PHY265 Lecture notes: Miscellaneous Topics

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Contents

1 Weak measurements

Following: *Introduction to Weak Measurements and Weak Values*, Boaz Tamir and Eliahu Cohen, 2013, **Quanta**, 2: 7-17.

1.1 The needle

The measurement device is described with a wave-function that depends on a one-dimensional continuous variable x;

$$|\phi\rangle = \int_{-\infty}^{\infty} dx\ \phi(x)\, |x\rangle\, . \tag{1}$$

The probability that the measurement device (or needle) is at position x depends on the function

$$\phi(x) = (2\pi\sigma^2)^{-\frac{1}{4}} e^{-x^2/(4\sigma^2)}. \tag{2}$$

The probability that the needle is it position x is

$$p(x) = \phi(x)\phi(x)^* = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/(2\sigma^2)}$$

which is a Gaussian distribution with a mean of zero and a variance of σ^2.

The position operator $\hat{X}_d$, when operating on $|x\rangle$ gives an eigenvalue of x. In other words $\hat{X}_d |x\rangle = x |x\rangle$.

It is helpful to introduce a momentum operator $\hat{P}_d$ that is conjugate to $\hat{X}_d$ such that

$$[\hat{X}_d, \hat{P}_d] = \hat{X}_d \hat{P}_d - \hat{P}_d \hat{X}_d = i\hbar. \tag{3}$$

1.2 Time dependent evolution – The Heisenberg picture

A wave function $|\psi(t)\rangle$ evolves as

$$|\psi(t)\rangle = e^{-iHt/\hbar} |\psi(0)\rangle \tag{4}$$

for a constant Hamiltonian $\hat{H}$.

For a time dependent Hamiltonian $H(t)$ equation 33 is more generally

$$|\psi(t)\rangle = e^{-i\int_0^t dt' \ H(t')/\hbar} |\psi(0)\rangle . \tag{5}$$

This expression follows if you consider Hamiltonian evolution during a series of small time intervals.

The Heisenberg picture is related to the Schrodinger picture via

$$\hat{A}(t) |\psi(0)\rangle = \hat{A}(0) |\psi(t)\rangle .$$

where $\hat{A}(0)$ is a time independent operator.

In the Heisenberg picture an operator $\hat{A}$ evolves as

$$\hat{A}(t) = U^\dagger(t)\hat{A}(0)U(t) \tag{6}$$

where

$$U(t) = e^{-iHt/\hbar} \qquad \text{for constant } H$$
$$U(t) = e^{-i\int_0^t dt' \ H(t')/\hbar} \qquad \text{otherwise.}$$

Equation 6 is consistent with

$$\frac{\partial \hat{A}(t)}{\partial t} = \frac{i}{\hbar}[H, \hat{A}(t)]. \tag{7}$$

Let's check that this is also true for the time dependent case. First we compute

$$\begin{aligned}\frac{dU}{dt} &= \frac{d}{dt} e^{-i\int_0^t dt' \ H(t')/\hbar} \\ &= -\frac{iH(t)}{\hbar} e^{-i\int_0^t dt' \ H(t')/\hbar} \\ &= -\frac{iH(t)}{\hbar} U(t).\end{aligned}$$

Recall that H is Hermitian.

$$\begin{aligned}
\frac{\partial \hat{A}(t)}{\partial t} &= \frac{d}{dt}\left(U^\dagger \hat{A}(0) U\right) \\
&= \frac{dU^\dagger}{dt}\hat{A}(0)U + U\hat{A}(0)\frac{dU}{dt} \\
&= \frac{iH(t)}{\hbar}U^\dagger \hat{A}(0)U + U^\dagger \hat{A}(0)U\frac{-iH(t)}{\hbar} \\
&= [\frac{iH(t)}{\hbar}, U^\dagger \hat{A}(0)U] \\
&= \frac{i}{\hbar}[H(t), \hat{A}(t)] \qquad (8)
\end{aligned}$$

We have shown that equation 7 holds for the time dependent Hamiltonian case.

Note that $H(t)$ commutes with itself so it commutes with $U(t)$. It does not matter whether we describe H in Heisenberg or Shrodinger picture. Equation 8 is equivalent to

$$\frac{\partial \hat{A}(t)}{\partial t} = U^\dagger [H(t), A(0)]U. \qquad (9)$$

1.3 Time dependent evolution for a momentum impulse

Consider a Hamiltonian

$$\hat{H}_n(t) = g(t)\hat{P}_d \qquad (10)$$

acting on wave function $|\phi(t=0)\rangle$, with $g(t)$ an impulse function that satisfies

$$\int_0^T dt\ g(t) = 1.$$

Equation 5 becomes

$$\begin{aligned}
|\psi(T)\rangle &= e^{-i\int_0^T dt\ \hat{H}_n(t)/\hbar}|\psi(0)\rangle \\
&= e^{-i\hat{P}_d/\hbar}|\psi(0)\rangle
\end{aligned}$$

A nice way to see the evolution is to compute

$$\begin{aligned}
\hat{X}_d(T) - \hat{X}_d(0) &= \int_0^T dt\ \frac{\partial \hat{X}_d(t)}{\partial t} \\
&= \int_0^T dt\ \frac{i}{\hbar}[\hat{H}_n, \hat{X}_d(t)] \qquad (11) \\
&= \int_0^T dt\ \frac{i}{\hbar}g(t)[\hat{P}_d, \hat{X}_d(t)] \\
&= \int_0^T dt\ \frac{i}{\hbar}g(t)[\hat{P}_d(t), \hat{X}_d(t)] \\
&= 1 \qquad (12)
\end{aligned}$$

and we have used equations 7 and 3. This means that the impulse shifts the mean position of the needle.

1.4 Needle and System

We consider a combined system that is the tensor product of a two state system S and the needle. Take operator $\hat{A}$ to operate on the two state system. Operator $\hat{A}$ has eigenvectors $|a_i\rangle$ and eigenvalues a_i, indexed by i so $\hat{A}|a_i\rangle = a_i|a_i\rangle$.

We take an interaction Hamiltonian

$$\hat{H}_{int} = g(t)\hat{A} \otimes \hat{P}_d \tag{13}$$

Because the Hamiltonian is a tensor product, the Heisenberg operators are also tensor products.

We recompute equation

$$\begin{aligned} \hat{X}_d(T) - \hat{X}_d(0) &= \int_0^T dt\ \frac{\partial}{\partial t} X_d(t) \\ &= \int_0^T dt\ \frac{i}{\hbar}[\hat{H}_{int}, \hat{X}_d(t)] \\ &= \int_0^T dt\ g(t)\frac{i}{\hbar} U^\dagger [\hat{A} \otimes \hat{P}_d(0), I \otimes \hat{X}_d(0)] U \\ &= \int_0^T dt\ g(t)\hat{A}(t) \\ &= \hat{A}(T). \end{aligned} \tag{14}$$

This implies that

$$\hat{X}_d(T)|a_j\rangle \otimes |\phi(x)\rangle = a_j|a_j\rangle \otimes |\phi(x)\rangle \tag{15}$$

However $\hat{X}_d$ is the position operator,

$$X_d(0)|\phi(x)\rangle = X_d(0)\int dx\ \phi(x)|x\rangle = \int dx\ x\phi(x)|x\rangle$$

gives the mean value of x. Equation 15 implies that Hamiltonian evolution by $\hat{H}_{int}$ over a time T shifts the mean value of x by a_j. This implies that

$$\hat{X}_d(T)|a_j\rangle \otimes |\phi(x)\rangle = |a_j\rangle \otimes |\phi(x - a_j)\rangle . \tag{16}$$

Suppose we have an initial state wave function $|\psi\rangle = \sum_j \alpha_j |a_j\rangle$ for the system and combined system and needle initial wave function

$$\begin{aligned} |\Psi(0)\rangle &= |\psi\rangle \otimes |\phi(x)\rangle \\ &= \sum_j \alpha_j |a_j\rangle \otimes |\phi(x)\rangle \end{aligned} \tag{17}$$

then

$$|\Psi(T)\rangle = e^{-\frac{i}{\hbar}\int_0^T dt\ \hat{H}_{int}} |\psi\rangle \otimes |\phi(x)\rangle \tag{18}$$

$$= \sum_j \alpha_j |a_j\rangle \otimes |\phi(x - a_j)\rangle \,. \tag{19}$$

Take $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ for a two state system and $\hat{A} = \frac{\hbar}{2}\sigma_z$ for a spin system. Following the interaction impulse the system would be in the state

$$|\Psi(T)\rangle = \int dx\ (\alpha|0\rangle \phi(x - \hbar/2) + \beta|1\rangle \phi(x + \hbar/2))\,|x\rangle \tag{20}$$

Using the two Gaussian functions

$$|\Psi(T)\rangle = \int dx\ (2\pi\sigma^2)^{-\frac{1}{4}} \Big(e^{-\frac{(x-\hbar/2)^2}{4\sigma^2}} \alpha|0\rangle \otimes |x\rangle + e^{-\frac{(x+\hbar/2)^2}{4\sigma^2}} \beta|1\rangle \otimes |x\rangle \Big). \tag{21}$$

1.5 Needle measurement for a two state system

Suppose the needle is measured for the two state system discussed in the previous section. If the needle position is measured to be at position x_0 then the wave function in equation 21 collapses.

$$\langle x_0|\,|\Psi(T)\rangle = (2\pi\sigma^2)^{-\frac{1}{4}} \Big(e^{-\frac{(x_0-\hbar/2)^2}{4\sigma^2}} \alpha|0\rangle + e^{-\frac{(x_0+\hbar/2)^2}{4\sigma^2}} \beta|1\rangle \Big). \tag{22}$$

The amplitude to for the system to be $|0\rangle$ would depend on the actual value of x_0. If x_0 is around 1 and σ is around 1 then the amplitude to post-select the $|0\rangle$ state is higher than that to post-select the $|1\rangle$ state. If σ is large then the bias is small toward one of the system states is small. There is a weak correlation between the value of the needle and the direction of the bias.

The setting where σ is large and the correlations between needle measurement and system state are weak is the setting for weak quantum measurement.

1.6 Weak measurement and Post-selection

Let $|\psi_{init}\rangle$ and $|\psi_{fin}\rangle$ be initial and final states for the system S which is weakly coupled to a needle. We assume that we can prepare the system initially in the state $|\psi_{init}\rangle$. We apply the weak interaction. Then we make a measurement of the system and discard all those that are not equal to $|\psi_{fin}\rangle$. We repeat this procedure many times, while counting the fraction of post selected states.

Post selection is a strong measurement with projection operator

$$P_f = |\psi_{fin}\rangle \langle\psi_{fin}| \otimes \hat{I}_d. \tag{23}$$

The interaction Hamiltonian is given by equation 13. Past time T the Hamiltonian is constant. We refer to $e^{i\hat{H}T/\hbar}$ as the evolution operator which takes into account evolution from $0 \to T$. The wave-function evolves as

$$|\Psi(T)\rangle = e^{-i\hat{H}T/\hbar} |\psi_{init}\rangle \otimes |\phi(x)\rangle \tag{24}$$

Let us expand both initial and final system states in terms the eigenstates of $\hat{A}$,

$$|\psi_{init}\rangle = \sum_j \alpha_{j,init} |a_j\rangle \tag{25}$$

$$|\psi_{fin}\rangle = \sum_j \alpha_{j,fin} |a_j\rangle \tag{26}$$

$$P_f = \sum_{k,l} \alpha_{k,fin}\alpha^*_{l,fin} |a_k\rangle \langle a_l| \,. \tag{27}$$

Applying the weak measurement

$$|\Psi(T)\rangle = \sum_j \alpha_{j,init} |a_j\rangle \otimes |\phi(x - a_j)\rangle \tag{28}$$

Projecting onto the final state in the system S only gives the post selected pointer wave-function which we arbitrarily call $|\gamma\rangle$

$$\begin{aligned} |\gamma\rangle = \langle\psi_{fin}|\Psi(T)\rangle &= \sum_{j,k} \langle a_k| \alpha^*_{k,fin}\alpha_{j,init} |a_j\rangle \otimes |\phi(x - a_j)\rangle \\ &= \sum_j \alpha^*_{j,fin}\alpha_{j,init} |\phi(x - a_j)\rangle \,. \end{aligned} \tag{29}$$

Note that $\sum_j \alpha^*_{j,fin}\alpha_{j,init} = \langle\psi_{fin}|\psi_{init}\rangle$. Taking the complex conjugate and integrating over space we find that the probability of measuring the final state in the system is

$$p_f = |\langle\psi_{fin}|\psi_{init}\rangle|^2. \tag{30}$$

Equation 29 is a sum of Gaussian functions, each with mean a_j. To compute the mean x value or expectation value of $\hat{X}_d$

$$\mu_x = \langle\gamma| \hat{X}_d |\gamma\rangle = \sum_k \langle\phi(x - a_k)| \alpha_{k,fin}\alpha^*_{k,init} \sum_j a_j \alpha^*_{j,fin}\alpha_{j,init} |\phi(x - a_j)\rangle \,.$$

This involves a bunch of products of overlapping Gaussians. To simplify further I think we need an approximation.

1.7 The weak value

It is convenient to define something called the **weak value** for the operator $\hat{A}$ (operating on the system S) that is a function of $|\psi_{init}\rangle$ and $|\psi_{fin}\rangle$, (state vectors in the system S)

$$\langle \hat{A} \rangle_w \equiv \frac{\langle \psi_{init}| \hat{A} |\psi_{fin}\rangle}{\langle \psi_{init}|\psi_{fin}\rangle}. \tag{31}$$

Notice that the weak value can be very large, particularly if $|\psi_{fin}\rangle$ is nearly perpendicular to $|\psi_{init}\rangle$. The weak value does not need to be a real number.

What does the needle distribution look like for the post selected states? We go back to the state after the interaction and after post selection and look at the needle's wave function which was given in equation 29.

$$\begin{aligned}
|\gamma\rangle = \langle \psi_{fin}|\Psi(T)\rangle &= \langle \psi_{fin}| e^{-i\hat{H}T/\hbar} |\psi_{init}\rangle \otimes |\phi(x)\rangle \\
&= \langle \psi_{fin}| e^{-i\hat{A}\otimes P_d T/\hbar} |\psi_{init}\rangle \otimes |\phi(x)\rangle \\
&\approx \langle \psi_{fin}| (I - i\hat{A} \otimes P_d T/\hbar) |\psi_{init}\rangle \otimes |\phi(x)\rangle
\end{aligned}$$

On the last step we assume that the measurement is weak and we approximate the exponential to first order. Apparently the approximation is justified if the needle wave-function has a wide variance and if T is small. Continuing,

$$\begin{aligned}
|\gamma\rangle &\approx \langle \psi_{fin}| (I - i\hat{A} \otimes P_d T/\hbar) |\psi_{init}\rangle \otimes |\phi(x)\rangle \\
&= \langle \psi_{fin}|\psi_{init}\rangle \otimes |\phi(x)\rangle - \langle \psi_{fin}| i\hat{A} \otimes P_d T/\hbar) |\psi_{init}\rangle \otimes |\phi(x)\rangle \\
&= \langle \psi_{fin}|\psi_{init}\rangle \left(I - \frac{\langle \psi_{fin}| i\hat{A} \otimes P_d T/\hbar |\psi_{init}\rangle}{\langle \psi_{fin}|\psi_{init}\rangle} \right) \otimes |\phi(x)\rangle \\
&= \langle \psi_{fin}|\psi_{init}\rangle \left(I - i\frac{\langle \psi_{fin}| \hat{A} |\psi_{init}\rangle}{\langle \psi_{fin}|\psi_{init}\rangle} P_d T/\hbar \right) |\phi(x)\rangle \\
&= \langle \psi_{fin}|\psi_{init}\rangle \left(I - i\langle \hat{A} \rangle_w P_d T/\hbar \right) |\phi(x)\rangle \\
&\approx \langle \psi_{fin}|\psi_{init}\rangle \, e^{-i\langle \hat{A} \rangle_w P_d T/\hbar} |\phi(x)\rangle .
\end{aligned} \tag{32}$$

Equation 32 (and without approximation equation 29) show the spatial x distribution of the needle. What does the probability distribution of the needle look like? The probability distribution is centered around the weak value $\langle A \rangle_w$. The needle is shifted to a large value if the weak value $\langle A \rangle_w$ is large.

The experimenter would not only be measuring the needle but would also be pre and post selecting the system with strong measurements.

If the experimenters can afford to run the experiment many many times, they could choose $|\psi_{init}\rangle$ and $|\psi_{fin}\rangle$ nearly perpendicular and put the probability p_f to a low value (via equation 30). This would mean that they would need to discard many measurements of the system, but then the needle distribution will show large deviations and be very sensitive to the coupling operator $\hat{A}$ in the interaction.

2 Quantum Random Walks

A popular way to do a random walk is to start with an initial state vector $|\psi\rangle$ in a product Hilbert space and then repetitively perform a series of unitary transformations. This is the type introduced by Aharonov, Davidovich, and Zagury 1993. Quantum random walks can also be called *quantum cellular automata*. Unlike a classical and stochastic random walk, all transformations are unitary and hence reversible. There is no actual stochasticity. The walk does not lose its recollection of the initial state and it cannot converge to a stationary distribution. For some problems, such as propagation along a random tree, quantum walks can give exponential speedups over classical walks. And some quantum algorithms look like quantum random walks.

We explore a discrete quantum random walk of a spin (a two state system) on a circle. The system can be at any location of a discrete set of points on the circle and it can have be in either spin up or spin down states.

Take a state space that is a tensor product of a space with two states (a qubit), $\mathcal{H}_2$, and a space with N states, $\mathcal{H}_N$ (the discrete points on the circle). A basis for this space is

$$|jn\rangle$$

where $j \in [0, 1]$ is spin up or down and $n \in [0, N-1]$. The N dimensional Hilbert space can be described as N possible particle positions.

The entire Hilbert space is a tensor product space $\mathcal{H} = \mathcal{H}_2 \otimes \mathcal{H}_N$.

We start with $|\psi\rangle = |00\rangle$ and alternate applying a spin mixing operator

$$\mathbf{H} \otimes \mathbf{I} \tag{33}$$

and a position change that depends on the spin

$$\mathbf{C} = \mathbf{P}_0 \otimes \mathbf{U}_+ + \mathbf{P}_1 \otimes \mathbf{U}_-. \tag{34}$$

$\mathbf{H}$ is the Hadamard operator on the spin state (in $\mathcal{H}_2$) and it takes

$$\mathbf{H}|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$
$$\mathbf{H}|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

$$\mathbf{H} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

The transformation $\mathbf{H} \otimes \mathbf{I}$ shifts the spin state but operates on a state in $\mathcal{H}$.

$\mathbf{P}_0$ projects the spin state onto $|0\rangle$ and $\mathbf{P}_1$ projects the spin state onto $|1\rangle$,

$$\begin{aligned} \mathbf{P}_0|0\rangle &= |0\rangle \\ \mathbf{P}_0|1\rangle &= 0 \\ \mathbf{P}_1|0\rangle &= 0 \\ \mathbf{P}_1|1\rangle &= |1\rangle \end{aligned}$$

These two operate on spin states, or those in $\mathcal{H}_2$. We can also write

$$\begin{aligned} P_0 &= |0\rangle\langle 0| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ P_1 &= |1\rangle\langle 1| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

The operators $\mathbf{U}_+$ and $\mathbf{U}_-$ raise and lower n

$$\begin{aligned} \mathbf{U}_+|n\rangle &= |n+1 \bmod N\rangle \\ \mathbf{U}_-|n\rangle &= |n-1 \bmod N\rangle \end{aligned}$$

These two operate on states in $\mathcal{H}_N$. We can also write

$$\mathbf{U}_+ = \sum_{j=0}^{N-2} |j+1\rangle\langle j| + |0\rangle\langle N-1| = \begin{pmatrix} 0 & 0 & 0 & .. & 0 & 1 \\ 1 & 0 & 0 & .. & 0 & 0 \\ 0 & 1 & 0 & .. & 0 & 0 \\ 0 & 0 & 1 & .. & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & .. & 1 & 0 \end{pmatrix}$$

$$\mathbf{U}_- = \sum_{j=1}^{N-1} |j-1\rangle\langle j| + |N-1\rangle\langle 0| = \begin{pmatrix} 0 & 1 & 0 & .. & 0 & 0 \\ 0 & 0 & 1 & .. & 0 & 0 \\ 0 & 0 & 0 & .. & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & .. & 0 & 1 \\ 1 & 0 & 0 & .. & 0 & 0 \end{pmatrix}$$

The operator $\mathbf{C}$ moves the particle to the right if the spin is up and moves the particle to left if the spin is down. It simulates a coin flip. This is why the procedure can be called a random walk. Because $\mathbf{U}_+|N-1\rangle = |0\rangle$ and $\mathbf{U}_-|0\rangle = |N-1\rangle$, the end of the N state space is connected with the beginning of it, so our position space is equivalent to N equidistant points on a circle.

We can think of operator $\mathbf{C}$ as defining a network of connections between states in the N-dimensional Hilbert space $\mathcal{H}_N$. The $\mathbf{C}$ operator entangles the spin with the particle position.

Combining the two operators

$$\mathbf{V} = \mathbf{C} * (\mathbf{H} \otimes \mathbf{I})$$

One can plot the probability of being in each state $|n\rangle$ at each iteration.

The resulting vector is has even/odd parity and spreads out ballistically (rather than diffusively) with more iterations. The shape of the probability distribution differs from that of a classical random walk.

Let's evaluate the first few states. Starting with $|\psi_0\rangle = |00\rangle$ we first apply the Hadamard to the spin state and we get

$$\mathbf{H} \otimes \mathbf{I} |\psi_0\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle)$$

We now apply the controlled raising and lower operator, $\mathbf{C}$. $|00\rangle \rightarrow |01\rangle$ and $|10\rangle \rightarrow |1, N-1\rangle$.

$$\mathbf{V} |\psi_0\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |1, N-1\rangle)$$

Let's operate again with the Hadamard

$$(\mathbf{H} \otimes \mathbf{I})\mathbf{V} |\psi_0\rangle = \frac{1}{2}(|01\rangle + |11\rangle + |0, N-1\rangle - |1, N-1\rangle)$$

Now we operate again with $\mathbf{C}$.

$$\mathbf{V}^2 |\psi_0\rangle = \frac{1}{2}(|02\rangle + 2|00\rangle - |0, N-2\rangle)$$

Notice the even/odd parity developing! Apply the Hadamard again

$$(\mathbf{H} \otimes \mathbf{I})\mathbf{V}^2 |\psi_0\rangle = \frac{1}{\sqrt{8}}(|02\rangle + |12\rangle + 2|00\rangle + 2|10\rangle - |0, N-2\rangle - |1, N-2\rangle)$$

Apply $\mathbf{C}$ again,

$$\mathbf{V}^3 |\psi_0\rangle = \frac{1}{\sqrt{8}}(|03\rangle + |11\rangle + 2|01\rangle + 2|1, N-1\rangle - |0, N-1\rangle - |1, N-3\rangle)$$

And so on.

A quantum random walk might be used to find a short path on a complicated network. They also are interesting as a class of cellular automata. One can study the influence of imperfections and external perturbations on the behavior of a quantum random walk. While static spatial random changes of the coin may lead to Anderson localization temporal randomness in the coin operator can cause decoherence resulting in a transition to classical random walking behavior. Their transport and percolation behavior is dependent on the network or graph of the shift operator. Some times people approximate continuous (in time) Hamiltonian evolution with a series of discretely (in time) applied operators. Quantum random walks can be considered relevant for numerical techniques aiming to approximate continuous systems.

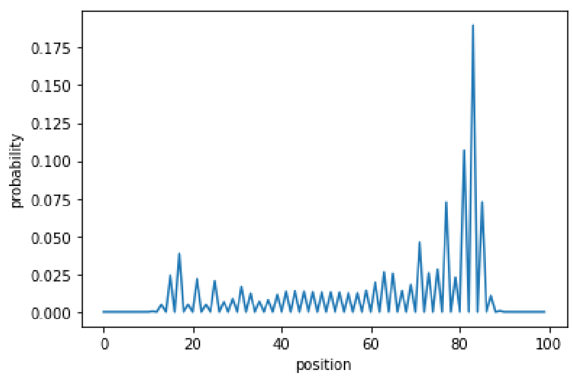

Figure 1: A quantum random walk after taking 100 steps. The initial state was $|0, N/2\rangle$ in a Hilbert space that is a tensor product of a two state system and an $N = 100$ state system. Then we alternatively applied a spin mixing (Hadamard) operation (equation 33) and a raising and lower operation that is sensitive to the spin (equation 34). After 100 of these operations the probability of being in state $|j\rangle$ is plotted (where $|j\rangle$ is for the N state system). The probability distribution spreads faster than it would in a classical random walk.