Homework # 6. Physics 265, Spring 2023

Topic: *On Error Correction*

Due date: Thursday April 13, 2023. This problem set may be long. If so choose a subset of problems to work on. You can also create your own problem and share it.

1. **Identities**

   a) Show that $HXH = Z$.

   b) Show that $HZH = X$.

   c) Show that $HXZH = -XZ$.

   d) Find a similar identity for Y.

   Here H is the Hadamard operator and X, Y, Z are the Pauli matrices.

2. **On Measuring $X_1X_2X_3X_4X_5X_6$ which is used in Shor's 9-bit code**

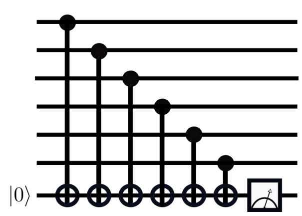

Figure 1: A circuit that computes the parity of 6 bits using an ancilla bit.

   In Figure 1 we show a circuit that computes the parity of 6 bits. The bottom bit is an ancilla bit that is measured at the end. A 1 is measured if there is an odd number of 1s in the 6 qubit state, which are in the top 6 lines in the circuit.

   Consider the operator $Z_1Z_2Z_3Z_4Z_5Z_6$. Here $Z_1 = Z \otimes I \otimes I \otimes I \otimes I \otimes I$ and the other operators are defined similarly so the index tells you which qubit the Z operates on. The operator $Z_1Z_2Z_3Z_4Z_5Z_6$ has eigenvalues 1, -1.

   In the circuit in Figure 1, a 1 is measured if the input in the first 6 qubits is an eigenstate of $Z_1Z_2Z_3Z_4Z_5Z_6$ with eigenvalue 1. In the circuit a 0 is measured if the input in the first 6 qubits is an eigenstate of $Z_1Z_2Z_3Z_4Z_5Z_6$ with eigenvalue 0.

   Because a different value is measured for each eigenstate of $Z_1Z_2Z_3Z_4Z_5Z_6$, the circuit in Figure 1 essentially measures this operator.

   a) Design a circuit that measures the operator $X_1X_2X_3X_4X_5X_6$.

   Hint: The states $|+\rangle, |-\rangle$ are eigenstates of X and you can transfer to the $|+\rangle, |-\rangle$ basis using a Hadamard operator.

   b) Design a circuit for a 9 bit system that uses two extra ancilla bits to measure $X_1X_2X_3X_4X_5X_6$ and $X_4X_5X_6X_7X_8X_9$.

3. **On correcting errors without measurement**

   We consider the 3-bit phase-flip error correction code that encodes

   $$|0\rangle \to |+++\rangle \qquad \text{and} \qquad |1\rangle \to |---\rangle$$

   A circuit that performs error detection is shown in Figure 2.

   a) Design a circuit that corrects the errors after measurement.

   b) Design a circuit that corrects the errors without measurement.

4. **Creating a projection operator for the encoding space**

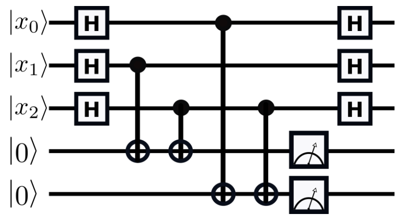

Figure 2: Detecting errors in the 3-bit phase-flip error correction code. The bottom two ancilla bits are measured to detect a single **phase** flip error in the top three bits.

Consider the stabilizer group S generated by $\langle X_1X_2, X_2X_3\rangle$ that gives the 3-bit phase-flip quantum error correcting code.

The encoded or stabilized space C contains vectors $|v\rangle$ such that $g|v\rangle = |v\rangle$ for all $g \in S$.

Show that

$$P = \frac{1}{4}(I + X_1X_2)(I + X_2X_3) \qquad (1)$$

is a projection operator in the 3 qubit space that projects to the encoding subspace C.

In other words show that

$$P = |+++\rangle\langle+++| + |---\rangle\langle---|.$$

It may be useful to write

$$X = |+\rangle\langle+| - |-\rangle\langle-|$$
$$I = |+\rangle\langle+| + |-\rangle\langle-|$$

Notice that X_1X_2 has eigenvalues ± 1 and $(I + X_1X_2)/2$ has eigenvalues $0, 1$.

5. **Creating a projection operator for the stabilized space**

Consider the stabilizer group S generated by k independent generators $\langle g_1, g_2, ...g_k\rangle$. with generators in the generalized Pauli group $\mathcal{G}_n$.

Show that a projector giving the stabilized or encoding space C can be constructed by

$$P = \frac{1}{2^k}\prod_{j=1}^{k}(I + g_j). \qquad (2)$$

A vector $|v\rangle$ that is stabilized by stabilizer group S satisfies

$$g|v\rangle = |v\rangle \text{ for all } g \in S.$$

The coding subspace C is the vector subspace stabilized by S.

A vector $|w\rangle$ that is perpendicular to subspace C satisfies $\langle w|v\rangle = 0$ for all $|v\rangle \in C$.

This can be done in two parts.

a) Show that for any $|v\rangle \in C$, the projection $P|v\rangle = |v\rangle$.

b) Show that for any $|w\rangle \perp C$, the projection $P|w\rangle = 0$.

Stabilizer groups contain operators that are in the generalized Pauli group $\mathcal{G}_n$. Stabilizer groups cannot contain $-I$ so all elements commute and all elements are Hermitian. The eigenvalues of any operator in $\mathcal{G}_n$ has eigenvalues ± 1. Consequently, the matrices $(I + g_j)/2$ have eigenvalues $0, 1$. As all members of S must commute, all members of $\{(I + g_j)\}$ must also commute. A set of matrices that commute are all simultaneously diagonalizable. That means that the eigenvalues of P must be $\in \{0, 1\}$.

6. **The 5-bit stabilizer code**

What is the minimum size for a quantum code which encodes a single qubit so that any error on a single qubit in the encoded state can be detected and corrected? The answer is 5-bits.

The stabilizer group S for the five bit error correcting code is generated by these four operators

$$\langle XZZXI, IXZZX, XIXZZ, ZXIXZ\rangle$$

that are also shown in Table 1. These could also be written as

$$\langle X_1 Z_2 Z_3 X_4, X_2 Z_3 Z_4 X_5, X_1 X_3 Z_4 Z_5, Z_1 X_2 X_4 Z_5 \rangle$$

or as

$$\langle X \otimes Z \otimes Z \otimes X \otimes I, I \otimes X \otimes Z \otimes Z \otimes X,$$
$$X \otimes I \otimes X \otimes Z \otimes Z, Z \otimes X \otimes I \otimes X \otimes Z \rangle$$

Table 1	
	Generators
g_1	$X\ Z\ Z\ X\ I$
g_2	$I\ X\ Z\ Z\ X$
g_3	$X\ I\ X\ Z\ Z$
g_4	$Z\ X\ I\ X\ Z$

Notice that the generators are cyclic permutations of each other.

Verify that the five qubit code can protect against an arbitrary single qubit error.

In other words check that the set

$$S_E = \{X_1, X_2, X_3, X_4, X_5, Z_1, Z_2, Z_3, Z_4, Z_5\}$$

is a correctible set of errors.

A correctable set of errors satisfies the following: For every pair E_i, E_j of errors in the set S_E, the operator $E_i^\dagger E_j$ either is in stabilizer group S or anticommutes with a generator of S. You will need to show that this is true for the set S_E and using the stabilizer generators in the table.

We don't need to include Y errors as if X and Z errors can be corrected and the product XZ for single qubits can be corrected, then the Y errors can also be corrected. This then means that any single qubit error can be corrected as any unitary transformation for a single qubit can be written as a linear combination of I, X, Y, Z operators.

7. **The 5-bit stabilizer error correcting code syndrome**

Describe a syndrome that allows correction of errors in the set

$$S_E = \{X_1, X_2, X_3, X_4, X_5, Z_1, Z_2, Z_3, Z_4, Z_5\}$$

for the 5 bit stabilizer code generated by the generators in Table 1.

You need to fill in the following table with 1s and -1s that represent measurements of the stabilizer generators. Elements that commute give 1.

	$XZZXI$	$IXZZX$	$XIXZZ$	$ZXIXZ$
X_1				
X_2				
X_3				
X_4				
X_5				
Z_1				
Z_2				
Z_3				
Z_4				
Z_5				

Does the syndrome allow you to also correct for additional errors in the set $\{Y_1, Y_2, Y_3, Y_4, Y_5\}$?

8. **The Quantum Hamming Bound**

Any non-degenerate quantum $[[n, k]]$ code that corrects all errors with weight t or less must satisfy the quantum Hamming bound

$$\sum_{i=0}^{t} 3^i \binom{n}{i} \leq 2^{n-k} \tag{3}$$

Suppose the number of qubits you want to encode is $k = 1$, but you want to be able to correct for all errors with weight $t = 2$ or less. That means you could correct all errors that affect 1 and 2 qubits. How many qubits n would you need in your error correction code?

You need to compute both sides of the equation 3 for various values of n and see for what n the inequality becomes satisfied.

9. **A fault tolerant and transversal phase gate for the 3 bit bit flip error correcting code**

   The phase gate

$$S = P_{\frac{\pi}{2}} = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} = \mathrm{diag}(1, i)$$

   Consider the 3-bit code with basis for its encoded states

$$\left|\tilde{0}\right\rangle = |000\rangle$$
$$\left|\tilde{1}\right\rangle = |111\rangle$$

   We would like to find a way to apply the phase gate transversally so that an operation is performed on each encoded qubit separately. The desired operation would give the same result as the phase gate but would be fault tolerant.

   The operation we desire looks like $Q^1 \otimes Q^2 \otimes Q^3$ where each operation is on a single qubit and it should send $|000\rangle \to |000\rangle$ and $|111\rangle \to i\,|111\rangle$.

   a) Show that $\bar{S} = S \otimes S \otimes S$ in the encoded 3 bit space does not carry out the single qubit phase gate.

   b) Find a different gate Q such that $\bar{Q} = Q \otimes Q \otimes Q$ does give the phase gate.

   c) Suppose a single bit flip error occurs prior to applying $\bar{Q}$. Show that the single bit flip error would still be corrected by the syndrome associated with this code, up to a global phase. Note that a phase flip error would be introduced and this would be a problem.

10. **Your Problem here**