Homework # 2. Physics 265, Spring 2025

Topic: On Measurements, Projections, Entanglement and Multiple Qubits

Due date: Tuesday Feb 18, 2025 (late). Choose a subset of problems of 8 problems to work on. You could also create and share your own problems or do problems from the previous problem set. Please upload your solutions to blackboard.

1. On changing basis for a 2 qubit system We defined

$$\begin{split} |+\rangle &= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \\ |-\rangle &= \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \end{split}$$

Show how the entangled state

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

can be written in the basis $\{|++\rangle, |+-\rangle |-+\rangle, |--\rangle\}.$

In other words find complex numbers a, b, c, d such that

$$\left|\psi\right\rangle = a\left|++\right\rangle + b\left|+-\right\rangle + c\left|-+\right\rangle + d\left|--\right\rangle$$

2. On partial measurement in a threequbit system

A three qubit system is in the state

$$|\psi\rangle = \frac{1}{\sqrt{3}} (|000\rangle + i |010\rangle - |111\rangle)$$

The first qubit is measured in the $\{|0\rangle, |1\rangle\}$ basis (for example using the Pauli Z gate). a) When measured, what is the probability that the first qubit is in the $|0\rangle$ state? b) If the first qubit is measured to be in the $|0\rangle$ state, what is the resulting quantum state vector afterward?

c) Suppose one qubit is measured and then immediately afterwards another qubit is measured, with both measurements using a single qubit Z gate. (By this I mean measuring in the $|0\rangle$, $|1\rangle$ basis and giving possible measurement values of ± 1 .) The final state depends upon the values found in each measurement. Does the final state depend upon which order the two measurements were done?

3. The Hadamard transform

A Hadamard transform is a $2^N \times 2^N$ matrix where N is an integer, $N \ge 1$. Starting with the 1×1 matrix $H_0 = 1$, the transforms can be defined recursively

$$\mathbf{H}_{m+1} \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbf{H}_m & \mathbf{H}_m \\ \mathbf{H}_m & -\mathbf{H}_m \end{pmatrix}$$

This gives

$$\mathbf{H}_1 = \frac{1}{\sqrt{2}} \left(\begin{array}{cc} 1 & 1\\ 1 & -1 \end{array} \right)$$

and

a) Show that \mathbf{H}_m is Hermitian.

b) Show that $\mathbf{H}_m^2 = \mathbf{I}_{2^m}$ where \mathbf{I}_{2^m} is the 2^m -dimensional identity matrix (and we take $I_0 = 1$).

c) Show that \mathbf{H}_m is unitary.

Hints: you can show these recursively. Start by showing that a,b,c are true for m = 1 and m = 2. Then show that if they are true for m, they are also true for m + 1.

4. A Hadamard transformation on a Bell pair state

Starting with a Bell pair,

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

in a two-qubit system, perform a Hadamard transformation on one of the qubits and then measure one of the qubits.

Show that the probability of measuring a 0 or 1 during the measurement does not depend on which bit is measured or on which bit you operated on with the Hadamard operation.

5. The Bell basis

The Bell basis is a set of 4 entangled states

$$\begin{split} \left| \Phi^+ \right\rangle &= \frac{1}{\sqrt{2}} (\left| 00 \right\rangle + \left| 11 \right\rangle) \\ \left| \Phi^- \right\rangle &= \frac{1}{\sqrt{2}} (\left| 00 \right\rangle - \left| 11 \right\rangle) \\ \left| \Psi^+ \right\rangle &= \frac{1}{\sqrt{2}} (\left| 01 \right\rangle + \left| 10 \right\rangle) \\ \left| \Psi^- \right\rangle &= \frac{1}{\sqrt{2}} (\left| 01 \right\rangle - \left| 10 \right\rangle) \end{split}$$

Write the following state in the Bell basis

$$\frac{1}{\sqrt{3}}(|00\rangle + |01\rangle + |10\rangle)$$

Your answer should look like this:

$$a\left|\Phi^{+}\right\rangle+b\left|\Phi^{-}\right\rangle+c\left|\Psi^{+}\right\rangle+d\left|\Psi^{-}\right\rangle$$

You need to find the complex numbers a, b, c, d.

6. On entanglement

Give an example of a two-qubit state that is a superposition with respect to the standard basis but that is not entangled.

The standard basis is $|00\rangle$, $|01\rangle$, $|10\rangle$, $|11\rangle$. A superposition is a state vector $|\psi\rangle = a |00\rangle + b |01\rangle + c |10\rangle + d |11\rangle$ where at least 2 of the complex numbers a, b, c, d are not zero.

A state that is **not** entangled can be written as a tensor product. In other words it can be written as $|\psi\rangle = |v\rangle \otimes |w\rangle$ where $|\psi\rangle$ is the 2 qubit state and $|v\rangle$, $|w\rangle$ are single qubit states.

7. Subspace decomposition and designing measurements

Your goal is to design a measurement on a three-qubit system that distinguishes between states in which the number of 1 bits is even, and those in which the number of 1 bits is odd, and gives no other information.

a) Describe the measurement in terms of a Hermitian operator **M**. Write your solution in bra/ket notation.

b) Find a set of complete and orthogonal and Hermitian projection operators \mathbf{P}_i such that $\mathbf{M} = \sum_{i} \mathbf{m}_i \mathbf{P}_i$, where m_i are the possible measurement values.

This is problem 4.10 from Rieffel & Polak.

8. On entanglement and measurement

Consider a two qubit system.

a) Show that if a measurement is performed on one of the bits of a two qubit **unentangled** state $|\psi\rangle$, the resulting 2 qubit state is **unentangled**. b) Can a measurement on an **unentangled** state $|\psi\rangle$ produce an **entangled** state? If so, give an example.

Hint: make use of the Bell states and construct projection operators.

c) Can an **unentangled** state be obtained by measuring a single qubit of an **entangled** state $|\phi\rangle$? If so, give an example.

Based on Problem 4.17 of Rieffel & Polak.

9. On the BB84 quantum communication protocol

Alice and Bob are sharing a key through the BB84 quantum communication protocol. Suppose Eve is eavesdropping on the line. Alice and Bob agree to send a test message to detect the presence of Eve. How many bits do Alice and Bob need to compare to have a 90 percent chance of detecting Eve's presence?

Hint: First find the probability that Alice and Bob detect an error with a single compared bit. Then estimate how many bits they need to compare to have a probability of greater than 0.9 that they detect an error. This involves numerically summing probabilities with the binomial distribution.

10. More on the Bell basis

Consider a two-qubit system. The Bell basis is comprised of

$$\begin{split} \left| \Phi^{+} \right\rangle &= \frac{1}{\sqrt{2}} (\left| 00 \right\rangle + \left| 11 \right\rangle) \\ \left| \Phi^{-} \right\rangle &= \frac{1}{\sqrt{2}} (\left| 00 \right\rangle - \left| 11 \right\rangle) \\ \left| \Psi^{+} \right\rangle &= \frac{1}{\sqrt{2}} (\left| 01 \right\rangle + \left| 10 \right\rangle) \\ \left| \Psi^{-} \right\rangle &= \frac{1}{\sqrt{2}} (\left| 01 \right\rangle - \left| 10 \right\rangle). \end{split}$$

For any Bell basis state

$$\left|\Phi\right\rangle \in \{\left|\Phi^{+}\right\rangle, \left|\Phi^{-}\right\rangle, \left|\Psi^{+}\right\rangle, \left|\Psi^{-}\right\rangle\},$$

and any \mathbf{V} a Pauli matrix

$$\mathbf{V} \in \{\mathbf{I}, \mathbf{X}, \mathbf{Y}, \mathbf{Z}\}$$

the following is true:

$$\mathbf{V}_1 \ket{\Phi} = \pm \mathbf{V}_2 \ket{\Phi}.$$

Here the index of \mathbf{V} denotes which qubit the Pauli matrix operates on. This relation is helpful for understanding quantum teleportation protocols.

If two states $|a\rangle$, $|b\rangle$ are equivalent up to a global phase then $\langle a|b\rangle = e^{i\delta}$ for some angle δ .

Find a unitary operation **U** on a single qubit and a Bell basis vector $|\Phi\rangle$ such that $\mathbf{U}_1 |\Phi\rangle$ cannot be written as a global phase times $\mathbf{U}_2 |\Phi\rangle$.

11. On consecutive measurements

Consider a single qubit system is in state $|\psi\rangle = a |0\rangle + b |1\rangle$.

The Pauli matrices we refer to as X, Y, Zgates. Note that ZX = -iY.

Suppose a Pauli Z gate is used to make a measurement and then a Pauli X gate is used to make a measurement.

a) What are the possible final states and what is the probability for each of them?

b) What are the possible different sets of measured values and what is the probability of set of measurements?

c) What if instead you measure with X first and then Z. What are the possible different sets of measured values and their probabilities?

d) What if instead of doing 2 measurements, you only measure with Y. What are the possible final states and what are their probabilities?



Figure 1: A recipe for teleporting a qubit $|\psi\rangle$ using two additional entangled qubits, a CNOT, a Hadamard operation and two measurements. Alice and Bob share an entangled state. The transmitter (Alice) applies the CNOT and the Hadamard and makes the two measurements. The transmitter then tells the receiver the results of the two measurements. The receiver (Bob) applies a transformation on the third qubit that is based on the measurements of the first two qubits. The receiver then holds the third qubit which has become identical to $|\psi\rangle$, the original state of the first qubit.

12. On Quantum Teleportation

A procedure for quantum teleportation of a qubit in state $|\psi\rangle$ from Alice to Bob when Alice and Bob share a Bell pair state is shown in Figure 1.

Instead of sharing the initial state $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ in the second and third qubits, Alice and Bob share the state $\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$. Alice and Bob can still teleport the first qubit. Alice opts to keep her procedures the same.

How should Bob vary what he does in response to Alice's measurements to ensure that the first qubit is teleported intact into Bob's (the third one)?

Draw a quantum circuit to illustrate your modified teleportation protocol.

13. The Chau02 Quantum Key Distribution Protocol

The Chau 02^1 Quantum Key Distribution

protocol is a variant of the BB84 protocol.

Alice randomly chooses 0 or 1. Then she randomly choose a basis from one of the following bases $\{|0\rangle, |1\rangle\}$, or $\{|+\rangle, |-\rangle\}$ or $\{|i\rangle, |-i\rangle\}$. If Alice chose a 0 she sends to Bob $|0\rangle$ or $|+\rangle$ or $|i\rangle$, depending upon her chosen basis.

Otherwise she sends to Bob $|1\rangle$ or $|-\rangle$ or $|-i\rangle$, depending upon her chosen basis.

Bob randomly chooses one of these bases for measurement.

Afterwards Alice and Bob share which bases they used and discard all bits in which their bits were not the same. The remaining bits are their shared key.

a) What fraction of Bob's measurements are eventually discarded?

Suppose Eve intercepts Alice's sent qubit, makes a measurement in one of the three bases and then sends the result to Bob.

b) What fraction of Alice and Bob's generated key would be incorrect due to Eve's interference?

14. A new or different problem

Propose and solve a different problem!

¹H. F. Chau 2002, Phys Rev. A, 66, 060302