# Homework # 1. Physics 265, Spring 2025

**Topic:** On linear algebra and notation commonly used in quantum mechanics and quantum computing

**Due date:** Thursday Feb. 4, 2025 at midnight. Choose a subset of 8 problems to work on. Please upload your solutions onto blackboard.

A unitary matrix is a square complex matrix that satisfies  $\mathbf{U}\mathbf{U}^{\dagger} = \mathbf{U}^{\dagger}\mathbf{U} = \mathbf{I}$ . A unitary matrix transforms one orthonormal basis to another. A Hermitian matrix is self-adjoint;  $\mathbf{A} = \mathbf{A}^{\dagger}$ .

# 1. On the definitions of unitary and Hermitian

a) Find an example of a non-identity 2x2 matrix that is both unitary and Hermitian.

b) Find an example of a non-identity 3x3 matrix that is both unitary and Hermitian.

# 2. On the rows and columns of a unitary matrix

a) Explain why each row of a unitary matrix is orthogonal to every other row and why each row has a norm of 1.

b) Explain why each column of a unitary matrix is orthogonal to every other column and why each column has a norm of 1.

c) For  $\alpha$  a real number, and for **U** a unitary matrix, is  $e^{i\alpha}$ **U** a unitary matrix?

d) Consider an  $N \times N$  unitary matrix **U** with eigenvalues  $\lambda_i$  with  $i \in 0....N-1$ . Show that the eigenvalues must be complex numbers on the unit circle. In other words  $|\lambda_i| = 1$ , they must be complex numbers with length 1.

e) A unitary operator  $\mathbf{Q}$  satisfies  $\mathbf{Q}^4 = \mathbf{I}$ where  $\mathbf{I}$  is the identity operator. The discrete quantum Fourier transform is an example of such an operator.

What are the possible eigenvalues of  $\mathbf{Q}$ ?

# 3. On commutators

An operator **a** satisfies

$$[\mathbf{a}, \mathbf{a}^{\dagger}] = \mathbf{a}\mathbf{a}^{\dagger} - \mathbf{a}^{\dagger}\mathbf{a} = \mathbf{I}$$

where **I** is the identity.

Show that **a** cannot be Hermitian or unitary.

Hint: Show that Unitary and Hermitian operators are normal matrices satisfying  $\mathbf{aa}^{\dagger} = \mathbf{a}^{\dagger}\mathbf{a}$ .

This commutator is obeyed by a raising operator but a raising operator is not Hermitian or unitary.

# 4. On the inner product, expectation value and uncertainty

For Hermitian operator **A** and state-vector  $|\psi\rangle$  the expectation value  $\langle \mathbf{A} \rangle = \langle \psi | \mathbf{A} | \psi \rangle$ .

The uncertainty  $\Delta A$  is a real number that is defined by

$$\Delta A = \sqrt{\langle \mathbf{A}^2 \rangle - \langle \mathbf{A} \rangle^2} \tag{1}$$

Show that the expression inside the square root is always greater or equal to zero.

Hints:

 $\langle \mathbf{A}^2 \rangle - \langle \mathbf{A} \rangle^2 = \langle (\mathbf{A} - \langle \mathbf{A} \rangle) \times (\mathbf{A} - \langle \mathbf{A} \rangle) \rangle.$ Use this fact to show that the expression for  $(\Delta A)^2$  can be written as  $\langle v | v \rangle$  for some vector  $|v\rangle$ .

# 5. On measuring a single qubit in mutually unbiased bases

You have a mechanism that can produce a qubit that is in a particular state

$$|\psi\rangle = a |0\rangle + b |1\rangle$$

for two complex numbers a, b. We assume this state is normalized, so  $aa^* + bb^* = 1$ . It may be convenient to work with  $|\psi\rangle$  on the Bloch sphere and in the form

$$\left|\psi\right\rangle=e^{i\gamma}\left(\cos\frac{\theta}{2}\left|0\right\rangle+e^{i\phi}\sin\frac{\theta}{2}\left|1\right\rangle\right)$$

where  $\gamma$  is a global phase.

The mechanism that you have access to can produce as many qubits in this particular state as you desire. Using many measurements you would like to find a, b and so determine what quantum state your mechanism produces. This procedure is called quantum state tomography. Quantum state tomography is when a quantum state is reconstructed using measurements on an ensemble of identical quantum states.

a) Suppose you make a series of many measurements in the  $\{|0\rangle, |1\rangle\}$  basis. What do you learn about the state  $|\psi\rangle$ ? In other words, what do you learn about *a* and *b* or equivalently about the Bloch state angles  $\theta, \phi, \gamma$ ?

b) Suppose you also make a series of many measurements in the  $\{|+\rangle, |-\rangle\}$  basis. What more do you learn about the state  $|\psi\rangle$ ?

c) Suppose you also make a series of measurements in the  $\{|i\rangle, |-i\rangle\}$  basis. Do you gain additional information?

The states

$$\begin{split} |+\rangle &= \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) \\ |-\rangle &= \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle) \\ |i\rangle &= \frac{1}{\sqrt{2}} (|0\rangle + i |1\rangle) \\ |-i\rangle &= \frac{1}{\sqrt{2}} (|0\rangle - i |1\rangle) \end{split}$$

# 6. On the cyclic nature of the trace

a) Show that the trace operator is cyclic.

$$\operatorname{tr}(\mathbf{ABC}) = \operatorname{tr}(\mathbf{BCA})$$

where **A**, **B**, **C** are two dimensional square matrices.

It may be convenient to use summation notation. If a matrix **A** has elements  $A_{ij}$  in an orthonormal basis, then  $\operatorname{tr} \mathbf{A} = \sum_j A_{jj}$  and the sum is over  $j \in 0, 1, \dots, N-1$  where N is the dimension of the vector space.

Matrix multiplication can also be done with indices. If  $\mathbf{D} = \mathbf{AB}$  then

$$D_{ik} = \sum_{j} A_{ij} B_{jk}$$

for matrices  $\mathbf{A}, \mathbf{B}, \mathbf{D}$ .

b) Suppose a matrix **A** can be diagonalized with a unitary operator **U**. In other words  $\mathbf{A} = \mathbf{U}\mathbf{D}\mathbf{U}^{\dagger}$  where  $\mathbf{D} = \text{diag}(\lambda_1, \lambda_2, \lambda_3...)$  is a diagonal matrix with values on the diagonal  $\lambda_i$ . Show that

$$\operatorname{tr} \mathbf{A} = \sum_{i} \lambda_{i}$$

It may be helpful to remember that a unitary matrix satisfies  $\mathbf{U}\mathbf{U}^{\dagger} = \mathbf{U}^{\dagger}\mathbf{U} = \mathbf{I}$  where  $\mathbf{I}$  is the identity matrix.

#### 7. Eigenvectors are perpendicular

a) Suppose  $\lambda_i, \lambda_j$  are eigenvalues of a Hermitian matrix **A** with associated eigenvectors  $|\mathbf{e}_i\rangle, |\mathbf{e}_j\rangle$  and suppose that  $\lambda_i \neq \lambda_j$ .

A vector  $|\mathbf{e}_{j}\rangle$  is an eigenvector of matrix **A** with eigenvalue  $\lambda_{j}$  if  $\mathbf{A} |\mathbf{e}_{j}\rangle = \lambda_{j} |\mathbf{e}_{j}\rangle$  and  $|\mathbf{e}_{j}\rangle$  is not zero.

Show that the two eigenvectors are perpendicular;  $\langle \mathbf{e_i} | \mathbf{e_j} \rangle = 0$ .

b) Show that any Hermitian operator **A** has eigenvalues that are real numbers.

#### 8. The Pauli spin matrices

Many quantum algorithms use simple single qubit transformations such as the Pauli spin matrices.

The Pauli spin matrices are

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$
$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

They satisfy  $\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = \mathbf{I}$  where  $\mathbf{I}$  is the identity matrix and

$$\sigma_x \sigma_y = -\sigma_y \sigma_x = i\sigma_z$$
  

$$\sigma_y \sigma_z = -\sigma_z \sigma_y = i\sigma_x$$
  

$$\sigma_z \sigma_x = -\sigma_x \sigma_z = i\sigma_y$$

The Pauli spin matrices are unitary.

a) What are the eigenvalues and eigenvectors of  $\sigma_z$ ?

b) Show that the unitary matrix  $\mathbf{H} = \frac{1}{\sqrt{2}}(\sigma_x + \sigma_z)$  diagonalizes  $\sigma_x$ . In other words show that  $\mathbf{H}\sigma_x\mathbf{H}^{\dagger}$  is a diagonal matrix.

**H** is known as the Hadamard gate and is

$$\mathbf{H} = \frac{1}{\sqrt{2}} \left( \begin{array}{cc} 1 & 1\\ 1 & -1 \end{array} \right)$$

c) Find the eigenvalues and eigenvectors of  $\sigma_x, \sigma_y$ .

d) Find a unitary matrix that diagonalizes  $\sigma_y$ . In other words, find **U** such that  $\sigma_y = \mathbf{U}\Lambda\mathbf{U}^{\dagger}$  where  $\Lambda$  is a diagonal matrix.

# 9. Using measurements to describe a unitary transformation

You have access to a quantum system consisting of a qubit. You can apply a unitary transformation

$$\mathbf{U} = e^{i\alpha\boldsymbol{\sigma}_x} = \cos\alpha\mathbf{I} + i\sin\alpha\boldsymbol{\sigma}_x$$

to the qubit, where  $\sigma_x$  is the Pauli X matrix, and  $\alpha$  is a phase that you would like to measure. The phase  $\alpha$  is a real number satisfying  $0 \leq \alpha < 2\pi$ .

**Quantum process tomography** is when known quantum states and measurements are used to probe the nature of quantum process so that it can be accurately described.

- You initialize your qubit in the  $|0\rangle$  state.
- You then operate on the qubit with the unitary operator U, giving you the state U |0>.
- Then you make a measurement in a basis of your choice.
- You can repeat this procedure as many times as you like.

a) Design a series of measurements to measure the mystery phase  $\alpha$ .

b) How many measurements would you need to determine  $\cos \alpha$  to an accuracy of 10%?

Hint: Use the mean and variance of the Bernoulli distribution https://en.wikipedia.org/wiki/

Bernoulli\_distribution. The variance of the Bernoulli distribution is a maximum if the probability of an outcome is 1/2 and you can use that value to estimate the number of measurements you would need to ensure that the standard deviation of combined measurements to be smaller than 0.1.

# 10. On qubit global phase

Two quantum states can be considered equivalent if they only differ by a complex phase. In other words  $|\phi\rangle \equiv |\psi\rangle$  if

$$|\phi\rangle = e^{i\alpha} |\psi\rangle$$

for a real number  $\alpha$ , which we call the phase.

This equivalence makes sense as there is no way to measure  $\alpha$  with conventional quantum measurements.

Below I give a list of pairs of states. Which pairs of states represent equivalent quantum states for a single qubit? If they are equivalent find the phase  $\alpha$ . If the states are not equivalent describe a measurement for which the probabilities of the outcomes of the two states differ. In other words, find an operator **A** that gives different  $\langle \mathbf{A} \rangle$  for each state.

a) 
$$\frac{1}{\sqrt{2}} (|+\rangle + |-\rangle)$$
 and  $|0\rangle$   
b)  $\frac{1}{\sqrt{2}} (|\mathbf{i}\rangle + |-\mathbf{i}\rangle)$  and  $|1\rangle$   
c)  $\frac{1}{\sqrt{2}} (|0\rangle + e^{-i\pi/4} |1\rangle)$  and  $\frac{1}{\sqrt{2}} (e^{i\pi/4} |0\rangle + |1\rangle)$   
d)  $|+\rangle$  and  $|\mathbf{i}\rangle$ 

$$\begin{split} |+\rangle &= \frac{1}{\sqrt{2}} \left( |0\rangle + |1\rangle \right) \\ |-\rangle &= \frac{1}{\sqrt{2}} \left( |0\rangle - |1\rangle \right) \\ |i\rangle &= \frac{1}{\sqrt{2}} \left( |0\rangle + i |1\rangle \right) \\ |-i\rangle &= \frac{1}{\sqrt{2}} \left( |0\rangle - i |1\rangle \right) \end{split}$$

#### 11. On the Bloch sphere

a) Show that no point  $|x\rangle$  on the Bloch sphere satisfies

$$\langle x | \sigma_x | x \rangle = \langle x | \sigma_y | x \rangle = \langle x | \sigma_z | x \rangle = 0$$

where  $\sigma_x, \sigma_y, \sigma_z$  are the Pauli spin matrices. b) Are there any points on the Block sphere that satisfy

$$\langle x | \sigma_x | x \rangle = \langle x | \sigma_z | x \rangle = 0?$$

If so find them.

Hints:

Any point on the Bloch sphere can be written with two angles  $\phi, \theta$  as

$$|x\rangle = \cos(\theta/2) |0\rangle + \sin(\theta/2)e^{i\phi} |1\rangle$$

The solution to  $\langle x | \sigma_x | x \rangle = 0$  is a great circle on the Bloch sphere. Similarly for  $\sigma_y$  and  $\sigma_z$ .

# 12. New problems!

Propose and solve your own problem.