Homework # 1. Physics 265, Spring 2024

Topic: *On linear algebra and notation commonly used in quantum mechanics and quantum computing*

Due date: Thursday Feb. 1, 2024. Choose a subset of about 8 problems to work on. Please upload your solutions to blackboard.

A unitary matrix is a square complex matrix that satisfies $\mathbf{U}\mathbf{U}^\dagger = \mathbf{U}^\dagger\mathbf{U} = \mathbf{I}$. A unitary matrix transforms one orthonormal basis to another. A Hermitian matrix is self-adjoint; $\mathbf{A} = \mathbf{A}^\dagger$.

1. **On the definitions of unitary and Hermitian**

   Find an example of a non-identity 2x2 matrix that is both unitary and Hermitian.

2. **On the rows and columns of a unitary matrix**

   a) Explain why each row of a unitary matrix is orthogonal to every other row and why each row has a norm of 1.

   b) Explain why each column of a unitary matrix is orthogonal to every other column and why each column has a norm of 1.

   c) For α a real number, and for $\mathbf{U}$ a unitary matrix, is $e^{i\alpha}\mathbf{U}$ a unitary matrix?

   d) Consider an $N\times N$ unitary matrix $\mathbf{U}$ with eigenvalues λ_i with $i \in 0....N-1$. Show that the eigenvalues must be complex numbers on the unit circle. In other words $|\lambda_i| = 1$, they must be complex numbers with length 1.

3. **On the inner product, expectation value and uncertainty**

   For Hermitian operator $\mathbf{A}$ and state-vector $|\psi\rangle$ the expectation value $\langle\mathbf{A}\rangle = \langle\psi|\,\mathbf{A}\,|\psi\rangle$.

   The uncertainty ΔA is a real number that is defined by

   $$\Delta A = \sqrt{\langle\mathbf{A}^2\rangle - \langle\mathbf{A}\rangle^2} \tag{1}$$

   Show that the expression inside the square root is always greater or equal to zero.

   Hints:

   $\langle\mathbf{A}^2\rangle - \langle\mathbf{A}\rangle^2 = \langle(\mathbf{A} - \langle\mathbf{A}\rangle) \times (\mathbf{A} - \langle\mathbf{A}\rangle)\rangle$. Use this fact to show that the expression for $(\Delta A)^2$ can be written as $\langle v|v\rangle$ for some vector $|v\rangle$.

4. **On uncertainty relations**

   For a state $|\psi\rangle$ and Hermitian operators $\mathbf{A}, \mathbf{B}$, let $\langle\mathbf{A}\rangle = \langle\psi|\,\mathbf{A}\,|\psi\rangle$ denote the expectation value of $\mathbf{A}$. Let ΔA denote the uncertainty

   $$\Delta A \equiv \sqrt{\langle\mathbf{A}^2\rangle - \langle\mathbf{A}\rangle^2}. \tag{2}$$

   (a) Show that

   $$\mathbf{A}\,|\psi\rangle = \langle\mathbf{A}\rangle\,|\psi\rangle + \Delta A\,|\psi_{\perp A}\rangle\,, \tag{3}$$

   where $|\psi_{\perp A}\rangle$ is a state orthogonal to $|\psi\rangle$.

   Note: $|\psi\rangle\,, |\psi_{\perp A}\rangle$ are normalized.

   (b) Use this result to prove the general uncertainty relation,

   $$\Delta A\ \Delta B \geq \frac{1}{2}|\langle[\mathbf{A},\mathbf{B}]\rangle|, \tag{4}$$

   where $[\mathbf{A},\mathbf{B}] = \mathbf{A}\mathbf{B} - \mathbf{B}\mathbf{A}$ is the commutator of $\mathbf{A}$ and $\mathbf{B}$.

   Hint: First show that

   $$\langle[\mathbf{A},\mathbf{B}]\rangle = \Delta A \Delta B(\langle\psi_{\perp A}|\psi_{\perp B}\rangle - \langle\psi_{\perp B}|\psi_{\perp A}\rangle)$$

   Since the interaction for measuring $\mathbf{A}$ disturbs the state-vector, an observable $\mathbf{B}$ that does not commute with $\mathbf{A}$ is affected by measurement of $\mathbf{A}$.

This problem is from the book *Quantum Paradoxes* by Aharonov and Rohrlich.

5. **On the cyclic nature of the trace**

a) Show that the trace operator is cyclic.

$$\mathrm{tr}(\mathbf{ABC}) = \mathrm{tr}(\mathbf{BCA})$$

where $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are two dimensional square matrices.

It may be convenient to use summation notation. If a matrix $\mathbf{A}$ has elements A_{ij} in an orthonormal basis, then $\mathrm{tr}\mathbf{A} = \sum_j A_{jj}$ and the sum is over $j \in 0, 1, ...N-1$ where N is the dimension of the vector space.

Matrix multiplication can also be done with indices. If $\mathbf{D} = \mathbf{AB}$ then

$$D_{ik} = \sum_j A_{ij} B_{jk}$$

for matrices $\mathbf{A}, \mathbf{B}, \mathbf{D}$.

b) Suppose a matrix $\mathbf{A}$ can be diagonalized with a unitary operator $\mathbf{U}$. In other words $\mathbf{A} = \mathbf{UDU}^\dagger$ where $\mathbf{D} = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3...)$ is a diagonal matrix with values on the diagonal λ_i. Show that

$$\mathrm{tr}\mathbf{A} = \sum_i \lambda_i$$

It may be helpful to remember that a unitary matrix satisfies $\mathbf{UU}^\dagger = \mathbf{U}^\dagger\mathbf{U} = \mathbf{I}$ where $\mathbf{I}$ is the identity matrix.

6. **Eigenvectors are perpendicular**

a) Suppose λ_i, λ_j are eigenvalues of a Hermitian matrix $\mathbf{A}$ with associated eigenvectors $|\mathbf{e}_i\rangle, |\mathbf{e}_j\rangle$ and suppose that $\lambda_i \neq \lambda_j$.

A vector $|\mathbf{e}_j\rangle$ is an eigenvector of matrix $\mathbf{A}$ with eigenvalue λ_j if $\mathbf{A}|\mathbf{e}_j\rangle = \lambda_j |\mathbf{e}_j\rangle$ and $|\mathbf{e}_j\rangle$ is not zero.

Show that the two eigenvectors are perpendicular; $\langle \mathbf{e_i} | \mathbf{e_j} \rangle = 0$.

b) Show that any Hermitian operator $\mathbf{A}$ has eigenvalues that are real numbers.

7. **The Pauli spin matrices**

Many quantum algorithms use simple single qubit transformations such as the Pauli spin matrices.

The Pauli spin matrices are

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$
$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

They satisfy $\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = \mathbf{I}$ where $\mathbf{I}$ is the identity matrix and

$$\sigma_x\sigma_y = -\sigma_y\sigma_x = i\sigma_z$$
$$\sigma_y\sigma_z = -\sigma_z\sigma_y = i\sigma_x$$
$$\sigma_z\sigma_x = -\sigma_x\sigma_z = i\sigma_y$$

The Pauli spin matrices are unitary.

a) What are the eigenvalues and eigenvectors of σ_z?

b) Show that the unitary matrix $\mathbf{H} = \frac{1}{\sqrt{2}}(\sigma_x + \sigma_z)$ diagonalizes σ_x. In other words show that $\mathbf{H}\sigma_x\mathbf{H}^\dagger$ is a diagonal matrix.

$\mathbf{H}$ is known as the Hadamard gate and is

$$\mathbf{H} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

c) Find the eigenvalues and eigenvectors of σ_x, σ_y.

d) Find a unitary matrix that diagonalizes σ_y.

8. **The Hadamard transform**

A Hadamard transform is a $2^N \times 2^N$ matrix where N is an integer. Starting with the 1×1 matrix $H_0 = 1$, the transforms can be defined recursively

$$\mathbf{H}_{m+1} \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbf{H}_m & \mathbf{H}_m \\ \mathbf{H}_m & -\mathbf{H}_m \end{pmatrix}$$

This gives

$$\mathbf{H}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

and

$$\mathbf{H}_2 = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

a) Show that $\mathbf{H}_m$ is Hermitian.

b) Show that $\mathbf{H}_m^2 = \mathbf{I}_{2^m}$ where $\mathbf{I}_{2^m}$ is the 2^m-dimensional identity matrix (and we take $I_0 = 1$).

c) Show that $\mathbf{H}_m$ is unitary.

9. **On indexing in base 2 for a vector space with dimension that is a power of 2**

Matrices used in quantum computing often have dimensions that are powers of two. For example they are often $2^n \times 2^n$ with n a non-zero positive integer. Indices for vectors or matrices run from 0 to $2^n - 1$ and so can be written as binary numbers with a series of digits that are 0 or 1.

For example suppose we have a 2^3 dimensional vector space. An index for a vector would be in the set $\{000, 001, 010, 011, 100, 101, 110, 111\}$.

A positive integer index i can be written as a series of digits where each digit is in the set { 0,1 }. We write i as a series $i_0, i_1, i_2....$ where i_k is in { 0,1 } and k is an index that refers to the digit. Here $k \in \{0, ..., n-1\}$ where n is the number of digits. Conventionally i_0 is the smallest digit and i_{n-1} is the largest one. For example with $i = 110$ in base 2 the digits are 1,1,0 and $i_0 = 0, i_1 = 1$ and $i_2 = 1$. In base 10, $i = 4 + 2 = 6$. This means that $i = i_2 2^2 + i_1 2^1 + i_0 2^0$. More generally

$$\begin{aligned} i &= i_{n-1}2^{n-1} + i_{n-2}2^{n-2} + ... \\ &\quad .... + i_3 2^3 + i_2 2^2 + i_1 2 + i_0 \\ &= \sum_{k=0}^{n-1} i_k 2^k \end{aligned}$$

Consider 2 integers i, j, both written in terms of digits in base 2.

$$\begin{aligned} i &= i_{n-1} i_{n-2} .... i_1 i_0 \\ j &= j_{n-1} j_{n-2} .... j_1 j_0 \end{aligned}$$

We can compute the sum

$$i \cdot j = \left(\sum_k i_k j_k \right) \mod 2.$$

Because it is modulo 2 the result is either equal to 1 or 0.

For example $0010 \cdot 1010 = 1$ and $1111 \cdot 1010 = 0$.

Compute the $2^2 \times 2^2$ matrix with components

$$H_{ij} = (-1)^{i \cdot j}$$

10. **On qubit global phase**

Two quantum states can be considered equivalent if they only differ by a complex phase. In other words $|\phi\rangle \equiv |\psi\rangle$ if

$$|\phi\rangle = e^{i\alpha} |\psi\rangle$$

for a real number α, which we call the phase.

This equivalence makes sense as there is no way to measure α with conventional quantum measurements.

Below I give a list of pairs of states. Which pairs of states represent equivalent quantum states for a single qubit? If they are equivalent find the phase α. If the states are not equivalent describe a measurement for which the probabilities of the outcomes of the two states differ. In other words, find an operator $\mathbf{A}$ that gives different $\langle \mathbf{A} \rangle$ for each state.

a) $\frac{1}{\sqrt{2}} \left(|+\rangle + |-\rangle\right)$ and $|0\rangle$

b) $\frac{1}{\sqrt{2}} \left(|\mathbf{i}\rangle + |-\mathbf{i}\rangle\right)$ and $|1\rangle$

c) $\frac{1}{\sqrt{2}} \left(|0\rangle + e^{-i\pi/4} |1\rangle\right)$ and $\frac{1}{\sqrt{2}} \left(e^{i\pi/4} |0\rangle + |1\rangle\right)$

d) $|+\rangle$ and $|\mathbf{i}\rangle$

We use the states

$$
\begin{aligned}
|+\rangle &= \frac{1}{\sqrt{2}} \left(|0\rangle + |1\rangle\right) \\
|-\rangle &= \frac{1}{\sqrt{2}} \left(|0\rangle - |1\rangle\right) \\
|\mathbf{i}\rangle &= \frac{1}{\sqrt{2}} \left(|0\rangle + i\,|1\rangle\right) \\
|-\mathbf{i}\rangle &= \frac{1}{\sqrt{2}} \left(|0\rangle - i\,|1\rangle\right)
\end{aligned}
$$

11. **On the Bloch sphere**

a) Show that no point $|x\rangle$ on the Bloch sphere satisfies

$$\langle x| \sigma_x |x\rangle = \langle x| \sigma_y |x\rangle = \langle x| \sigma_z |x\rangle = 0$$

where $\sigma_x, \sigma_y, \sigma_z$ are the Pauli spin matrices.

b) Are there any points on the Block sphere that satisfy

$$\langle x| \sigma_x |x\rangle = \langle x| \sigma_z |x\rangle = 0?$$

If so find them.

Hints:

Any point on the Bloch sphere can be written with two angles ϕ, θ as

$$|x\rangle = \cos(\theta/2)\,|0\rangle + \sin(\theta/2) e^{i\phi}\,|1\rangle$$

The solution to $\langle x| \sigma_x |x\rangle = 0$ is a great circle on the Bloch sphere. Similarly for σ_y and σ_z.

12. **On eigenvalues of fermionic raising and lowering operators**

A set of N fermions can be described with a set of N operators $\{\hat{a}_j\}$ where index j goes from 1 to N. The operators satisfy

$$
\begin{aligned}
&\hat{a}_j \hat{a}_k^\dagger + \hat{a}_k^\dagger \hat{a}_j = \delta_{jk} \hat{I} \\
&\hat{a}_j \hat{a}_k + \hat{a}_k \hat{a}_j = 0 \qquad \forall j, k
\end{aligned}
$$

where $\hat{I}$ is the identity operator.

a) For any operator $\hat{A}$, show that $\hat{A}\hat{A}^\dagger$ is Hermitian.

b) Find the possible eigenvalues for the operator $\hat{N}_j = \hat{a}_j \hat{a}_j^\dagger$.

Hint: Show that $\hat{N}_j^2 - \hat{N}_j = 0$.

13. **Antipodal points on the Bloch Sphere**

For (x, y, z) a point on the sphere, $(-x, -y, -z)$ is its antipodal point. With wave vector

$$|\psi\rangle = \cos(\theta/2)\,|0\rangle + \sin(\theta/2) e^{i\phi}\,|1\rangle$$

the point on the Bloch sphere is

$$(x, y, z) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

a) Show that antipodal points on the surface of the Bloch sphere represent orthogonal states.

b) Optional: Show that any two orthogonal states correspond to antipodal points on surface of the Bloch sphere.

14. **Transformations of eigenbases**

    In an N dimensional quantum space, we find an orthogonal basis $\mathcal{V}$ consisting of eigenvectors of a Hermitian operator $\mathbf{A}$.

    Describe the set S of unitary operators, where $\mathbf{U} \in S$ if $\mathbf{U} |v_j\rangle$ is an eigenvector of $\mathbf{A}$, for all $|v_j\rangle \in \mathcal{V}$.

    Hint: this problem is straightforward if you assume that the eigenvalues differ from each other.

15. **Eigenvalues of the QFT**

    A unitary operator $\mathbf{Q}$ satisfies $\mathbf{Q}^4 = \mathbf{I}$ where $\mathbf{I}$ is the identity operator.

    What are the possible eigenvalues for $\mathbf{Q}$?

16. **Finding the dimension of a vector space - 2×2 Hermitian matrices**

    A 2×2 complex matrix can be written as

    $$\mathbf{A} = \begin{pmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{pmatrix}$$

    or

    $$\mathbf{A} = A_{00} |0\rangle \langle 0| + A_{01} |0\rangle \langle 1| + A_{10} |1\rangle \langle 0| + A_{11} |1\rangle \langle 1|$$

    where $A_{00}, A_{01}, A_{10}, A_{11}$ are independent complex numbers.

    We can think of $|0\rangle \langle 0|, |1\rangle \langle 0|, |0\rangle \langle 1|, |1\rangle \langle 1|$ as orthonormal basis elements. With that in mind, the space of 2×2 complex matrices is a 4 dimensional complex vector space.

    If we divide each matrix element into a real and complex part, we can consider the space of 2×2 complex matrices as an 8 dimensional real vector space with basis vectors $|0\rangle \langle 0|, |1\rangle \langle 0|, |0\rangle \langle 1|, |1\rangle \langle 1|$ and $i|0\rangle \langle 0|, i|1\rangle \langle 0|, i|0\rangle \langle 1|, i|1\rangle \langle 1|$.

    a) What is the dimension of the vector space of 2×2 Hermitian matrices?

    b) What is the dimension of the vector space of $n \times n$ Hermitian matrices?

17. **On the BB84 quantum communication protocol**

    Alice and Bob are sharing a key through the BB84 quantum communication protocol. Suppose Eve is eavesdropping on the line. Alice and Bob agree to send a test message to detect the presence of Eve. How many bits do Alice and Bob need to compare to have a 90 percent chance of detecting Eve's presence?

    Hint: involves numerically summing probabilities with the binomial distribution.

18. **New problems!**

    Propose and solve your own problem.