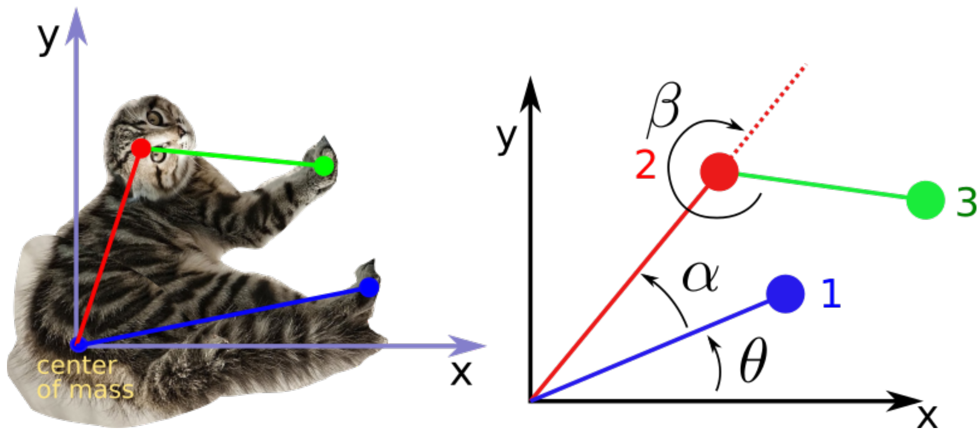

Figure 5: For falling cats. The center of mass is assumed to be at the origin. The cat is modeled with an additional three masses, one at the hind feet, the other at the front feet and the third one at the head. The three masses are connect with massless stiff rods. Body deformation space is described by α, β but body orientation with θ. By increasing α then β, then decreasing α then decreasing β, the cat can return to its original body shape. However θ will not be the same. The cat's orientation is changed by this series of body deformations.

1.8 A Falling Cat

Consider a two dimensional cat in the xy plane that is in the air. We neglect gravitational acceleration as the cat falls and assume that the cat has no angular momentum. The center of mass is at the origin; see Figure 5. The hind legs are at x_1, y_1. We assume that they are a distance R from the center of mass. The head is at x_2, y_2 and is a distance R from the center of mass. The right legs are at x_3, y_3 and are a distance R from the head. Using the angles θ, α, β as shown in Figure 5 the positions of each mass,

$$\begin{aligned}
(x_1, y_1) &= (R\cos\theta, R\sin\theta) \\
(x_2, y_2) &= (R\cos(\alpha+\theta), R\sin(\alpha+\theta)) \\
(x_3, y_3) &= (R\cos(\alpha+\theta) + R\cos(\alpha+\beta+\theta), R\sin(\alpha+\theta) + R\sin(\alpha+\beta+\theta))
\end{aligned} \tag{48}$$

A constraint is that the z-component of angular momentum is conserved and we set it to zero assuming that the cat is initially not rotating

$$L = \sum_i m_i \left(x_i \frac{dy_i}{dt} - y_i \frac{dx_i}{dt} \right) = 0. \tag{49}$$

Multiplying by dt we find

$$\sum_i (x^i dy^i - y^i dx^i) = 0.$$

It's a non-holonomic constraint. Equations 48 when differentiated and the constraint on angular momentum are consistent with

$$(4 + 2\cos\beta)d\theta + (3 + 2\cos\beta)d\alpha + (1 + \cos\beta)d\beta = 0. \tag{50}$$

This equation can be inverted with

$$\begin{aligned} d\theta &= -A_\alpha(\alpha, \beta)d\alpha - A_\beta(\alpha, \beta)d\beta \\ A_\alpha(\alpha, \beta) &= \frac{3 + 2\cos\beta}{4 + 2\cos\beta} \\ A_\beta(\alpha, \beta) &= \frac{1 + \cos\beta}{4 + 2\cos\beta} \end{aligned} \tag{51}$$

Because we can get changes in θ from changes in α, β we can describe the shape space with α, β alone. The configuration space for α, β are a torus as they are two angles. The angle θ describes the overall orientation of the cat. While the cat is falling it can twist its legs by changing α, β. By a series of transformations, it can change its orientation and land on its feet even if it originally was falling with its feet upward.

We can define a one form in α, β space,

$$\omega = A_\alpha d\alpha + A_\beta d\beta = -d\theta. \tag{52}$$

Its derivative

$$d\omega = \left(\frac{\partial A_\beta}{\partial \alpha} - \frac{\partial A_\beta}{\partial \beta} \right) d\alpha \wedge d\beta \tag{53}$$

Consider moving on a loop γ in α, β space. Along this loop path, $d\theta$ is given by equations 51. We can integrate

$$\Delta\theta = \oint_\gamma d\theta = \oint_\gamma -\omega.$$

Here we are integrating on a path $\alpha(t), \beta(t)$. Using Stoke's theorem this integral is equal to

$$\Delta\theta = \int_D -d\omega$$

where D is the area inside the loop. As $d\omega$ integrated in an area is not necessarily zero, a loop in α, β space can give a change in the orientation angle θ.

A can be called a *vector potential* or a *connection*. The two form $d\omega$ can be called a *curvature tensor*.

Note: conservation of angular momentum can be considered a non-holonomic constraint, however it is an ideal constraint as it does no work.

In terms of infinitesimal transformations, a small change in α while fixing β is accomplished by this operator

$$L_\alpha = A_\alpha \partial_\theta + \partial_\alpha$$

Likewise a small change in β while fixing β is accomplished by this operator

$$L_\beta = A_\beta \partial_\theta + \partial_\beta$$

Computing the commutators of these

$$[L_\alpha, L_\beta] = (\partial_\alpha A_\beta - \partial_\beta A_\alpha)\partial_\theta$$

Going around a small loop in α, β space gives a change in the direction of ∂_θ. And we recognize the commutator as the curvature in equation 53.

We can construct a mechanical connection

$$\Gamma = (A_\alpha d\alpha + A_\beta d\beta - d\theta)\partial_\theta \tag{54}$$

which projects like this; $\Gamma L_\alpha = 0$, $\Gamma L_\beta = 0$, $\Gamma L_\theta = \partial_\theta$. Taking $d\Gamma$ and projecting onto the horizontal subspace, we again get the curvature in equation 53.

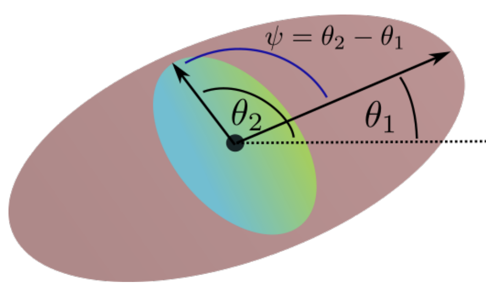

Figure 6: Elroy's Beanie is a nice example illustrating geometric phase and flat mechanical connection. Assuming angular momentum conservation, a full rotation in ψ of the inner oval with respect to the outer one, gives a non-zero change in $\Delta\theta_1$, the orientation of the outer oval.

1.9 Elroy's Beanie

We consider two oval solid bodies connected with a pivot point at their center of mass, see Figure 6. The angle of the outer oval body, with moment of inertia I_1 is oriented with respect to the inertial frame with angle θ_1. The inner oval body, with moment of inertia I_2, is oriented with respect to the inertial frame with angle θ_2. The angle between the two ovals is $\psi = \theta_2 - \theta_1$. The total configuration space is described by two angles. The two bodies can be described by ψ alone. So we can think of the internal freedom as described by angle ψ. We describe the orientation of the body with respect to the outside with $\theta = \theta_1$. The total angular momentum

$$L = I_1\dot{\theta}_1 + I_2\dot{\theta}_2 = I_1\dot{\theta} + I_2(\dot{\psi} + \dot{\theta})$$

$$dL = I_1 d\theta + I_2(d\psi + d\theta)$$

If angular momentum is conserved

$$d\theta = -\frac{I_1}{I_1 + I_2} d\psi$$

If we move around a loop in ψ

$$\Delta\theta = \int_0^{2\pi} -\frac{I_1}{I_1 + I_2} d\psi = -\frac{2\pi I_1}{I_1 + I_1} \tag{55}$$

So an internal rotation of the smaller oval gives a partial rotation of the entire mechanism.

Geometrically this can be described with a flat connection

$$A = d\theta + \frac{I_1}{I_1 + I_2} d\psi$$

This can be called a mechanical connection. The fibre is S^1 and with varying θ, The projection from the entire space $S^1 \times S^1 \to S^1$ is $\pi(\theta, \psi) \to \psi$.

2 Control Systems

Two main drivers. Can the system be moved from one state to another one. Which states are accessible?

2.1 Control of an inverted pendulum on a cart

The configuration space of a pendulum on a cart (see Figure 7) is $\mathcal{R} \times S^2$ and described by horizontal position of the cart s and pendulum angle ϕ. The velocity of m is

$$(\dot{x}, \dot{z})_m = (\dot{s} + l\cos\phi\ \dot{\phi}, -l\sin\phi\ \dot{\phi})$$

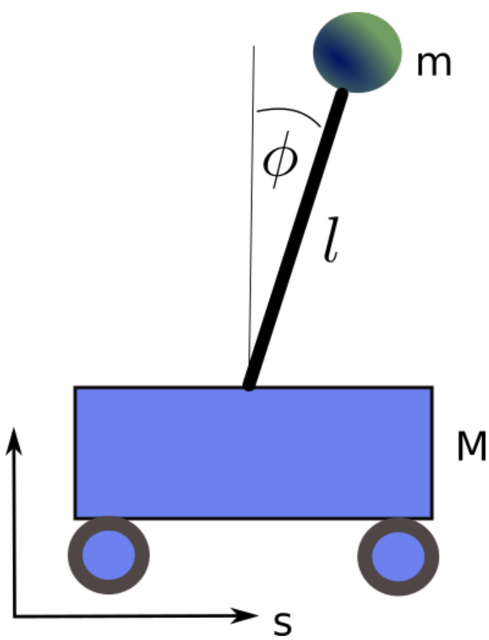

Figure 7: An inverted pendulum on a cart. The cart mass is M, the pendulum has length l, the pendulum bob mass is m. The angle ϕ describes the orientation of the pendulum with respect to the cart. The cart horizontal position is described with s. Gravity acceleration g pulls downward.

giving a total kinetic energy

$$T = \frac{1}{2}M\dot{s}^2 + \frac{1}{2}m\left((\dot{s} + l\cos\phi\ \dot{\phi})^2 + l^2\sin^2\phi\ \dot{\phi}^2\right)$$

The Lagrangian

$$L(s, \phi, \dot{s}, \dot{\phi}) = \frac{1}{2}(M+m)\dot{s}^2 + \frac{1}{2}ml^2\dot{\phi}^2 + ml\cos\phi\ \dot{s}\dot{\phi} - mgl\cos\phi. \tag{56}$$

We apply an external force u on the cart alone

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{s}} = u \tag{57}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} = \frac{\partial L}{\partial \phi} \tag{58}$$

We can define momenta

$$p_s = \frac{\partial L}{\partial \dot{s}} = (M+m)\dot{s} + ml\cos\phi\ \dot{\phi} \tag{59}$$

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = ml^2\dot{\phi} + ml\cos\phi\ \dot{s} \tag{60}$$

We can write this in matrix form

$$\begin{pmatrix} p_s \\ p_\phi \end{pmatrix} = \begin{pmatrix} (M+m) & ml\cos\phi \\ ml\cos\phi & ml^2 \end{pmatrix} \begin{pmatrix} \dot{s} \\ \dot{\phi} \end{pmatrix} \tag{61}$$