PHY141 Lectures 8,9 notes

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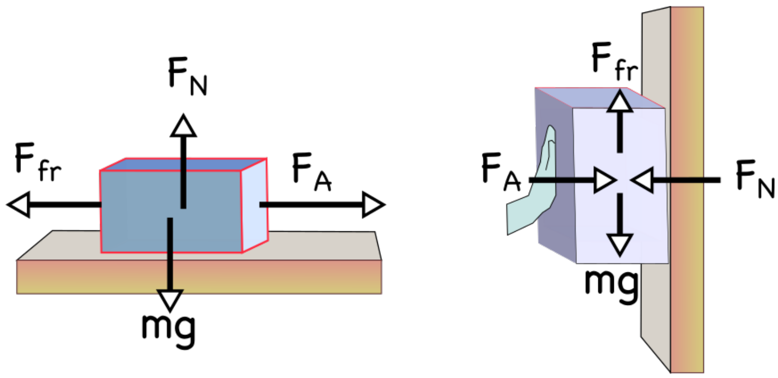

Figure 1: The friction force is exerted parallel to the surface and its strength depends on the normal force.

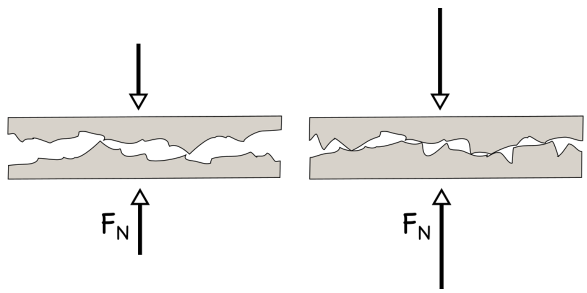

Figure 2: The friction force is remarkably independent of the contact physics. A larger normal force gives a larger friction force.

1 Friction and Drag

1.1 Energy loss

Newton's first law posits that a body in motion keeps moving. From changes in momentum we can infer the existence or nature of interaction forces. Alternatively interactions causes changes in momentum.

However, we live in a world where masses don't usually keep coasting. Exceptions are pucks on an air hockey table, or on ice, or projectiles moving through vacuum.

Friction is a dissipative force between surfaces. Drag is due to hydrodynamic forces. Both of these depend upon velocity, unlike gravity or the electric force.

1.2 Friction

Consider a block on a horizontal surface. A force downward is exerted onto the block F_N. A horizontal force is also applied to the block F_A. If the horizontal component of the force is too low, the block will not move. If the force pushing it down is high, the block will not move. Because the block does not move, the friction force exactly opposes the applied horizontal force,

$$|\mathbf{F}_{fr}| = |\mathbf{F}_A|. \tag{1}$$

This is known as **static friction**. There is a maximum value for the static friction

$$F_{fr} \leq \mu_s F_N \tag{2}$$

defined with static friction coefficient μ_s. The friction force adjusts itself to match F_A up to a maximum value of $\mu_s F_N$ so as to keep the block from moving. Once the friction force reaches this maximum value, the block can start to slide.

We assume the block/surface contact is flat. We define a coordinate system with directions perpendicular and parallel to the surface. We construct a unit vector $\hat{\mathbf{n}}$ that is perpendicular to the surface and a unit vector parallel to the surface $\hat{\mathbf{s}}$. The force exerted by the surface on the block due is decomposed into normal and parallel components

$$\mathbf{F} = F_N \hat{\mathbf{n}} + F_{fr} \hat{\mathbf{s}}. \tag{3}$$

Kinetic friction or dynamic friction is the horizontal friction force when the block slides. It is a constant force

$$F_{fr} = \mu_k F_N \tag{4}$$

with dynamic friction coefficient μ_k. The force is applied in the opposite direction of the direction of motion

$$\mathbf{F}_{fr} = -\mu_k F_N \ \mathrm{sign}(v) \hat{\mathbf{s}} \tag{5}$$

with velocity v. Here $\hat{\mathbf{s}}$ is in the direction of motion.

This model for friction forces is known as the **Coulomb model**. The details of the surface are ignored in the Coulomb model for friction. The details of the number of contacts, the area of contacts and forces between contacts, surface deformation and how these depend on the normal force and speed are all ignored.

The coefficient of static friction is expected to be somewhat larger than the coefficient of kinetic friction.

The commonly assumed Coulomb friction description is not smooth! It is non trivial to numerically integrate a dynamical system with Coulomb friction because the transitions are not smooth.

Friction is required for stopping a car, turning a car or motor cycle and many forms of locomotion.

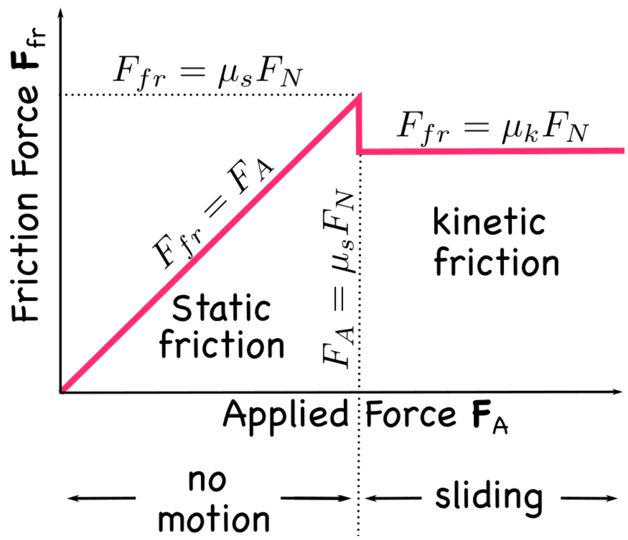

Figure 3: The friction force F_{fr} as a function of applied horizontal force, F_A. We show static and kinetic friction regimes.

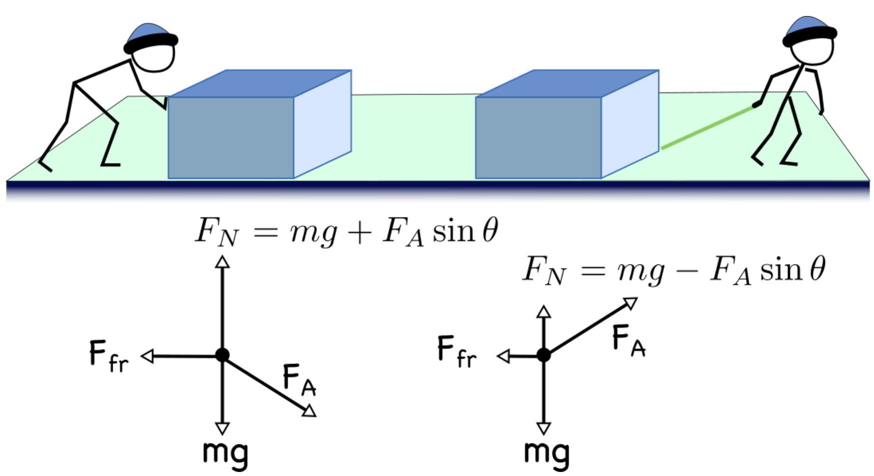

Figure 4: The friction force is stronger if there a force pushing the box downward (on left) rather than pulling it upward (on right).

1.3 Drag

Hydrodynamic or aerodynamic drag depend on the cross sectional area A, the velocity with respect to the fluid (or air) v, and the density ρ of the fluid that is being displaced.

$$F_D = \frac{1}{2} C_D \rho A v^2. \tag{6}$$

The rate that mass is swept up by the object is $\rho A v = dM/dt$. However momentum per unit volume is ρv so the rate that momentum is changed depends on $\rho A v^2$. This accounts for the v^2 dependence. The unit-less drag coefficient C_D is sensitive to body shape and fluid viscosity (or Reynolds number). Drag is applied in the direction opposite to the motion (with respect to the fluid).

Figure 5: The rate that mass of the fluid is swept up $\frac{dM}{dt} = \rho A v$ where A is the cross sectional area, ρ is the density of the fluid and v is the velocity. The rate of momentum change is $\frac{dp}{dt} \sim \frac{dM}{dt} v = \rho A v^2$. The drag force scales with these physical quantities.

For a falling raindrop of mass M we balance the drag force against the gravitational force

$$\frac{1}{2} C_D \rho_{air} A v^2 = Mg \tag{7}$$

giving terminal velocity

$$v_{term} = \sqrt{\frac{Mg}{\frac{1}{2} C_D \rho_{air} A}}. \tag{8}$$

If the radius of the drop is R, then $M \propto R^3$ and $A \propto R^2$ and we expect a higher terminal velocity for larger raindrops.

Hydrodynamic drag limits the speed of boats and fish, and sets a terminal velocity for raindrops, where the drag force exactly balances the gravitational force.

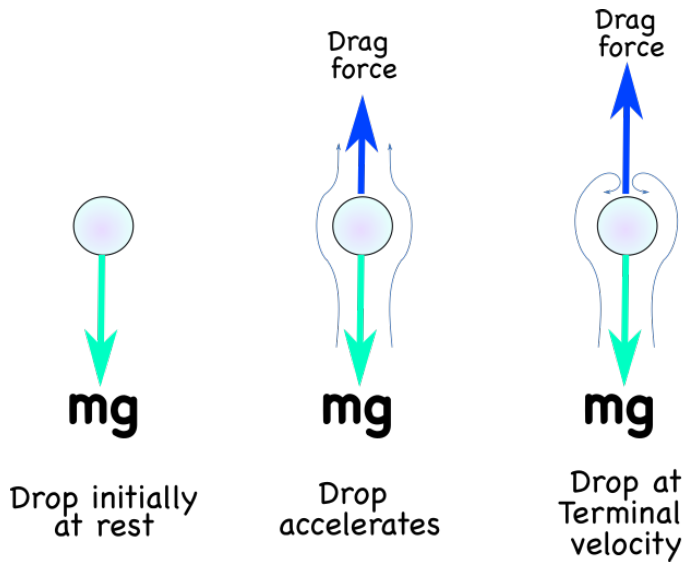

Figure 6: A falling raindrop reaches terminal velocity when the aerodynamic drag force balances gravity.

1.4 Dashpots

Another common frictional type of force is with a dashpot. Here the force is proportional to the velocity and it is applied opposite to the velocity

$$\mathbf{F} = -\alpha \mathbf{v}. \tag{9}$$

Dashpots can be hydraulic or contain a viscous fluid or an air piston.

The coefficient α has units of mass/time.

Dashpots, drag and friction are velocity dependent (not position dependent) forces. Why does friction depend on velocity? It depends on velocity because it is always exerted in the direction that is opposite the motion.

2 Friction and damping examples

2.1 Damped motion

Consider a point mass with mass m that is damped with a dashpot. The dashpot exerts a force $F = -\alpha v$. Newtons law give

$$ma = F = -\alpha v$$
$$m\frac{d^2x}{dt^2} = -\alpha\frac{dx}{dt}$$

The acceleration $a = \frac{dv}{dt}$ so we can write Newton's equation as

$$\frac{dv}{dt} = -\frac{\alpha}{m} v. \tag{10}$$

We solve this equation by integrating it

$$\begin{aligned} \frac{dv}{v} &= -\frac{\alpha}{m} dt \\ \ln v &= -\frac{\alpha}{m} + \text{constant} \\ v &= e^{-\frac{\alpha}{m}t} C' \end{aligned}$$

where C' is a constant. Using $t = 0$ we find that $C' = v_0$ the initial velocity.

The solution to the first order differential equation (equation 10) is an exponential decay

$$v(t) = v_0 e^{-\frac{\alpha}{m}t} \tag{11}$$

where v_0 is an initial velocity. The velocity decays to zero. We can integrate a second time to find

$$x(t) = \frac{v_0 m}{\alpha} \left(1 - e^{-\frac{\alpha}{m}t}\right) + x_0 \tag{12}$$

where x_0 is the initial value of x.

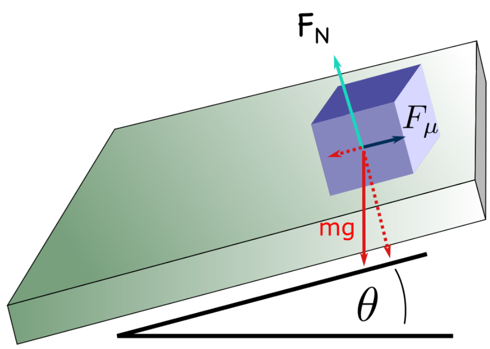

Figure 7: A block on an inclined plane. The normal and tangential components of the gravitational force are shown in red. The friction force F_μ opposes the tangential component of the gravitational force, but its strength depends on the normal force F_N.

2.2 A block on an inclined plane with friction and sandpiles

A block of mass m rests on an inclined plane (see Figure 7). The angle between the plane and horizontal is θ. There is friction between the block base and the surface of the inclined plane.

The angle θ is slowly raised. When θ is high enough the block begins to slide! We denote θ_* as the critical angle, that is just high enough that the block starts to slide.

The component of the gravitational force that is normal to the surface is

$$F_N = mg\cos\theta$$

The component of the gravitational force that is tangential to the surface is

$$F_T = mg\sin\theta$$

The maximum static friction force is

$$F_{\mu_s} = \mu_s F_N = \mu_s mg\cos\theta \tag{13}$$

where μ_s is the static coefficient of friction.

When $F_T > F_\mu$ then gravity exceeds the maximum static friction force and the block can slide. At the critical angle when the block starts to slide

$$\begin{aligned} F_T &= F_{\mu_s} \\ mg\sin\theta_* &= \mu_s mg\cos\theta_* \\ \tan\theta_* &= \mu_s \\ \theta_* &= \arctan \mu_s \end{aligned}$$

We can solve for the coefficient of friction by measuring the critical angle θ_*.

A friction coefficient between particles of sand can be measured from the maximum slope of a sand pile. Sand piles are often conical shapes (see Figure 8). If the slope exceeds the friction coefficient, then the sand starts to flow. It flows until the critical slope is reached. We have found that many granular materials (agricultural grains, sand, gravel, plastic beeds) have a critical angle of about 30°. In our lab, an exception in our lab was polystyrene beads. They liked to slide and the angle is lower.

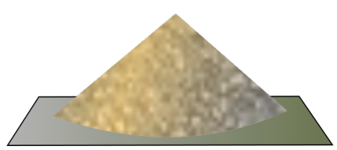

Figure 8: A sand pile. The slope is related to the static coefficient of friction.

2.3 Stopping depth for drag

Consider a mass that experiencing a drag force, $F_{Drag} \propto v^2$. Because drag forces depend on the square of velocity we can write the equation of motion as

$$\frac{dv}{dt} = -\alpha v^2. \tag{14}$$

The coefficient (from equation 6)

$$\alpha = \frac{1}{2m} C_D \rho A^2. \tag{15}$$

The coefficient α has units of 1/length. We solve equation 14 by integrating it

$$\begin{aligned}
\frac{dv}{v^2} &= -\alpha dt \\
-\frac{1}{v} &= -\alpha t + \text{constant} \\
v(t) &= \frac{1}{\alpha t + v_0^{-1}}.
\end{aligned}$$

The time for the velocity to drop depends on the time $\frac{1}{\alpha v_0}$. Physically $1/\alpha$ is a **stopping depth** and is the time it takes the moving object to sweep up about its own mass in fluid.

3 Work, Energy and Power

The energy principle states that

$$\Delta E_{\text{system}} = W_{\text{surr}} + Q$$

where ΔE_{system} is the energy of the system and W_{surr} is the **work** done by the surroundings on the system. The **heat**, Q is the energy flow (or heat flow) from surroundings into the system due to a difference in temperature.

When a force $\mathbf{F}$ is applied to an object and it produces a displacement $\mathbf{d}$ the **work** done by the force is

$$W \equiv \mathbf{F} \cdot \mathbf{d} = F d \cos\theta$$

where θ is the angle between the displacement and force vectors. Work is a scalar, is in units of energy or Joules (J) and can be positive or negative or zero.

A Joule is N m is kg m^2/s^2.

3.1 The sign of work

If the force and displacement are in the same direction the work is positive. The system gains energy.

3.2 Power

Power is the rate of work or work per unit time

$$P = \frac{dW}{dt}$$

The units of power are J/s or W (Watts) or kg m^2/s^3.

Consider a force F that is exerted over a small distance dx. The work done is

$$dW = F dx.$$

If we divide both sides by dt we find

$$\begin{aligned} dW &= F dx \\ \frac{dW}{dt} &= F\frac{dx}{dt} \\ P &= Fv \end{aligned}$$

More generally

$$P = \mathbf{F} \cdot \mathbf{v}.$$

Instantaneously the power depends on force times velocity.

3.3 A circular orbit

Consider an object in a circular orbit due to a gravitational interaction. The force is radial but the velocity is tangential. Because $\mathbf{F} \cdot \mathbf{v} = 0$, no work is done by the radial force. This means the total energy is conserved.

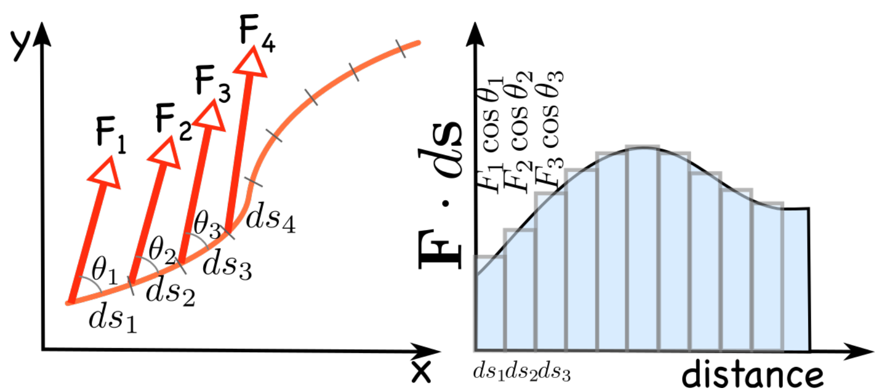

Figure 9: Integrating the work done by a varying force.

3.4 Kinetic energy

We first discuss the non-relativistic setting and then generalize to the relativistic one.

3.4.1 Kinetic energy in the non relativistic limit

Consider a particle of mass m initially at rest at $t = t_0$. We apply a constant force $\mathbf{F} = m\mathbf{a}$ for time $\Delta t = t_1 - t_0$. During the acceleration,

$$\mathbf{v} = \mathbf{a}(t - t_0).$$

The final velocity is

$$\mathbf{v}_1 = \mathbf{a}(t_1 - t_0).$$

We take $\mathbf{x}_0$ and $\mathbf{x}_1$ to be initial and final positions. To find the total work we integrate force times distance

$$\begin{aligned}
W &= \int_{\mathbf{x}_0}^{\mathbf{x}_1} \mathbf{F} \cdot d\mathbf{x} \\
&= \int_{t_0}^{t_1} \mathbf{F} \cdot \frac{d\mathbf{x}}{dt}\, dt \\
&= \int_{t_0}^{t_1} m\mathbf{a} \cdot \mathbf{v}\, dt \\
&= \int_{t_0}^{t_1} m\mathbf{a} \cdot \mathbf{a}(t - t_0)\, dt \\
&= ma^2 \left(\frac{t^2}{2} - t_0 t \right) \Big|_{t_0}^{t_1} \\
&= ma^2 \frac{(t_1 - t_0)^2}{2}
\end{aligned}$$

If the initial velocity $v_0 = 0$ then

$$W = \frac{mv_1^2}{2}$$

We recognize this as the kinetic energy. When applying a constant force to an initially stationary object, the work done is equal to the kinetic energy.

What if the initial velocity is not zero? Under constant acceleration

$$\mathbf{v}_1 = \mathbf{a}(t_1 - t_0) + \mathbf{v}_0$$

We insert this into our expression for the work and find that

$$\begin{aligned}
W &= \int_{t_0}^{t_1} m\mathbf{a}\cdot\mathbf{v}\ dt \\
&= \int_{t_0}^{t_1} m\mathbf{a}\cdot(\mathbf{a}(t_1-t_0)+\mathbf{v}_0)\,dt \\
&= \int_{t_0}^{t_1} m\left[\mathbf{a}\cdot\mathbf{a}(t-t_0)+\mathbf{a}\cdot\mathbf{v}_0\right]\ dt \\
&= m\left[a^2\left(\frac{t^2}{2}-t_0t\right)+m\mathbf{a}\cdot\mathbf{v}_0 t\right]\Bigg|_{t_0}^{t_1} \\
&= ma^2\frac{(t_1-t_0)^2}{2}+m\mathbf{a}\cdot\mathbf{v}_0(t_1-t_0).
\end{aligned}$$

The difference or gain in kinetic energy is

$$\begin{aligned}
\Delta K &= \frac{m}{2}\left[v_1^2-v_0^2\right] \\
&= \frac{m}{2}\left[(\mathbf{a}(t_1-t_0)+\mathbf{v}_0)^2-v_0^2\right] \\
&= \frac{m}{2}\left[a^2(t_1-t_0)^2+2\mathbf{a}\cdot\mathbf{v}_0(t_1-t_0)\right] \\
&= ma^2\frac{(t_1-t_0)^2}{2}+m\mathbf{a}\cdot\mathbf{v}_0(t_1-t_0).
\end{aligned}$$

We recognize the work done on the mass by the force as equivalent to the gain in kinetic energy, $W = \Delta K$.

In summary, if a constant force is applied to a particle, the work done is equal to the change in kinetic energy.

With a changing force, we integrate force times displacement to find the total work. The total work done is also equal to the change in kinetic energy. This follows because we can sum the changes that take place during each time interval.

3.4.2 An example of the work done by friction on a sliding block

A block of mass m is on a horizontal surface with kinetic friction coefficient μ_k. It is initially sliding with velocity v_0.

It slides a distance d at which point it stops.

What is the distance d?

The initial kinetic energy is $\frac{1}{2}mv_0^2$. The change in kinetic energy is equal to $\Delta K = \frac{1}{2}mv_0^2$ since the final kinetic energy is zero.

The friction force is $F_{fr} = \mu_k F_N = \mu_k mg$ and is applied a distance d.

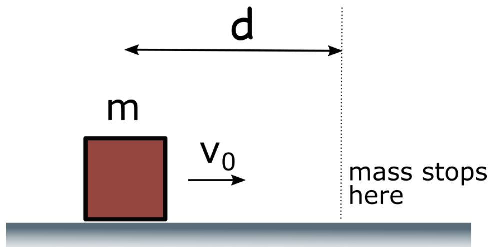

Figure 10: A block slides and comes to rest at a distance d from its initial position. The block initially has velocity v_0. The horizontal surface has a kinetic friction coefficient μ_k. The friction force on the block is $F_{fr} = \mu_k F_N = \mu_k mg$. Friction is responsible for decellerating the block.

The work done by the friction force is

$$W = \mu_k mgd$$

The work is equal to the change in kinetic energy

$$\Delta K = W$$
$$\frac{1}{2}mv_0^2 = \mu_k mgd$$

giving distance

$$d = \frac{v_0^2}{2g\mu_k}.$$

3.4.3 Energy for relativistic particles

We now generalize for the relativistic setting. To make this calculation simpler we work in 1 dimension only. As before the work done across a distance dx is

$$W = F\ dx.$$

This means that the change in energy

$$dE = F\ dx = \frac{dp}{dt}\ dx.$$

This means that

$$\frac{dE}{dx} = \frac{dp}{dt} \tag{16}$$

Recall the relativistic relation for momentum

$$\mathbf{p} = \gamma m \mathbf{v}$$

with

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

Let us compute the time derivative

$$\begin{aligned}
\frac{dp}{dt} &= \frac{d}{dt}\left(\frac{mv}{\sqrt{1 - v^2/c^2}}\right) && (17)\\
&= \frac{m\frac{dv}{dt}}{\sqrt{1 - v^2/c^2}} - \frac{(mv)}{(1 - v^2/c^2)^{\frac{3}{2}}}(-\frac{v}{c^2}\ \frac{dv}{dt}) \\
&= \frac{m\frac{dv}{dt}}{(1 - v^2/c^2)^{\frac{3}{2}}}\left(1 - \frac{v^2}{c^2} + \frac{v^2}{c^2}\right) \\
&= \frac{m}{(1 - v^2/c^2)^{\frac{3}{2}}}\frac{dv}{dt} && (18)
\end{aligned}$$

With

$$\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = \frac{dv}{dx}v$$

we can write equation 18 as

$$\frac{dp}{dt} = \frac{mv}{(1 - v^2/c^2)^{\frac{3}{2}}}\frac{dv}{dx} \tag{19}$$

We want a definition for E such that equation 16 is satisfied. It turns out that

$$E = \frac{mc^2}{\sqrt{1 - v^2/c^2}} = \gamma mc^2 \tag{20}$$

satisfies this condition. Let's compute dE/dx to check

$$\begin{aligned}
\frac{dE}{dx} &= \frac{mc^2}{(1 - v^2/c^2)^{\frac{3}{2}}}(-1/2)(-2)\frac{v}{c^2}\frac{dv}{dx} \\
&= \frac{mv}{(1 - v^2/c^2)^{\frac{3}{2}}}\frac{dv}{dx}
\end{aligned}$$

and this is equivalent to our expression for dp/dt above (equation 19). Because we only considered one direction of motion (in the x direction) we have shown that

$$\frac{dE}{dx} = \frac{dp_x}{dt} \tag{21}$$

is consistent with the definition for energy $E = \gamma mc^2$.

It is more work to show that the same relation is true if the momentum has components in the other directions so that

$$\frac{dE}{dx} = \frac{dp_x}{dt}$$
$$\frac{dE}{dy} = \frac{dp_y}{dt}$$
$$\frac{dE}{dz} = \frac{dp_z}{dt}$$

are consistent with the definition for energy $E = \gamma mc^2$. Using the gradient operator

$$\boldsymbol{\nabla} E = \left(\frac{\partial E}{\partial x}, \frac{\partial E}{\partial y}, \frac{\partial E}{\partial z} \right)$$

we can write these three expressions as

$$\boldsymbol{\nabla} E = \frac{d\mathbf{p}}{dt}. \tag{22}$$

3.5 Rest mass, rest energy and kinetic energy

Consider our definition for energy $E = \gamma mc^2$ that is consistent with

$$\frac{dE}{dx} = \frac{dp}{dt}$$

with momentum $p = \gamma mv$. With $v = 0$, $\gamma = 1$ and

$$E_0 = \lim_{v \to 0} \gamma mc^2 = mc^2 \tag{23}$$

This is known as the **rest energy**.

What is the kinetic energy, K?

We can update our definition of kinetic energy

$$K \equiv E - mc^2 = mc^2 \left(\frac{1}{\sqrt{1 - v^2/c^2}} - 1 \right). \tag{24}$$

It is useful to expand in a **Taylor series** about $x = 0$. Here is an example of a Taylor expansion of $\sqrt{1+x}$:

$$\begin{aligned} f(x) &= \sqrt{1+x} \\ f(0) &= 1 \\ f'(x) &= \frac{1}{2\sqrt{1+x}} \\ f'(0) &= \frac{1}{2} \\ f(x) &\sim f(0) + f'(0)x + ... = 1 + \frac{x}{2} + ... \\ \sqrt{1+x} &\approx 1 + \frac{x}{2} \end{aligned}$$

Likewise we can do a similar expansion for

$$\begin{aligned} f(x) &= \frac{1}{\sqrt{1-x}} \\ f(0) &= 1 \\ f'(x) &= \frac{1}{2(1-x)^{\frac{3}{2}}} \\ f'(0) &= \frac{1}{2} \\ f(x) &\sim f(0) + f'(0)x + ... = 1 + \frac{x}{2} + ... \\ \frac{1}{\sqrt{1-x}} &= 1 + \frac{x}{2} + ... \end{aligned}$$

We use this expansion to rewrite equation 24.

In the limit of small v

$$\begin{aligned} K \equiv E - mc^2 &= mc^2 \left(\frac{1}{\sqrt{1 - v^2/c^2}} - 1 \right) \\ &\approx mc^2 \left(1 + \frac{v^2}{2c^2} - 1 \right) \\ &= \frac{mv^2}{2} \end{aligned} \tag{25}$$

where I took the first term in a Taylor expansion. We recover the non-relativistic form for kinetic energy.

Notice that our definitions for energy and momentum both depend on velocity v. This means that both energy and momentum depend on the observer reference frame. The rest mass is a constant and is the same in any reference frame.

We can compute

$$\begin{aligned} E^2 - p^2c^2 &= \gamma^2 m^2 c^4 - \gamma^2 m^2 v^2 c^2 \\ &= m^2 c^4 \gamma^2 (1 - v^2/c^2) \\ &= (mc^2)^2. \end{aligned} \tag{26}$$

Because $E^2 - p^2c^2$ depends only on the rest energy, it is frame independent. Also if you know the rest mass and momentum p you can compute energy E and vice versa.

This relation follows from the Minkowski metric and considering $(E, \mathbf{p})$ as a four-vector. The length of the four vector is a *relativistic invariant*.

Because rest mass or rest energy is an invariant, in high energy astrophysics or particle physics, particle masses are often given in units of rest energy or $E_{rest} = mc^2$.

3.5.1 Work on relativistic particles

What happens if a force F is applied to a relativistic particle over a distance d? The work done is equal to the change in energy ΔE.

For example, if the particle initially has energy $E_{init} = \gamma_{init} mc^2$ and the final particle energy is $E_{final} = \gamma_{final} mc^2$ then the work

$$W = Fd = \Delta E = E_{final} - E_{init} = (\gamma_{final} - \gamma_{init}) mc^2.$$

4 Summary

- Work $W = \int \mathbf{F} \cdot d\mathbf{x}$.
- Power $P = \frac{dW}{dt} = \mathbf{F} \cdot \mathbf{v}$.
- Lorenz factor $\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$.
- Rest mass energy $E_0 = mc^2$.
- Energy, momentum for relativistic particles.
  $E = \gamma mc^2$, $\mathbf{p} = \gamma m \mathbf{v}$.
- Relativistic invariant: $E^2 - p^2c^2 = (mc^2)^2$.
- Kinetic energy $K = E - mc^2$ and $K = mv^2/2$ for $v \ll c$.
- Energy principle (valid for relativistic and non-relativistic particles).

  $$\frac{dE}{d\mathbf{x}} = \frac{d\mathbf{p}}{dt}$$

  This is equivalent to $\Delta E = W$ because $dE = \frac{d\mathbf{p}}{dt} \cdot d\mathbf{x} = \mathbf{F} \cdot d\mathbf{x} = dW$.