Homework # 4. Physics 265, Spring 2024

Topic: *On the density matrix, generalized measurement, entropy and quantum channels.*

Due date: Thursday Mar 28, 2024. Choose a subset of about 7 problems to work on. You can also create and share your own problems or do problems from a different problem set or book.

1. **On 2×2 density matrices**

   A density matrix $\boldsymbol{\rho}$ of a 2 state system is a 2x2 Hermitian matrix $\boldsymbol{\rho}$ that is positive definite and has trace 1.

   A Hermitian matrix satisfies $H = H^\dagger$ where † is the complex transpose. With

   $$H = \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix} \qquad H^\dagger = \begin{pmatrix} a_{00}^* & a_{10}^* \\ a_{01}^* & a_{11}^* \end{pmatrix}$$

   (a) Show that any 2x2 Hermitian matrix with trace 1 can be written in the following form

   $$\boldsymbol{\rho} = \frac{1}{2}\left(\mathbf{I} + \mathbf{p}\cdot\boldsymbol{\sigma}\right)$$

   where $\mathbf{p}$ is a vector of real numbers $\mathbf{p} = (p_x, p_y, p_z)$ and $\mathbf{I}$ is the identity matrix. The vector of Pauli spin matrices $\boldsymbol{\sigma} = (\boldsymbol{\sigma}_x, \boldsymbol{\sigma}_y, \boldsymbol{\sigma}_z)$. The vector $\mathbf{p}$ is sometimes called a polarization.
   The Pauli matrices

   $$\boldsymbol{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
   $$\boldsymbol{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$
   $$\boldsymbol{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

   The Pauli spin matrices obey

   $$\boldsymbol{\sigma}_x^2 = \boldsymbol{\sigma}_y^2 = \boldsymbol{\sigma}_z^2 = \mathbf{I}.$$

   The Pauli spin matrices obey

   $$\mathrm{tr}\boldsymbol{\sigma}_x = \mathrm{tr}\boldsymbol{\sigma}_y = \mathrm{tr}\boldsymbol{\sigma}_z = 0.$$

   (b) Show that

   $$\boldsymbol{\rho}^2 = \frac{1}{4}\left((1+p^2)\mathbf{I} + 2\mathbf{p}\cdot\boldsymbol{\sigma}\right)$$

   Note that the Pauli matrices do not commute: $\sigma_x\sigma_z \neq \sigma_z\sigma_x$

   (c) Show that

   $$\mathrm{tr}(\boldsymbol{\rho}^2) = \frac{1}{2}\left(1+p^2\right)$$

   (d) With $\boldsymbol{\rho} = \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{01}^* & \rho_{11} \end{pmatrix}$ show that

   $$\rho_{00} = \frac{1}{2}(1+p_z)$$
   $$\rho_{11} = \frac{1}{2}(1-p_z)$$
   $$\rho_{01} = \frac{1}{2}(p_x - ip_y)$$

   (e) Show that

   $$\det\boldsymbol{\rho} = \frac{1}{4}\left(1-p^2\right).$$

2. **On eigenvalues of 2×2 density matrices and meaning of positive definite**

   (a) The eigenvalues of a 2×2 matrix $\mathbf{A}$ can be written in terms the trace and determinant of the matrix;

   $$\lambda_\pm = \frac{1}{2}\left(\mathrm{tr}\mathbf{A} \pm \sqrt{(\mathrm{tr}\mathbf{A})^2 - 4\det\mathbf{A}}\right).$$

   Compute the eigenvalues of a density matrix $\boldsymbol{\rho}$ with polarization $\mathbf{p}$.

   (b) A matrix $\boldsymbol{\rho}$ is **positive semidefinite** if for all possible $|\psi\rangle$ state vectors, $\langle\psi|\,\boldsymbol{\rho}\,|\psi\rangle \geq 0$.
   Show that for a 2×2 density matrix $\boldsymbol{\rho}$ to be positive semidefinite, the polarization $|p| \leq 1$.

3. **On pure and mixed states for a two state system**

An example of a pure state is

$$|\psi\rangle = a|0\rangle + b|1\rangle$$

with $aa^* + bb^* = 1$.

The density matrix of this pure state is

$$\begin{aligned}\boldsymbol{\rho} &= |\psi\rangle\langle\psi| \\ &= aa^*|0\rangle\langle 0| + ab^*|0\rangle\langle 1| \\ &\quad + a^*b|0\rangle\langle 1| + bb^*|1\rangle\langle 1| \\ &= \begin{pmatrix} aa^* & ab^* \\ a^*b & bb^* \end{pmatrix}\end{aligned}$$

(a) Show that the eigenvalues of the density matrix for a pure state are $1, 0$.

(b) Show that the polarization of the density matrix for a pure state has $|p| = 1$ and so lies on a sphere (the Bloch sphere!)

An example of a mixed state is one with probability p_0 that it is in state $|0\rangle$ and $p_1 = 1 - p_0$ that it is in state $|1\rangle$. The density matrix is weighted by the probabilities

$$\boldsymbol{\rho} = p_0|0\rangle\langle 0| + p_1|1\rangle\langle 1|.$$

The density matrix of a **mixed state** with probability p_0 that it is in state $|\psi_0\rangle$ and probability p_1 that it is in state $|\psi_1\rangle$ is

$$\boldsymbol{\rho} = p_0|\psi_0\rangle\langle\psi_0| + p_1|\psi_1\rangle\langle\psi_1|.$$

(c) Compute the density matrix for a mixed state that has probability $p_0 = 1/2$ that it is in state $|\psi_0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and probability $p_1 = 1/2$ that it is in state $|\psi_1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$.

(d) Find the polarization and the eigenvalues of this density matrix.

4. **On distinguishability of pure state decompositions**

Consider a system that has probability p_i of being in pure state with density operator ρ_i. Here i is an integer index and $\sum_{i=1}^N p_i = 1$. We have a set of N probabilities $\{p_i\}$ and a set of pure state density operators $\{\rho_i\}$. We assume $\rho_i \neq \rho_j$ for all $i \neq j$.

The density matrix for the full system is described by the sum $\rho = \sum_{i=1}^N p_i\rho_i$.

If we have a density matrix, ρ, a sum $\rho = \sum_{i=1}^N p_i\rho_i$, where $\{\rho_i\}$ are pure states, and $\{p_i\}$ are a set of probabilities, is a decomposition of the density matrix in terms of pure states. The decomposition is non-trivial if $\rho_i \neq \rho_j$ for all pairs i, j.

a) Is a non-trivial decomposition of a density matrix into a sum of pure states unique?

b) If not, find an example of two systems that have the same density matrix but have different decompositions in terms of a sum of pure states.

Hints: It is enough to consider 2 state systems, and sums that contain only two terms ($N = 2$).

This has some bizarre implications for any non-linear model for quantum evolution.

5. **On traced density matrices**

Consider the Bell state

$$|\psi\rangle_{AB} = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$

a) What is the density matrix ρ_{AB} for the full 2-qubit system?

b) Compute the density matrix

$$\rho_A = \mathrm{tr}_B\, \rho_{AB}.$$

This is the reduced density matrix for the first qubit.

c) Compute the density matrix

$$\rho_B = \mathrm{tr}_A\, \rho_{AB}.$$

This is the reduced density matrix for the second qubit.

6. **On von Neumann entropy**

   The **von Neumann entropy** of a density matrix $\boldsymbol{\rho}$ is

   $$S(\boldsymbol{\rho}) = -\mathrm{tr}(\boldsymbol{\rho} \log_2 \boldsymbol{\rho}) = -\sum_i \lambda_i \log_2 \lambda_i$$

   where λ_i are the eigenvalues. (Note instead of $\log_2$ often people define this with the natural log, giving $S = \mathrm{tr}(\rho \ln\ \rho)$. You can use whichever you prefer).

   (a) What is the maximum possible von Neumann entropy for a 2×2 density matrix? Find a 2×2 density matrix that has the maximum possible von Neumann entropy.

   Note: the eigenvalues are constrained by $\sum_i \lambda_i = 1$.

   (b) What are the eigenvalues of an $N \times N$ density matrix with the maximum possible von Neumann entropy?

   (c) Show that a 2×2 density matrix that has the maximum possible von Neumann entropy also has the mimimum possible value of $\mathrm{tr}\boldsymbol{\rho}^2$ and minimum possible polarization $|p|$.

   (d) What is the von Neumann entropy of a pure state (in any dimension)?

   (e) A system starts in a pure state $\psi = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. Its density matrix is

   $$\boldsymbol{\rho}_{init} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

   What are the eigenvalues of the initial density matrix, what is the von Neumann entropy and what is the polarization $\mathbf{p}$?

   The state is then measured in the $|0\rangle$, $|1\rangle$ basis but the result of the measurement is **not recorded**. The probability that $|0\rangle$ is measured is $1/2$ and the probability that $|1\rangle$ is measured is $1/2$. After measurement the state has $p_0 = 1/2$ that it is in state $|0\rangle$ and $p_1 = 1/2$ that it is in state $|1\rangle$. The density matrix after measurement is

   $$\boldsymbol{\rho}_{after} = p_0 \,|0\rangle \langle 0| + p_1 \,|1\rangle \langle 1| = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

   What are the eigenvalues of the new density matrix, what is the von Neumann entropy and what is the polarization?

   The entropy should have increased due to uncertainty introduced by measurement. We say quantum interference information is 'erased'.

   (f) What happens to the entropy if the measurement value is recorded?

7. **On a generalized measurement (POVM)**

   Consider the following five states:

   $$|e_k\rangle = \cos(2\pi k/5)\,|0\rangle + \sin(2\pi k/5)\,|1\rangle$$

   where $k \in \{0, ..., 4\}$. These five states form a pentagon on the XZ-plane of the Bloch sphere. Consider the operators

   $$E_k = \frac{2}{5}\,|e_k\rangle \langle e_k|$$

   a) show that

   $$\begin{aligned} E_k &= \frac{2}{5}\,|e_k\rangle \langle e_k| \\ &= \frac{1}{5}\begin{pmatrix} 1 + \cos(4\pi k/5) & \sin(4\pi k/5) \\ \sin(4\pi k/5) & 1 - \cos(4\pi k/5) \end{pmatrix} \end{aligned}$$

   b) Show that the operators E_k are all positive semi-definite.

c) Show that the set of operators $\{E_k\}$ for $k \in \{0, ..., 4\}$ forms a valid POVM.

Elements of a POVM must satisfy $\sum_k E_k = I$ and they must be positive semi-definite.

Hints: It is useful to write $\sin a \cos a = \frac{1}{2}\sin(2a)$ and use similar relations for the square of cosines and sines.

It is useful to know that the sum of complex roots of unity is zero.

$$\sum_{k=0}^{4} e^{i2\pi k/5} = 0$$

Also the real and imaginary parts of this sum also sum to zero. Also

$$\sum_{k=0}^{4} e^{i4\pi k/5} = 0$$

A positive semi-definite Hermitian operator has non-negative eigenvalues.

8. **The amplitude damping channel**

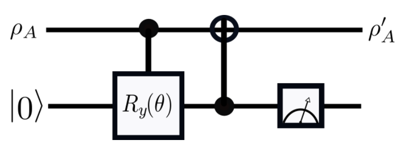


Figure 1: A circuit model for amplitude damping.

A circuit model for amplitude damping is shown in Figure 1. The rotation

$$\begin{aligned} R_y(\theta) &= e^{-i\theta\sigma_y/2} = \cos(\theta/2)\mathbf{I} - i\mathbf{Y}\sin(\theta/2) \\ &= \cos(\theta/2)\mathbf{I} + \sin(\theta/2)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \cos(\theta/2) & \sin(\theta/2) \\ -\sin(\theta/2) & \cos(\theta/2) \end{pmatrix}. \end{aligned}$$

The input into the top qubit bit is a density matrix which can be described with a set of probabilities and pure states. Measurement of the bottom qubit is not necessary.

a) What is the unitary transformation U performed by the two controlled gates? This unitary transformation acts on the full 2 qubit space $\mathcal{H}_A \otimes \mathcal{H}_B$ where $\mathcal{H}_A$ is the Hilbert space for the top qubit and $\mathcal{H}_B$ is the Hilbert space for the bottom qubit. You can write this as a matrix or in a ket/bra form.

b) Assume the first qubit is described with pure state

$$|\phi\rangle_A = a|0\rangle + b|1\rangle .$$

Show that the final density matrix

$$\rho'_A = \begin{pmatrix} aa^* + bb^*\gamma & ab^*\sqrt{1-\gamma} \\ a^*b\sqrt{1-\gamma} & bb^*(1-\gamma) \end{pmatrix}$$

with $\gamma = \sin^2(\theta/2)$.

c) Suppose that a 0 is measured in the second qubit in the circuit in Figure 1. Compute the matrix

$$\mathbf{M}_0 = \sum_{ij} M_{0,ij} |i\rangle_A \langle j|_A$$

that acts on vectors in $\mathcal{H}_A$ and has components

$$M_{0,ij} = \langle i|_A \langle 0|_B U |j\rangle_A |0\rangle_B .$$

d) Suppose that a 1 is measured in the second qubit. Compute the matrix

$$\mathbf{M}_1 = \sum_{ij} M_{0,ij} |i\rangle_A \langle j|_A$$

with components

$$M_{1,ij} = \langle i|_A \langle 1|_B U |j\rangle_A |0\rangle_B .$$

e) Show that the amplitude damping channel can be described with

$$\rho'_A = \mathbf{M}_0\rho_A\mathbf{M}_0^\dagger + \mathbf{M}_1\rho_A\mathbf{M}_1^\dagger.$$

f) Show that the set $\{\mathbf{M}_0, \mathbf{M}_1\}$ are Kraus operators giving a POVM generalized measurement. (The operators $\mathbf{M}_0^\dagger \mathbf{M}_0, \mathbf{M}_1^\dagger \mathbf{M}_1$ are positive semi-definite and together they are complete).

9. **More on amplitude damping channels**

In the previous problem, the bottom qubit is initialized at $|0\rangle$. What if we set the bottom qubit in the state $|1\rangle$?

a) Using the unitary operation we computed in the previous problem, compute two more operators

$$\mathbf{M}_2 = \langle 0|_B \, U \, |1\rangle_B \quad \text{and} \quad \mathbf{M}_3 = \langle 1|_B \, U \, |1\rangle_B \, .$$

The set $\{\mathbf{M}_2, \mathbf{M}_3\}$ also are Kraus operators that form a POVM measurement.

b) Show that the four operators

$$\mathbf{N}_0 = \sqrt{p} \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix}$$
$$\mathbf{N}_1 = \sqrt{p} \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix}$$
$$\mathbf{N}_2 = \sqrt{1-p} \, \mathbf{M}_2$$
$$\mathbf{N}_3 = \sqrt{1-p} \, \mathbf{M}_3$$

also are a set of Kraus operators that form a POVM measurement.

c) Create a circuit that gives four possible measurement outcomes, N_0, N_1, N_2, N_3? (I am not all that sure about answers to this).

I think if you specify the state of the second quit you would get a pair of measurement operators, one for each possible measurement. For example if you set the second qubit to $\sqrt{p}\,|0\rangle + \sqrt{1-p}\,|1\rangle$ you would get two measurement operators that depend on p that give a transition between the pair $\{M_0, M_1\}$ (if $p = 1$) to $\{M_2, M_3\}$ (with $p = 0$). This is 2 measurement operators, not 4! You could instead have one bit in a mixed state with density matrix $p\,|0\rangle\langle 0| + (1-p)\,|1\rangle\langle 1|$. If we include extra bits we can measure 2 bits giving 4 possible measurements.

10. **Orthogonality of operators via the Frobenius (trace) inner product**

The Frobenius inner product is the trace of the product of two complex matrices;

$$\langle \mathbf{A}, \mathbf{B} \rangle_F = \operatorname{tr}\big(\mathbf{A}^\dagger \mathbf{B}\big)$$

Using this inner product we can find an orthonormal basis in which to decompose 2×2 complex matrices.

a) Show that the set

$$\{\mathbf{I}, \boldsymbol{\sigma}_x, \boldsymbol{\sigma}_y, \boldsymbol{\sigma}_z\} \tag{1}$$

forms an orthogonal basis with respect to the Frobenius inner product for 2×2 dimensional operators. Here $\boldsymbol{\sigma}_x, \boldsymbol{\sigma}_y, \boldsymbol{\sigma}_z$ are the Pauli matrices and $\mathbf{I}$ is the identity operator.

b) This basis of equation 1 is not orthonormal. Modify it so that it is normalized. We would like $\operatorname{tr}\big(\mathbf{A}\mathbf{A}^\dagger\big) = 1$ for each operator $\mathbf{A}$ in the basis set.

c) Is the basis from part b large enough to span the entire space of 2×2 complex matrices?

d) How would you describe traceless 2×2 complex matrices in terms of your basis in part b? (Show how would you write a matrix as a sum of coefficients times basis elements).

e) How would you describe Hermitian 2×2 complex matrices in terms of your basis in part b?

f) How would you describe Hermitian 2×2 complex matrices with trace of 1 in terms of your basis in part b?

g) Find an orthonormal basis with respect to the Frobenius inner product for 3×3 complex matrices.

11. **On constructing unitary operators in bipartite systems**

Consider a tensor product space $\mathcal{H}_1 \otimes \mathcal{H}_2$ that is comprised of two finite dimensional Hilbert spaces. The space $\mathcal{H}_1$ has a projective measurement consisting of a complete set of n orthogonal Hermitian projection operators $\{P_0, P_1, P_2....\}$. We also have a set of n unitary operators $\{U_0, U_1, U_2, ...\}$ operating on $\mathcal{H}_2$.

We construct an operator

$$V = \sum_{i=0}^{n-1} P_i \otimes U_i$$

a) Show that V is unitary.

b) Can a similar unitary operator be constructed from operators of a POVM measurement?

This problem is relevant for construction of unitary operators used in quantum random walks.

12. **The number of Kraus operators in the operator sum decomposition of a quantum channel**

a) A density operator ρ is constructed for a 4 qubit system. The system is not in a pure state. The density matrix is written as a sum

$$\rho = \sum_i p_i \rho_i$$

where pure states $|a_i\rangle$ give $\rho_i = |a_i\rangle \langle a_i|$ and p_i are probabilities that sum to 1. The pure states are not necessarily orthogonal but they are linearly independent.

What is the maximum number of terms in the sum?

b) Consider a quantum channel $\mathcal{E}$ that operates on density operator σ describing a 2 qubit system.

What is the maximum number of Kraus operators needed to describe the channel?

Hint: Channel-state duality states that a channel operating in a Hilbert space of dimension d can be mapped to a 'state' (or actually a density matrix) ρ_{AB} in a Hilbert space of dimension d^2. The maximum number of Kraus operators in the operator sum decomposition of the channel is equal to the maximum number of pure states needed to describe ρ_{AB}.

13. **On The Quantum Zeno effect**

Consider a near identity single qubit unitary transformation

$$\mathbf{U} = e^{i\epsilon\mathbf{H}} \sim \mathbf{I} + i\epsilon\mathbf{H}$$

where $\mathbf{H}$ is Hermitian and ϵ is real, positive and small.

Our goal is to contrast the behavior of the two circuits shown in Figures 2 and 3.

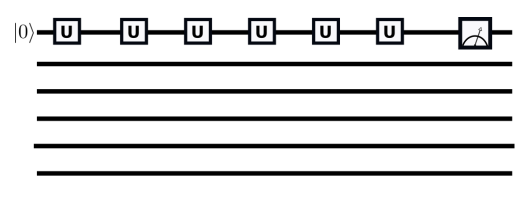


Figure 2: A unitary operation is done 6 times on the first qubit.

Consider the limit of n operations, either n unitary transformations as shown in Figure

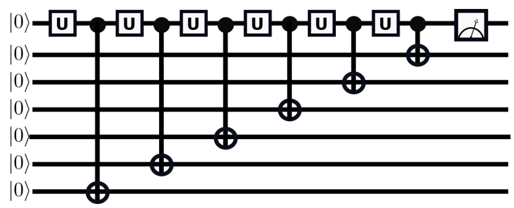


Figure 3: After each unitary transformation on the first qubit a CNOT is applied with control bit the first qubit. After each CNOT is applied, the target qubit is left untouched.

2 or n unitary transformations with alternating CNOT gates as shown in Figure 3.

a) Estimate the probability p_1 that $|1\rangle$ is measured in the first qubit at the end of the circuit in Figure 2 and as a function of n the number of iteractions. You can do the computation approximately to first order in ϵ, assuming ϵ is small; (though the probability should scale with ϵ^2).

b) Estimate the probability p_1 that $|1\rangle$ is measured in the first qubit at the end of the two circuit in Figure 3 as a function of n.

From p_1 you can compute the probability $p_0 = 1 - p_1$ that the first qubit is measured to be in the $|0\rangle$ state. If p_0 is near 1 then the system acts like it is frozen in the $|0\rangle$ state.

The CNOTs in Figure 3 give the effect of measuring the top qubit between each of the unitary operations on the top qubit. The first qubit is more likely to be measured to be in the $|0\rangle$ state at the end of the circuit in Figure 3 than at the end of the circuit in Figure 2. This is in analogy to the Quantum Zeno effect where measurements freeze a state. Here the state is frozen because information is transferred to other parts of the system.

This example was inspired by a blog by Cameron Calcluth.

14. **Comparison between Shannon and von Neumann entropy**

Consider a density matrix that is a mixture of two states $|0\rangle$ and $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ where the probability the qubit is in the $|0\rangle$ state is $p \in [0,1]$ and the probability it is in the $|+\rangle$ state is $1-p$. The density matrix is

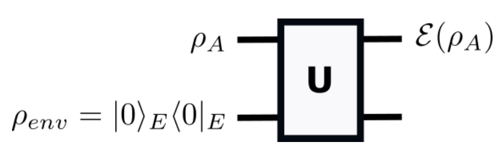
$$\rho(p) = p\,|0\rangle\langle 0| + (1-p)\frac{1}{2}(|0\rangle + |1\rangle)(\langle 0| + \langle 1|)$$

a) Compute the von Neumann entropy $S(\rho(p))$ for this density operator. Plot $S(\rho(p))$ as a function of p.

b) Compute the Shannon entropy $H(p)$ for a system with probability distribution $p, 1-p$. Plot $H(p)$ as a function of p.

Hints: I found analytical formulas for both and plotted them as a function of probability $p \in [0,1]$. The shapes are similar but the peak values are different.

15. **The dephasing (or phase-damping) channel**



We consider a bipartite system $\mathcal{H}_A \otimes \mathcal{H}_E$ where $\mathcal{H}_A$ is a single qubit and the environment $\mathcal{H}_E$ has more than two states. The dephasing channel operates on the density operator ρ_A for the single qubit. The dephasing channel can be described with a unitary transformation U that transforms

$$|0\rangle_A|0\rangle_E \to \sqrt{1-q}\,|0\rangle_A|0\rangle_E + \sqrt{q}\,|0\rangle_A|1\rangle_E$$
$$|1\rangle_A|0\rangle_E \to \sqrt{1-q}\,|1\rangle_A|0\rangle_E + \sqrt{q}\,|1\rangle_A|2\rangle_E$$

Assumed is that the environment starts in the state $\rho_E = |0\rangle_E\langle 0|_E$. Note that the qubit A does not change state via the unitary transformation. Instead it 'scatters' off the environment with probability q. A preferred basis $|0\rangle_A, |1\rangle_A$ is set by this channel as it is the only basis in which bit flips do not occur in the qubit due to interaction with the environment.

To ensure that U is unitary we complete the transformations of the other basis vectors

$$
\begin{aligned}
&|0\rangle_A |1\rangle_E \to \sqrt{1-q}\,|0\rangle_A |1\rangle_E - \sqrt{q}\,|0\rangle_A |0\rangle_E \\
&|1\rangle_A |2\rangle_E \to \sqrt{1-q}\,|1\rangle_A |2\rangle_E - \sqrt{q}\,|1\rangle_A |0\rangle_E \\
&|0\rangle_A |2\rangle_E \to |0\rangle_A |2\rangle_E \\
&|1\rangle_A |1\rangle_E \to |1\rangle_A |1\rangle_E .
\end{aligned}
$$

a) Show that the unitary transformation induces a channel $\mathcal{E}$ operating on ρ_A that is described with the following three Kraus operators

$$
\begin{aligned}
\mathbf{M}_0 &= \sqrt{1-q}\,\mathbf{I} \\
\mathbf{M}_1 &= \sqrt{q}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\
\mathbf{M}_2 &= \sqrt{q}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}
\end{aligned}
$$

You will need to assume that the environment is initially in the $\rho_{\rm env} = |0\rangle_E \langle 0|_E$ state.

b) Show that the dephasing channel can be written as

$$
\begin{aligned}
\mathcal{E}(\rho_A) &= \sum_a \mathbf{M}_a \rho_A \mathbf{M}_a \\
&= \left(1 - \frac{q}{2}\right)\rho_A + \frac{q}{2}\sigma_z \rho_A \sigma_z .
\end{aligned}
$$

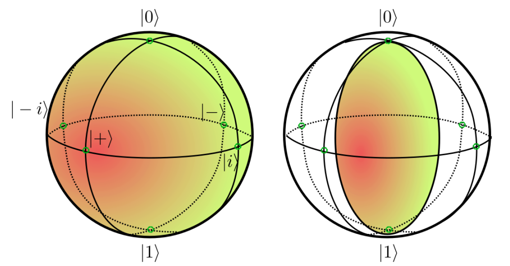
The density operator ρ_A can be written in terms of a polarization vector $\mathbf{p} = (p_x, p_y, p_z)$, and

$$
\rho_A = \frac{1}{2}(\mathbf{I} + \mathbf{p}\cdot\boldsymbol{\sigma}).
$$

c) Show that the dephasing channel gives new polarization

$$
\begin{aligned}
\mathbf{p}' &= (p'_x, p'_y, p'_z) \\
&= ((1-q)p_x, (1-q)p_y, p_z).
\end{aligned}
$$

The z component of the polarization is not affected.



The Bloch sphere shrinks to a prolate spheroid aligned with the z axis. The polarization in z is associated with the diagonal elements of the density operator. As the x and y polarization decreases, the density matrix becomes increasingly diagonal.

The dephasing channel picks out a preferred basis for decoherence. A possible physical setting might be interactions of a dust grain with the microwave background radiation. In this case the preferred basis is somewhat mysterious, but might be related to the spatial locality of the interaction.

16. **The Lindlblad master equation consistent with the dephasing channel**

Neglecting unitary evolution from a Hamiltonian, the Lindblad master equation if governed by a single operator gives

$$
\frac{d\rho}{dt} = \Gamma\left(\mathbf{L}\rho\mathbf{L}^\dagger - \frac{1}{2}\mathbf{L}^\dagger\mathbf{L}\rho - \frac{1}{2}\rho\mathbf{L}^\dagger\mathbf{L}\right).
$$

The dephasing channel on a single qubit

$$
\mathcal{E}(\rho) = \left(1 - \frac{q}{2}\right)\rho + \frac{q}{2}\sigma_z\rho\sigma_z .
$$

Show that for $\mathbf{L} = \sigma_z$, small probability q, and $q \propto \Gamma t$, the dephasing channel is consistent with evolution via the Lindblad master equation on a single qubit

$$
\frac{d\rho}{dt} = \Gamma\left(\sigma_z\rho\sigma_z^\dagger - \frac{1}{2}\sigma_z^\dagger\sigma_z\rho - \frac{1}{2}\rho\sigma_z^\dagger\sigma_z\right).
$$

17. **Ways to make new quantum channels**

A quantum channel $\mathcal{E}$ is described with a set of n_K Kraus operators $\{K_j\}$, giving the map $\mathcal{E}(\rho) = \sum_j K_j \rho K_j^\dagger$.

a) Consider a set of n_K unitary operators $\{V_j\}$.

Show that the set $\{\tilde{K}_j\}$ with $\tilde{K}_j = V_j K_j$ for every j is a set of Kraus operators that also gives a quantum channel.

Hint: Show that $\sum_j \tilde{K}_j^\dagger \tilde{K}_j = I$.

b) Show that $\mathcal{E}_b$ where

$$\mathcal{E}_b(\rho) = \alpha(\operatorname{tr}\rho)\rho_0 + (1-\alpha)\mathcal{E}(\rho)$$

is a quantum channel, for any channel $\mathcal{E}$, any density operator ρ_0 and $\alpha \in [0,1]$.

Hint: Note that $\rho \to (\operatorname{tr}\rho)\rho_0$ is a CPTP map. It might be useful to recall that every CPTP linear map from the space of operators onto itself has a Kraus representation and vice versa, every Kraus representation gives a CPTP linear map. Use a composition to show that the resulting combination is a channel. I did not find it very easy to find a set of Kraus operators for $\rho \to \rho_0$, but you need not find one if you can show that one exists.

18. **An initialization/reset channel**

An initialization or reset channel is one that puts a system into a particular state.

In an N dimensional quantum space, consider the channel described with a set of N Kraus operators $\{K_{reset,j}\}$ with

$$K_{reset,j} = |0\rangle\langle j|\,.$$

The operation of the channel

$$\mathcal{E}_{reset}(\rho) = \sum_{j=0}^{N-1} K_{reset,j}\,\rho K_{reset,j}^\dagger.$$

Here we use basis states $|j\rangle$ with $j = \{0, 1, ..., N-1\}$.

a) Show that this reset channel returns the $|0\rangle$ state, when applied to any pure quantum state.

Consider a bipartite system $\mathcal{H}_A \otimes \mathcal{H}_B$. The density matrix is initially ρ_{AB} for the full system. The dimension of $\mathcal{H}_A$ we denote n_A.

A channel $\mathcal{E}_{reset,A}$ that resets only the $\mathcal{H}_A$ sub-system is described by n_A Kraus operators $\{K_{reset,j}\}$ with $j \in \{0, ..., N-1\}$ and

$$K_{reset,j} = |0\rangle\langle j| \otimes I_B$$

where I_B is the identity in the $\mathcal{H}_B$ subspace.

b) Show that this channel is equivalent to the map

$$\mathcal{E}_{reset,A}(\rho_{AB}) = |0\rangle\langle 0| \otimes \rho_B$$

where $\rho_B = \operatorname{tr}_A \rho_{AB}$.

19. **Bob's POVM for quantum communication between Alice and Bob**

Alice sends Bob a series of qubits. Each qubit is either $|0\rangle$ or $|+\rangle$, and with each state equally likely. Here $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

Bob chooses to make measurements with the following three POVM operators

$$\begin{aligned}\mathbf{E}_1 &= a\,|1\rangle\langle 1| \\ \mathbf{E}_- &= a\,|-\rangle\langle -| \\ \mathbf{E}_N &= \mathbf{I} - \mathbf{E}_1 - \mathbf{E}_0.\end{aligned}$$

Note that the set of measurement operators is complete as $\mathbf{E}_1 + \mathbf{E}_- + \mathbf{E}_N = \mathbf{I}$. The coefficient $a > 0$ is positive and real. Here $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$.

If Bob gets a measurement associated with $\mathbf{E}_1$ then he knows that Alice did not send $|0\rangle$ and so she must have sent him a $|+\rangle$ state. Bob is then 100% sure that Alice sent him a $|+\rangle$ state.

If Bob gets a measurement with $\mathbf{E}_-$ then he knows that Alice did not send $|+\rangle$ and so she must have sent a $|0\rangle$ state. Bob is then 100% sure that Alice sent him a $|0\rangle$ state.

If Bob gets a measurement with $\mathbf{E}_N$ then he is not sure which state Alice sent.

a) What is the probability that Bob measures $\mathbf{E}_1$ or $\mathbf{E}_-$ and so knows exactly what Alice sent?

b) What value of a maximizes Bob's ability to get as much information as possible from Alice's transmitted qubits?

Hints: In the $|0\rangle, |1\rangle$ basis

$$\mathbf{E}_N = \begin{pmatrix} 1-a/2 & a/2 \\ a/2 & 1-3a/2 \end{pmatrix}. \tag{2}$$

For the POVM measurement to be good, E_N must be a positive operator and that means that both its eigenvalues must be positive. This means that a cannot be too large.

The eigenvalues of $\mathbf{E}_N$ are

$$\lambda_\pm = \frac{1}{2}(\pm\sqrt{2}a - 2a + 2)$$

(I typed the following into wolfram alpha: eigenvalues [[1-x/2,x/2],[x/2,1-3x/2]]

c) Is this POVM measurement optimal according to the Holevo bound?

Hints: The Holevo bound is

$$H(X:Y) \leq S(\rho)$$

because Alice sends pure states, each which have entropy of zero. On the left we have mutual information (what Bob can learn from his measurements about Alice's sent information) and on the right we have the von-Neumann entropy of Alice's mixture, $\rho = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|+\rangle\langle +|$. The eigenvalues of this density operator are $\cos^2\left(\frac{\pi}{8}\right)$ and $\sin^2\left(\frac{\pi}{8}\right)$, and the entropy $S(\rho) = 0.6008760366928562$.

20. **On how noise causes contraction in a quantum channel**

Consider $\mathcal{E}_a$, a quantum channel which is a trace preserving and completely positive map $\rho \to \rho'$ from the space of operators to the space of operators in a d-dimensional quantum space.

A small amount of noise is added giving a new channel $\mathcal{E}_b$. The noise is represented by the depolarization channel

$$\mathcal{E}_{\text{depol}}(\rho) = \alpha\frac{\mathbf{I}}{d} + (1-\alpha)\rho,$$

with $\alpha \in [0,1)$. The new channel $\mathcal{E}_b$ is defined by

$$\begin{aligned}\mathcal{E}_b(\rho) &= \mathcal{E}_{\text{depol}}(\mathcal{E}_a(\rho)) \\ &= \alpha\frac{\mathbf{I}}{d} + (1-\alpha)\mathcal{E}_a(\rho).\end{aligned}$$

With α small, only a small amount of noise is added.

Show that no matter what the properties of $\mathcal{E}_a$, no matter how small $\alpha > 0$, the noisy channel $\mathcal{E}_b$ has a unique fixed point and all orbits converge onto it. The noisy channel is called *ergodic* and *mixing*.

In other words show that

$$\lim_{t\to\infty} ||\mathcal{E}_b^t(\rho) - \rho_*|| = 0$$

for all density operators ρ. Here ρ_* is the fixed point of $\mathcal{E}_b$ which satisfies $\mathcal{E}_b(\rho_*) = \rho_*$. By $\mathcal{E}_b^t(\rho)$ we mean $\mathcal{E}_b(\mathcal{E}_b(\mathcal{E}_b(.....\mathcal{E}_b(\rho))))$ where $\mathcal{E}_b$ is applied t times. Here $||A|| = \text{tr}\left(\sqrt{AA^\dagger}\right)$ is the trace norm.

Hints: It helps to know the following:

Any quantum channel has at least one fixed point, ρ_* satisfying $\mathcal{E}(\rho_*) = \rho_*$.

All quantum channels are contractive in the sense that

$$||\mathcal{E}(\rho) - \mathcal{E}(\sigma)|| \leq ||\rho - \sigma||$$

for any two density operators ρ, σ.

21. **On eigenmatrices of a quantum channel**

We have a quantum channel $\mathcal{E}$. Take $\mathbf{x}$ to be an operator, not necessarily Hermitian or with trace of 1. You can think of the channel as a linear operator $\mathcal{L}$ that operates on a vectorized version of a density operator, or any operator. The linear operator $\mathcal{L}$ is not necessarily normal or Hermitian, so it may not be diagonalizable.

We can find eigenvalues for $\mathcal{L}$. For example, a fixed point satisfies $\mathcal{E}(\rho) = \rho$. Here the density operator ρ is an eigenvector of $\mathcal{L}$ with an eigenvalue of 1.

Take the operator $\mathbf{x}$ to be an a right eigenvector of $\mathcal{L}$ with eigenvalue λ. It satisfies

$$\mathcal{E}(\mathbf{x}) = \lambda \mathbf{x}.$$

a) Show that λ^* is also an eigenvalue of $\mathcal{L}$.

This implies that right-eigenvalues of $\mathcal{L}$, come in pairs, if they are complex.

b) Show that eigenvectors of $\mathcal{L}$ are traceless unless their associated eigenvalue is 1. In other words show that $\mathrm{tr}(\mathbf{x}) = 0$ if $\mathcal{E}(\mathbf{x}) = \lambda \mathbf{x}$ and $\lambda \neq 1$.

Hints: For (a) use an operator sum decomposition with a set of Kraus operators and take the adjoint. For (b) use the fact that channels are trace preserving.

22. **Your problem here!**