Homework set 11. Physics 141, Fall 2022

Due date: Monday Dec 2, 2022 at noon.

Total of 12 points. On entropy, statistical physics and gases.

1. On multiplicity of states (2 points)

   A nano-system $\mathcal{S}_1$ with four harmonic oscillators and 1 quantum of energy is sitting next to another system $\mathcal{S}_2$ with four harmonic oscillators and 2 quanta of energy. The quanta of energy are the same size for both systems.

   a. Initially what is the multiplicity of states for $\mathcal{S}_1$ and what is the multiplicity of states for $\mathcal{S}_2$? What is the entropy of $\mathcal{S}_1$ and what is the entropy of $\mathcal{S}_2$?

   b. With the two systems isolated and separated and out of thermal contact, what is the multiplicity of states for the total system? (Hint: Multiply the multiplicities $g_1 g_2$). What is the entropy?

   c. The two systems are put in thermal contact. What is the multiplicity of states for the combined system once it reaches equilibrium? What is the entropy of the combined system? The entropy should have increased.

   Hints: Entropy is log of the multiplicity of states.

2. On temperature and entropy (3 points)

   Consider a system of N particles with up/down spin values. Each particle has magnetic moment $\pm m$. Under a magnetic field B the total energy is

   $$U = 2smB$$

   where $2s$ is the spin excess (the number of spin ups subtracted by the number of spin downs). Notice that U can be negative. The multiplicity of spin states

   $$g(N,s) \approx g(N,0)e^{-\frac{2s^2}{N}}$$

   for large N and $s \ll N$.

   a. What is the entropy $\sigma(N,U)$ in terms of the total energy?

   b. Show that the temperature of the system

   $$T = -\frac{m^2B^2N}{k_BU}$$

   where U is the average total energy.

   c. Two spin systems are brought in thermal contact. One has N_1 particles and initially has total energy U_1 and the other has N_2 particles and initially has total energy U_2. What is the temperature when they reach thermal equilibrium?

   Hint: Temperature is defined as

   $$\frac{1}{k_BT} = \left(\frac{\partial \sigma}{\partial U}\right)_N$$

3. On the Boltzmann factor (3 points)

   A linear cavity of length L can support photon modes.

   The lowest energy mode has energy

   $$\epsilon_0 = \hbar\omega_0$$

   and frequency

   $$\omega_0 = \frac{\pi c}{L}$$

   so that it fits in the cavity.

   More than one photon with the same frequency can be present at the same time in the cavity. A state with n photons in the lowest energy mode has energy $E_n = n\hbar\omega_0$.

   a. With only one possible cavity mode (the lowest energy one) and the cavity in contact with a thermal reservoir at temperature T, show that the probability there are n photons in the cavity (all of the lowest energy mode) is

   $$P(n) = \frac{1}{Z}e^{-\frac{n\hbar\omega_0}{k_BT}}$$

with partition function

$$Z = \frac{1}{1 - e^{-\frac{\hbar\omega_0}{k_B T}}} \tag{1}$$

What do you need to do? Assume that the probability that there are n photons depends on the Boltzmann factor. To make sure that the sum of all the probabilities adds to 1, you need to compute the partition function, Z which is the sum of all the Boltzmann factors.

Hint: To compute the partition function use the geometric series

$$\sum_{i=0}^{\infty} x^i = \frac{1}{1-x}$$

b. The average energy is a sum over all states of energy times probability

$$\bar{E} = \sum_i P(\epsilon_i)\epsilon_i = \frac{\sum_i e^{-\epsilon_i \beta}\epsilon_i}{\sum_j e^{-\epsilon_j \beta}}$$

where ϵ_i is the energy of state i and

$$\beta \equiv \frac{1}{k_B T}.$$

Show that

$$-\frac{1}{Z}\frac{dZ}{d\beta} = \bar{E} \tag{2}$$

where $\bar{E}$ is the average energy at temperature T.

Hint: $\frac{d}{d\beta}\sum_i e^{-\epsilon_i \beta} = \sum_i -\epsilon_i e^{-\epsilon_i \beta}$.

This is in general true, not only for this particular problem and is convenient for problems where the partition function is known.

c. Compute the average energy $\bar{E}$ of photons in the cavity at temperature T.

Hint: use equations 1 and 2.

Show that the answer is

$$\bar{E} = \frac{\hbar\omega_0}{e^{\frac{\hbar\omega_0}{k_B T}} - 1}$$

d. What is the average number of photons in the cavity?

Hint: the average energy is directly related to the average number of photons.

e. Show that at high temperatures or when $k_B T \gg \hbar\omega_0$, the average energy of photons in the cavity is

$$\bar{E} \approx k_B T$$

Hint: if x is small $e^x \sim 1 + x + ...$

This calculation is similar to that used to derive the black body (Planck) radiation spectrum but in that case many photon modes are taken into account.

4. On heat capacity of a harmonic oscillator (2 points)

   Consider a quantum mechanical harmonic oscillator with vibrational energy states described by

   $$E_n = \left(\frac{1}{2} + n\right)\hbar\omega_0$$

   where n is a quantum number and ω_0 a frequency.

   a. At high temperature or when $k_B T \gg \hbar\omega_0$, what is the average energy in vibrations?

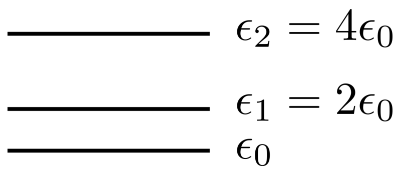
   Hint: use the calculations you did in the previous problem. The systems are equivalent! At large n, the energy $E_n \sim n\hbar\omega_0$.

   b. At high temperature or when $k_B T \gg \hbar\omega_0$, what is the specific heat C_V?

   Hint: $C_V = \frac{d\bar{E}}{dT}$.

5. On the Boltzmann factor for a three state system (2 points)

   We consider a collection of particles that have three possible energy states, ϵ_0, $\epsilon_1 = 2\epsilon_0$, and $\epsilon_2 = 4\epsilon_0$. Here ϵ_0 is the ground state energy. The particles are in contact with a thermal reservoir at temperature T.



a. What temperature T gives twice as many particles in the middle energy state as in the highest energy state? (T can be a function of ϵ_0).

b. At this temperature what is the fraction of particles in the ground state?

Hints. The fraction of particles in the ground state is

$$n_0 = \frac{1}{Z} e^{-\frac{\epsilon_0}{k_B T}}$$

and similar expressions give the fractions in the other states. The partition function is

$$Z = e^{-\frac{\epsilon_0}{k_B T}} + e^{-\frac{\epsilon_1}{k_B T}} + e^{-\frac{\epsilon_2}{k_B T}}$$

It is helpful to show that you can write $Z = x + x^2 + x^4$ with a particular choice of x. The first part lets you solve for x.