PHY141 Lectures 3,4,5 notes

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1 Matter and elementary particles

Elementary particles of the **standard model** are point like. They carry spin, mass, and charge (electron and color). See for example, The standard model.

Quarks are confined in hadrons. Hadrons are comprised of 2 or three quarks and are colorless.

Protons and neutrons are hadrons and they are the building blocks of atomic nuclei.

Nuclei and electrons are building blocks of atoms.

Atoms are building blocks of molecules, solids and liquids, plasmas, etc.

Baryons are hadrons, they contain 3 quarks and they are fermionic. That means their spins are half integers.

Mesons contain a quark and an anti-quark and they have integer spins and so are bosonic.

Quarks have electric charge of $\pm 2/3$ or $\pm 1/3$.

The electron and electron neutrino are leptons. Ditto for the muon and tau leptons. Leptons have spin 1/2. The neutrinos have zero charge. The muon, electron and tau all have charge of -1.

The quarks, electrons and neutrinos all have antiparticles with opposite charges.

The recent detected Higgs boson is also part of the standard model.

1.1 Fundamental Forces

There are four fundamental forces: gravity, the electromagnetic force, the weak force and the strong force.

On large scales we can ignore short range forces. Charges averages to zero on large scales. Gravity wins on large distances even though it is usually the weakest force.

Gravity is always attractive. Mass is always positive. Mass determines the strength of the gravitational interaction. Gravity might have a force carrier called the graviton, but these have not been detected.

Interactions involving leptons and neutrinos occur via the weak force. Often timescales are long (e.g., neutron decay). The weak force is carried by $\pm W, Z$ bosons.

The electromagnetic force can either be attractive or repulsive depending upon the signs of the charges. The strength of the interaction between two particles depends on the charges. Photons are also electromagnetic waves and are the force carrier for the electromagnetic force.

The strong force keeps atomic nuclei together. It is mediated by gluons. It is short range and relevant for atomic fusion and processes of nuclear burning that light up stars.

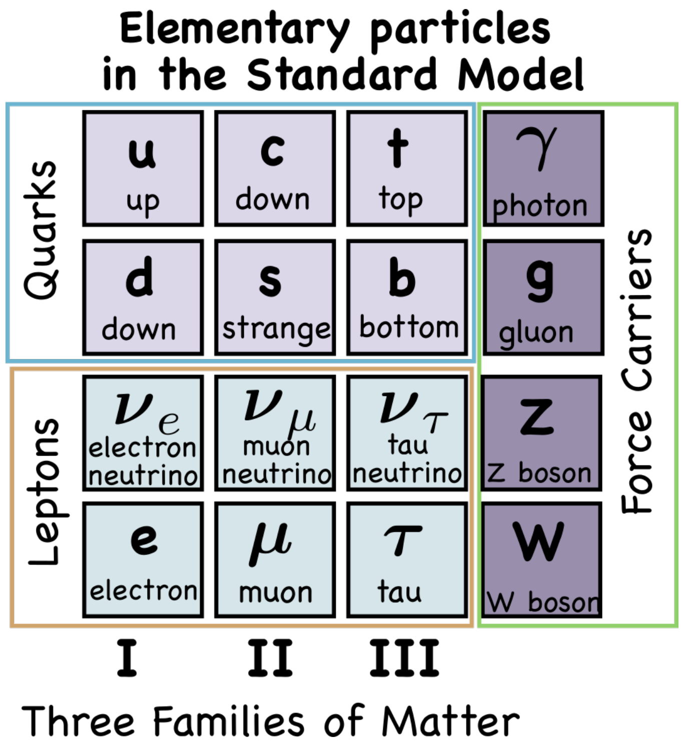

Figure 1: The particles of the standard model. The Higgs boson should also be included!

What is missing in the standard model? While the strong, weak and electromagnetic forces can be described in a single unifying theory, gravity is not part of this theory. There are numerous constants needed to specify the properties of the standard model. It might be nice to have a theory that would predict them. Dark matter and dark energy are not predicted by the standard model. So we lack a theory for most of the mass in the universe.

2 Motion

A particle without any forces acting on it has a constant velocity. The velocity is constant in direction and in magnitude. (We are assuming flat space time!).

If a particle has a changing velocity then it is accelerating (or decelerating).

A force acting on a particle will change its velocity. We can infer that a force is acting on a particle by the acceleration on the particle.

2.1 One dimension

In one dimension a point particle has position $x(t)$.

The particle velocity

$$v(t) = \frac{dx}{dt}.$$

Its acceleration

$$a(t) = \frac{dv}{dt} = \frac{d^2x}{dt^2}.$$

If the acceleration is constant we can integrate

$$\begin{aligned} a &= \frac{d^2x}{dt^2} = \frac{d}{dt}\left(\frac{dx}{dt}\right) \\ at &= \frac{dx}{dt} + \text{constant} \\ v &= at - \text{constant} \\ v(t) &= at + v_0 \end{aligned}$$

where v_0 is the velocity at $t = 0$. We can integrate again

$$\begin{aligned} \frac{dx}{dt} &= at + v_0 \\ x &= \frac{at^2}{2} + v_0 t + \text{constant} \\ x(t) &= \frac{at^2}{2} + v_0 t + x_0 \end{aligned}$$

where x_0 is the position at $t = 0$.

2.2 Motion in 3 dimensions

The particle position is now given by 3 coordinates

$$\mathbf{x}(t) = (x(t), y(t), z(t))$$

The position $\mathbf{x}$ also defines a vector from the origin to the position of the particle. The particle velocity is also a vector

$$\mathbf{v}(t) = \frac{d\mathbf{x}}{dt}$$

The velocity has both length and direction. The speed $v = |\mathbf{v}|$ is the magnitude of this vector. Acceleration is also a vector

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{x}}{dt^2}$$

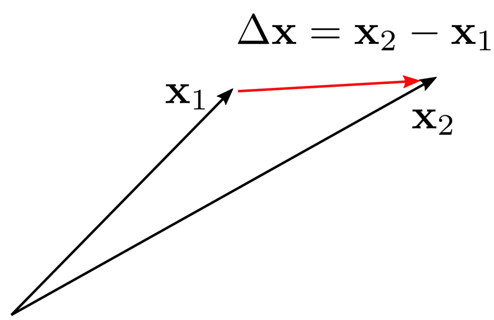

Figure 2: If a particle is at $\mathbf{x}_1$ at $t = 0$ and at $\mathbf{x}_2$ at $t = \Delta t$ then the velocity is $\mathbf{v} = \Delta\mathbf{x}/\Delta t$. The red arrow shows that this gives the direction of motion.

Consider a constant acceleration $\mathbf{a} = (a_x, a_y, a_z)$ We integrate and find velocity components

$$
\begin{aligned}
v_x(t) &= a_x t + v_{x0} \\
v_y(t) &= a_y t + v_{y0} \\
v_z(t) &= a_z t + v_{z0}
\end{aligned}
$$

where $\mathbf{v}_0 = (v_{x0}, v_{y0}, v_{z0})$ is the velocity at time $t = 0$. The position as a function of time

$$
\begin{aligned}
x(t) &= a_x \frac{t^2}{2} + v_{x0} t + x_0 \\
y(t) &= a_y \frac{t^2}{2} + v_{y0} t + y_0 \\
z(t) &= a_z \frac{t^2}{2} + v_{z0} t + z_0
\end{aligned}
$$

where $\mathbf{x}_0 = (x_0, y_0, z_0)$ is the position at $t = 0$. We could also write

$$
\begin{aligned}
\mathbf{v}(t) &= \mathbf{a}t + \mathbf{v}_0 \\
\mathbf{x}(t) &= \mathbf{a}\frac{t^2}{2} + \mathbf{v}_0 t + \mathbf{x}_0
\end{aligned}
$$

2.3 Projectile motion

Gravitational acceleration is to a good approximation downward and constant on the surface of the Earth; $\mathbf{a} = (0, 0, -g)$ or $\mathbf{a} = -g\hat{\mathbf{z}}$ where $\hat{\mathbf{z}}$ is a unit vector pointing upward. The equation of motion for a projectile is

$$
\begin{aligned}
\mathbf{v}(t) &= -gt\hat{\mathbf{z}} + \mathbf{v}_0 \\
\mathbf{x}(t) &= -g\frac{t^2}{2}\hat{\mathbf{z}} + \mathbf{v}_0 t + \mathbf{x}_0
\end{aligned}
$$

We can also write

$$
\begin{aligned}
x(t) &= v_{0x}t + x_0 \\
y(t) &= v_{0y}t + y_0 \\
z(t) &= -g\frac{t^2}{2} + v_{0z}t + z_0
\end{aligned}
$$

The x, y directions coast. There is only acceleration in the z direction.

Even though it looks simple for an individual particle, collective projectile motion is not always simple.

For example: splash craters and pulses into gravel

A movie of a splash crater

Another movie of a splash crater

A high speed video of a bucket of gravel hit from underneath with a hammer

A splash ejecta curtain is only affected by gravity, yet particles land at different times and different places. The curtain looks V-shaped. The pulse into gravel launches particles into the air. When they land the largest particles are left on top.

Let's manipulate our equation of motion

$$
\begin{aligned}
v^2(t) &= (-gt\hat{\mathbf{z}} + \mathbf{v}_0)^2 = g^2t^2 + v_0^2 - 2v_{0z}gt \\
gz(t) &= -g^2\frac{t^2}{2} + v_{0z}gt + gz_0 \\
\frac{v^2}{2} + gz &= g^2\frac{t^2}{2} - g^2\frac{t^2}{2} + \frac{v_0^2}{2} - v_{0z}gt + v_{0z}gt + gz_0 \\
&= \frac{v_0^2}{2} + gz_0 \\
&= \text{constant.}
\end{aligned}
$$

We recognize a sum of kinetic and potential energy (per unit mass) that remains constant. This is equivalent to conservation of energy.

2.4 Averaging

Consider a particle that is at $x = 0$ at $t = 0$ and is at $x = L$ at $t = T$. In the time interval, the particle's velocity is $v(t)$.

$$
\begin{aligned}
L &= \int_0^L \, dx \\
&= \int_0^T \frac{dx}{dt} \, dt \\
&= \int_0^T v(t) dt
\end{aligned}
$$

The average velocity in the interval

$$\bar{v} = \frac{1}{T}\int_0^T v(t)\ dt$$
$$= \frac{L}{T}$$

Consider dividing the time interval into N bins, each dt long. The number of bins is $N = T/dt$ so the average velocity

$$\bar{v} = \frac{1}{N}\sum v_i = \frac{1}{T}\sum v_i dt = \frac{1}{T}\int v\ dt$$

The average velocity $\bar{v} = L/T = \Delta x/\Delta t$ does not depend upon the particle trajectory path or how velocity depends on time, it only depends on the path end points (their positions and times).

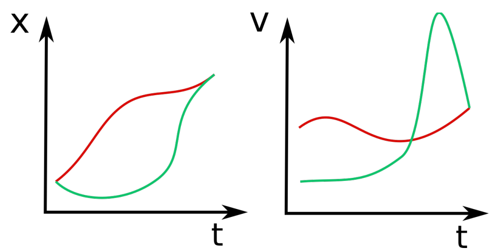

Figure 3: The average velocity across an interval only depends on the positions at the beginning and end of the path taken. On the left we show two paths that have the same starting and end points. In the right hand plot, the integrated area or $\Delta x = \int v\ dt$ under the two curves is the same.

A similar computation can be done for the average acceleration. Consider an interval starting at t_1 and ending at t_2.

$$a = \frac{dv}{dt}$$
$$\int_{t_1}^{t_2} adt = \int_{t_1}^{t_2} \frac{dv}{dt}\ dt = \int_{v_1}^{v_2} dv = v_2 - v_1 = \Delta v$$

The area under an acceleration curve is the change in velocity. If we divide both sides by the length of the interval, the average acceleration

$$\bar{a} = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} adt = \frac{\Delta v}{\Delta t}$$

where $\Delta t = t_2 - t_1$.

3 Vectors

Vectors are used to describe displacements or directions. Vectors have both direction and length. Things described with vectors are: velocity, displacement, position from origin, acceleration, force, tangent vector, and momentum.

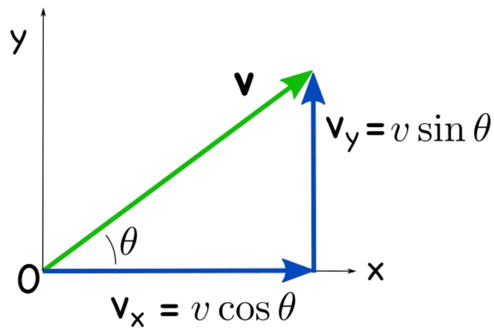

Figure 4: Vector components. The length $V = \sqrt{V_x^2 + V_y^2}$. Also $V_x = V\cos\theta$, $V_y = V\sin\theta$, $\tan\theta = V_y/V_x$.

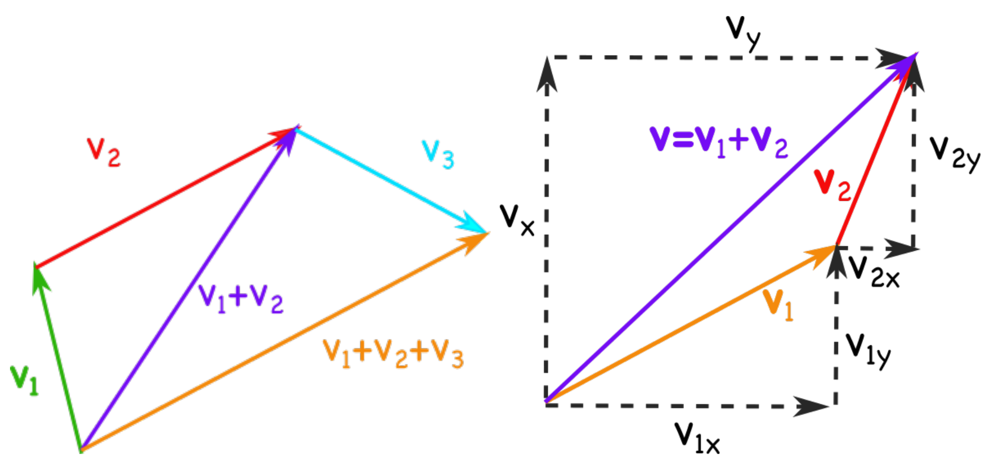

Figure 5: Vector addition.

3.1 Unit vectors

If we have a vector $\mathbf{A} = (A_x, A_y, A_z)$, then the vector

$$\hat{\mathbf{A}} = \frac{\mathbf{A}}{|A|} = \left(\frac{A_x}{A}, \frac{A_y}{A}, \frac{A_z}{A}\right)$$

is a vector of length 1 that is in the same direction as A. We call this vector a **unit** vector because it has length unity. We recall that $|A| = \sqrt{A_x^2 + A_y^2 + A_z^2}$. Let's check that the length $|\hat{\mathbf{A}}| = 1$.

$$|\hat{\mathbf{A}}| = \sqrt{\frac{A_x^2}{A^2} + \frac{A_y^2}{A^2} + \frac{A_z^2}{A^2}} = \sqrt{\frac{A^2}{A^2}} = 1. \tag{1}$$

3.2 Dot products

Dot products

$$\mathbf{a} \cdot \mathbf{b} = a_x b_x + a_y b_y + a_z b_z = ab\cos\theta$$

where θ is the angle between the vectors. It is useful to know that

$$(\mathbf{a} + \mathbf{b})^2 = a^2 + b^2 + 2\mathbf{a} \cdot \mathbf{b}.$$

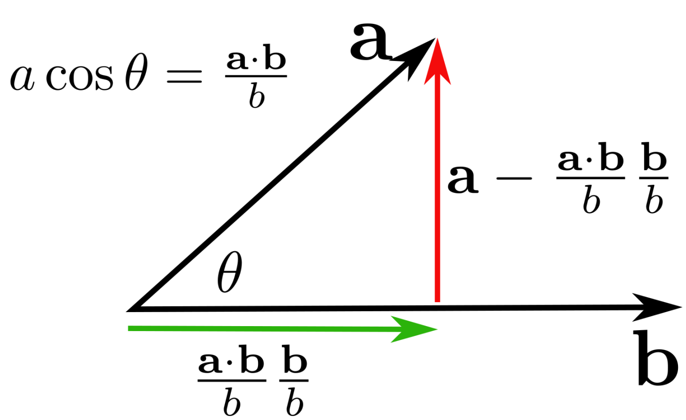

Figure 6: Dot product and components of one vector that are parallel and perpendicular to another vector. This picture will become useful when we consider force components. The green vector has length $a\cos\theta$ and is in the direction of $\mathbf{b}$. The red vector has length $a\sin\theta$ and is in a direction perpendicular to $\mathbf{b}$. The vector $\mathbf{a}$ is decomposed into components perpendicular to and parallel to $\mathbf{b}$.

4 Circular motion and polar coordinates

For problems involving rotation (like Keplerian motion) the solutions are sometimes much neater in polar coordinates. In 2d

$$\mathbf{r} = (x, y) = (r\cos\theta, r\sin\theta).$$

The inverse transformation

$$R = \sqrt{x^2 + y^2}$$
$$\theta = \mathrm{atan2}\,(y, x)\,.$$

Here atan2 is a function that depends on atan. It returns an angle in $[0, 2\pi]$ or $[-\pi, \pi]$ and in the correct quadrant. Specifically atan2(y, x) returns atan(y/x) plus a multiple of π depending upon the signs of x and y.

Polar coordinates can be extended to 3D with r, θ, z and this system known as cylindrical coordinates.

4.1 Unit vectors in polar coordinates

With a particle at x, y or R, θ, it is often useful to define two unit vectors, $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$.

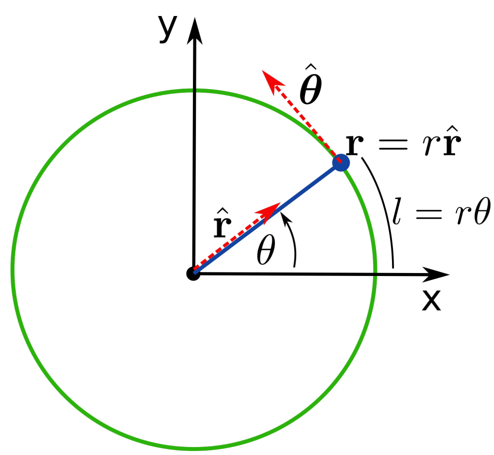

Figure 7: Illustrating polar coordinates for a point mass undergoing uniform circular motion.

$$\hat{\mathbf{r}} = \frac{\mathbf{r}}{r} = \frac{(x, y)}{\sqrt{x^2 + y^2}} = (\cos\theta, \sin\theta)$$
$$\hat{\boldsymbol{\theta}} = \frac{(-y, x)}{\sqrt{x^2 + y^2}} = (-\sin\theta, \cos\theta)$$

These two vectors are perpendicular as $\hat{\mathbf{r}} \cdot \hat{\boldsymbol{\theta}} = 0$. The sign for $\hat{\boldsymbol{\theta}}$ is chosen so that it is pointing in the direction of rotation if the angle θ is increasing.

The position

$$\mathbf{r} = r\hat{\mathbf{r}}$$

The velocity

$$\mathbf{v} = \dot{r}\hat{\mathbf{r}} + r\frac{d}{dt}\hat{\mathbf{r}} \tag{2}$$

where $\dot{r} = \frac{dr}{dt}$ and

$$\begin{aligned} \frac{d\hat{\mathbf{r}}}{dt} &= \frac{d}{dt}(\cos\theta, \sin\theta) \\ &= (-\sin\theta, \cos\theta)\dot{\theta} \\ &= \dot{\theta}\hat{\boldsymbol{\theta}} \end{aligned}$$

It will also be useful to compute

$$\begin{aligned} \frac{d\hat{\boldsymbol{\theta}}}{dt} &= \frac{d}{dt}(-\sin\theta, \cos\theta) \\ &= (-\cos\theta, -\sin\theta)\dot{\theta} \\ &= -\hat{\mathbf{r}}\dot{\theta}. \end{aligned}$$

We update equation 2

$$\begin{aligned} \mathbf{v} &= \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}} \\ &= v_r\hat{\mathbf{r}} + v_\theta\hat{\boldsymbol{\theta}}. \end{aligned}$$

With components in polar coordinates $v_r = \dot{r}$ and $v_\theta = r\dot{\theta}$. Here v_r is the radial velocity component and v_θ is the tangential velocity component. An object undergoing uniform circular motion has $v_r = 0$. Because $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ are perpendicular and unit vectors we can use them as a basis to give components of vectors, including velocity. However, $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ are not fixed as they move with the particle and we need to remember that when taking derivatives w.r.t. time.

Inspecting Figure 7, if θ is increasing the instantaneous tangential velocity is $r\dot{\theta}$.

The **angular rotation rate**

$$\Omega = \dot{\theta} = \frac{d\theta}{dt}$$

Equations such as $l = r\theta$ or $v_\theta = R\dot{\theta}$ are incorrect unless θ is in radians. The units of angular rotation or angular velocity are rad/s.

4.2 Uniform rotation and acceleration in polar coordinates

We take the time derivative of our equation for velocity

$$\begin{aligned}
\mathbf{v} &= \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}} \\
\mathbf{a} &= \frac{d\mathbf{v}}{dt} \\
&= \ddot{r}\hat{\mathbf{r}} + \dot{r}\frac{d\hat{\mathbf{r}}}{dt} + \dot{r}\dot{\theta}\hat{\boldsymbol{\theta}} + r\ddot{\theta}\hat{\boldsymbol{\theta}} + r\dot{\theta}\frac{d\hat{\boldsymbol{\theta}}}{dt} \\
&= \ddot{r}\hat{\mathbf{r}} + \dot{r}\dot{\theta}\hat{\boldsymbol{\theta}} + \dot{r}\dot{\theta}\hat{\boldsymbol{\theta}} + r\ddot{\theta}\hat{\boldsymbol{\theta}} - r\dot{\theta}^2\hat{\mathbf{r}} \\
&= (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\boldsymbol{\theta}}
\end{aligned}$$

The acceleration vector in terms of radial and tangential components

$$\mathbf{a} = a_r\hat{\mathbf{r}} + a_\theta\hat{\boldsymbol{\theta}}$$

$$a_r = \ddot{r} - r\dot{\theta}^2 \tag{3}$$

$$a_\theta = 2\dot{r}\dot{\theta} + r\ddot{\theta}. \tag{4}$$

The polar coordinate frame is rotating so in equation 4 there is a Coriolis term.

The angular acceleration

$$\ddot{\theta} = \frac{d^2\theta}{dt^2} = \frac{d\Omega}{dt}$$

The angular acceleration is the rate of change of the angular rotation rate.

In uniform circular motion $\dot{r} = 0$, $\ddot{\theta} = 0$ and

$$\mathbf{a} = -r\dot{\theta}^2\hat{\mathbf{r}}. \tag{5}$$

The acceleration is radially inward and this is sometimes called **centripetal** acceleration. As the tangential velocity component $v_\theta = r\dot{\theta}$ we can also write the acceleration as

$$\mathbf{a} = -\frac{v^2}{r}\hat{\mathbf{r}}.$$

Let's check our relation for acceleration using Cartesian coordinates. Uniform circular motion with constant radius r and angular rotation rate Ω

$$(x, y) = (R\cos(\Omega t), R\sin(\Omega t))$$

where initial conditions are $(x_0, y_0) = (R, 0)$ at $t = 0$. The velocity

$$(v_x, v_y) = (-R\Omega\sin(\Omega t), R\Omega\cos(\Omega t))$$

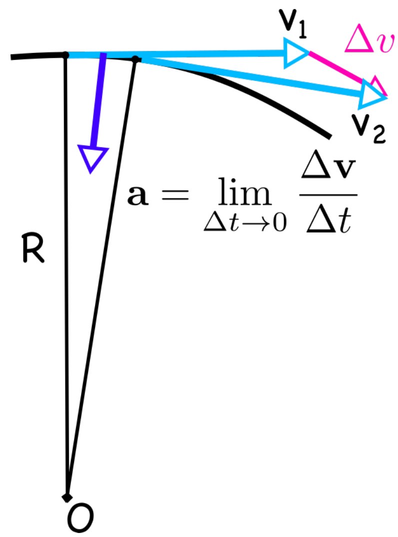

Figure 8: For an object undergoing uniform circular motion around the origin, the acceleration vector points toward the origin.

The acceleration
$$(a_x, a_y) = (-R\Omega^2 \cos(\Omega t), -R\Omega^2 \sin(\Omega t))$$
The magnitude of the acceleration is
$$a = \sqrt{a_x^2 + a_y^2} = R\Omega^2(\cos^2(\Omega t) + \sin^2(\Omega t)) = R\Omega^2$$
To find the radial component of acceleration
$$\begin{aligned} a_r &= \mathbf{a} \cdot \hat{\mathbf{r}} \\ &= (-R\Omega^2 \cos(\Omega t), -R\Omega^2 \sin(\Omega t)) \cdot (\cos(\Omega t), \sin(\Omega t)) \\ &= -R\Omega^2(\cos^2(\Omega t) + \sin^2(\Omega t)) \\ &= -R\Omega^2 \end{aligned}$$
As the absolute value of this is equivalent to the magnitude of the acceleration (or the length of the acceleration vector), we know that the tangential component of the acceleration is zero. This is equivalent to the expression we derived in polar coordinates (equation 5).

4.3 Rotation period, angular frequency and frequency

Consider
$$x(t) = R\cos(\Omega t) \qquad y(t) = R\sin(\Omega t)$$

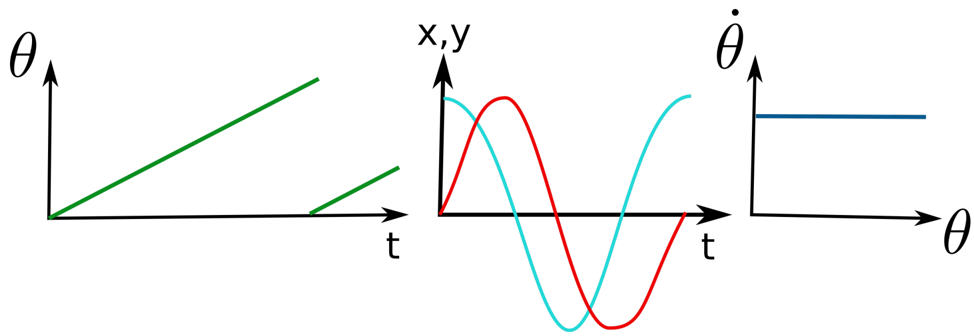

Figure 9: For uniform circular motion, we plot $\theta(t), x(t), y(t)$ vs t and $\dot\theta$ vs θ. In the middle plot x is a blue line and y is a cyan line.

with $\Omega = \dot\theta$ is the angular rotation rate or the angular frequency.

What is the rotation period P?

The angle Ωt must advance from 0 to 2π during a full rotation period. We set $\Omega P = 2\pi$. We solve for rotation period

$$P = \frac{2\pi}{\Omega}.$$

Angular frequency is not the same as frequency. Often people write $x = R\cos(2\pi f t)$ where f is a frequency that is cycles per second. With frequency f equal to cycles per second $P = 1/f$. It may be useful to remember that

$$\Omega = 2\pi f.$$

5 Momentum changes and force

A force can be applied for a small time interval. In this case it is an impulse and it causes a change in the particle's momentum and velocity. Alternatively, a change in momentum $\Delta\mathbf{p}$ over a small duration of time Δt can be used to estimate the force $\mathbf{F}$ on the particle.

$$\Delta\mathbf{p} = \mathbf{F}\Delta t$$

or equivalently

$$\mathbf{F} = \frac{\Delta\mathbf{p}}{\Delta t}$$

This is sometimes called the **momentum principle**. In other words

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} \tag{6}$$

Force is a rate of change of momentum.

The non-relativistic limit is

$$v \ll c$$

or the speed v is much lower than the speed of light c. In the non-relativistic limit, momentum is proportional to velocity

$$\mathbf{p} = m\mathbf{v}$$

$$\frac{d\mathbf{p}}{dt} = m\frac{d^2\mathbf{x}}{dt^2} = m\mathbf{a}.$$

Equation 6 reduces to the well known

$$\mathbf{F} = m\mathbf{a}.$$

Force is mass times acceleration is one of Newton's laws.

Force is given in N or Newtons. In the MKS system, N = kg m/s^2.

At high speeds $\mathbf{p} \neq m\mathbf{v}$. Nothing can go faster than the speed of light, c. However the closer the speed gets to the speed of light, the more momentum it has. A relativistic generalization of momentum is

$$\mathbf{p} \equiv \gamma m\mathbf{v} \tag{7}$$

with

$$\gamma \equiv \frac{1}{\sqrt{1-\beta^2}}$$

$$\beta \equiv \frac{v}{c}$$

As $v \to c$ or $\beta \to 1$, and we find $\gamma \to \infty$.

As $v \to 0$, $\beta \to 0$ and $\gamma \to 1$.

The factor γ is known as the **Lorenz factor**. The momentum can get infinitely large as speed gets closer to the speed of light. Equation 7 has a factor of mass in it, which is now called the **rest mass**. By modifying our definition for momentum, the relation between force and momentum change $\mathbf{F} = \frac{d\mathbf{p}}{dt}$ is valid even at relativistic velocities.

5.1 Principles of relativity and momentum

- In flat space time and in the absence of external forces, and as viewed from a non-accelerating reference frame, particles move at constant velocity and in a straight line. This is Newton's first law.
- The laws of physics should be independent of the observer velocity. Physical laws work in the same way for an observer in uniform motion as for an observer at rest. This is the principle of relativity.
- Forces or interactions cause changes in momentum. (The momentum principle). The momentum principal is consistent with Newton's law $F = ma$ in the non-relativistic limit.

5.2 Predicting positions and velocities

We have a point particle of rest mass m, initial velocity $\mathbf{v}_0$ and initial position $\mathbf{x}_0$. An external force is applied on the particle, $\mathbf{F}$. The force could be time, position and possibly even velocity dependent.

What is the initial momentum? If the velocity is small compared to the speed of light then the initial momentum is $\mathbf{p}_0 = m\mathbf{v}_0$.

If the velocity is large then equation 7 gives the initial momentum

$$\mathbf{p}_0 = \gamma_0 m \mathbf{v}_0.$$

After a time Δt what are the particle position, velocity and momentum?

$$\begin{aligned} \Delta\mathbf{p} &= \mathbf{F}\Delta t \\ \mathbf{p}_{new} - \mathbf{p}_0 &= \mathbf{F}\Delta t \\ \mathbf{p}_{new} &= \mathbf{p}_0 + \mathbf{F}\Delta t \end{aligned}$$

This gives us a formula for the new momentum in terms of the old one (or initial one), the force and the time step Δt.

What is the new velocity?

We want to find the new velocity v_{new} from its momentum. Using equation 7 ($\mathbf{p} = \gamma m\mathbf{v}$), we solve for v in terms of p.

$$\begin{aligned} p^2 = \gamma^2 m^2 v^2 &= \frac{m^2 v^2}{1 - v^2/c^2} \\ p^2(1 - v^2/c^2) &= m^2 v^2 \\ p^2 &= (p^2/c^2 + m^2)v^2 \\ v^2 &= \frac{p^2}{p^2/c^2 + m^2} \end{aligned}$$

Restoring the direction

$$\mathbf{v} = \frac{\mathbf{p}/m}{\sqrt{1 + \frac{p^2}{m^2c^2}}}.$$

We plug in the new momentum to find the new velocity

$$\mathbf{v}_{new} = \frac{\mathbf{p}_{new}/m}{\sqrt{1 + \frac{p_{new}^2}{m^2c^2}}}$$

What is the new position?

$$\frac{d\mathbf{x}}{dt} = \mathbf{v}$$
$$\frac{\mathbf{x}_{new} - \mathbf{x}_0}{\Delta t} = \mathbf{v}_{new}$$
$$\mathbf{x}_{new} = \mathbf{x}_0 + \mathbf{v}_{new}\Delta t.$$

We can repeat these steps, starting with a new initial condition (using the new momentum and position instead of the initial momentum and position). This is a first order or Eulerian numerical integration technique. It's not all that accurate. However, if you use a small time step and don't integrate for very long, it can give decent numerical results.

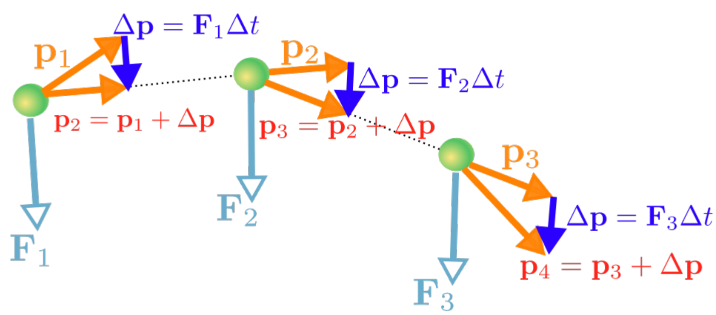

Figure 10: An integration of position and momentum for a particle that is affected by an external force.

6 The gravitational force

We have two point masses, m_1 at position $\mathbf{x}_1$ and m_2 at position $\mathbf{x}_2$. The vector between the two is

$$\mathbf{r}_{21} = \mathbf{x}_2 - \mathbf{x}_1.$$

This vector goes from $\mathbf{x}_1$ to $\mathbf{x}_2$. The length of this vector is $r_{21} = |\mathbf{r}_{21}|$. We can again define a unit vector

$$\hat{\mathbf{r}}_{21} = \frac{\mathbf{x}_2 - \mathbf{x}_1}{r_{21}}.$$

The gravitational force from m_2 onto m_1 is

$$\mathbf{F}_{21} = \frac{Gm_1m_2}{r_{21}^2}\hat{\mathbf{r}}_{21} = \frac{Gm_1m_2}{r_{21}^3}\mathbf{r}_{21}.$$

This points in the direction of m_2 and so the force is an attractive force.

The gravitational force from m_1 onto m_2

$$\mathbf{F}_{12} = \frac{Gm_1m_2}{r_{21}^2}\hat{\mathbf{r}}_{12} = -\mathbf{F}_{21}$$

This is equal and opposite to the force from m_2 onto m_1.

This is Newton's third law and the rule is sometimes called *reciprocity*.

Both forces are along the direction connecting the two point masses.

The gravitational constant

$$G = 6.7 \times 10^{-11}\ \frac{\text{N m}^2}{\text{kg}^2} \qquad \text{or} \qquad \frac{\text{kg}^{-1}\ \text{m}^3}{\text{s}^2}$$

where N is a Newton.

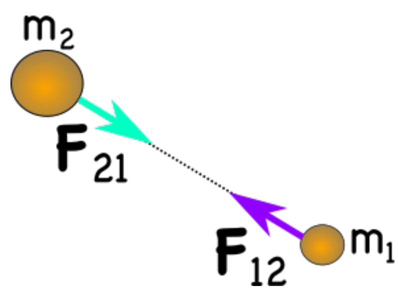

Figure 11: The force exerted by m_1 onto m_2 is equal and opposite to that exerted by m_2 onto m_1.

If there are more than one mass near m_1 then the forces from each one just add on to the force exerted on to m_1.

In an N-body system the force on M with position $\mathbf{X}$ due to a bunch of masses m_i, each at position $\mathbf{x}_i$ is

$$\mathbf{F}_M = \sum_i \frac{GMm_i}{|\mathbf{x}_i - \mathbf{X}|^3}(\mathbf{x}_i - \mathbf{X})$$

When computing this sum, you need to keep track of the vector directions of each particle.

6.1 The shell theorem

Consider a uniform density spherical shell with total mass M_{shell}, radius R shell thickness h (see Figures 13 and 14). The position of its center is $\mathbf{x}$. We now consider another point mass M at position $\mathbf{y}$. If $\mathbf{y}$ is outside the shell, the gravitational force from the massive

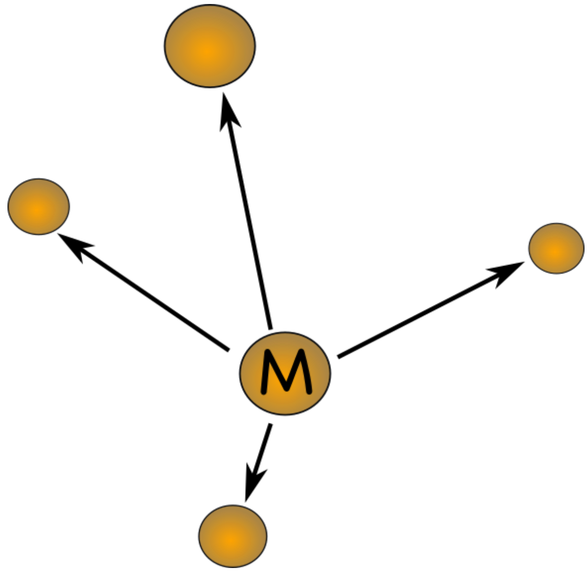

Figure 12: In an N-body simulation, a particle with mass M feels the gravitational attractive forces from all other massive particles.

shell on the point mass m is the same as that from a point mass of mass M_{shell} at position $\mathbf{x}$. If $\mathbf{y}$ is inside the shell the gravitational force from the shell on m is zero.

We consider a uniform density sphere with total mass M_{sphere} and central position $\mathbf{x}$. A uniform density sphere is a sum of shells. Outside the sphere, the gravitational force it is equal to that of that of a point mass with mass M_{sphere} and position $\mathbf{x}$.

How is the shell theorem proved? It takes a only few lines to show that it is true using Gaus' law which is taught as part of multivariable calculus.

Suppose we have a uniform density circular ring of mass. Sometimes the force from a planet is estimated by assuming its mass is spread evenly about its orbit into a ring (this is known as a secular approximation). Consider another particle in the same plane as the massive ring and inside the ring.

Is the gravitational force from the ring equal to zero at points within the ring? Recall that the gravitational force from a uniform density shell is zero for points inside the shell.

The answer is no. The shell would pull the particle radially outward. There is no handy theorem that gives a simple formula for the force. The force can be written in terms of a special function known as a Laplace integral.

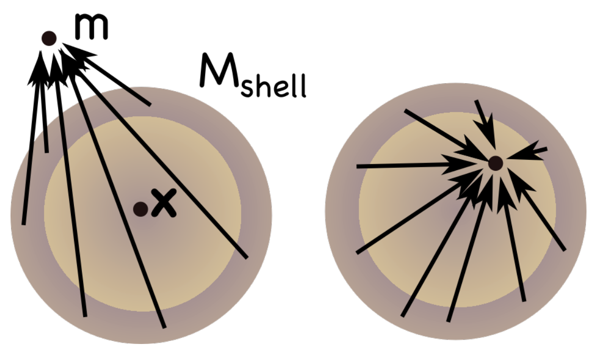

Figure 13: A uniform density shell. On the left the force from the mass on the shell on a smaller mass outside the shell is the integral of the forces from each parcel of mass in the shell. This integral is equal to the force exerted by a point mass of the same mass as the shell that is located in the center of the shell. On the right the forces from each mass parcel cancel out. The shell does not exert any gravitational force on points inside the shell.

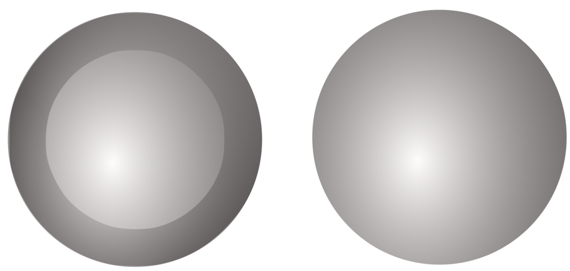

Figure 14: Outside a uniform density spherical shell and outside a uniform density sphere, both objects exert the same gravitational force as a point mass with the same mass centered at their centers. At points inside the shell, the gravitational force from the shell is zero.

6.2 Relating the gravitational constants: G and g

We consider a mass m on the surface of the Earth and take M and R to be the mass and radius of the Earth. The gravitational force on m is

$$\mathbf{F} = -\frac{GMm}{R^2}\hat{\mathbf{r}}$$

where $\hat{\mathbf{r}}$ is the vector pointing away from the center of the Earth. Over small distances we ignore the curvature of the surface and associate $\hat{\mathbf{r}} = \hat{\mathbf{z}}$. Here we have oriented our Cartesian coordinate system so that $+z$ is upward. Force per unit mass is an acceleration

$$\frac{\mathbf{F}}{m} = -\frac{GM}{R^2}\hat{\mathbf{z}}$$

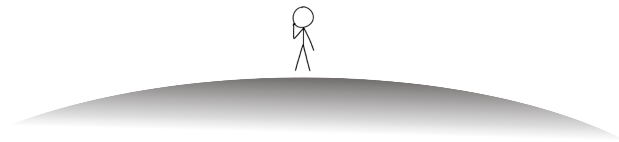

Figure 15: On large scales the gravitational force from the Earth points radially towards the center of the Earth. To a pretty good approximation and on small scales we can work in a Cartesian coordinate system with a constant gravitational acceleration pointing downward.

Locally the distance to the center of the Earth does not significant change giving an acceleration

$$g = \frac{GM}{R^2}$$

Using the radius and mass of the Earth, and the gravitational constant G we can compute g. On Earth the gravitational acceleration $g = 9.8$ m/s^2 or N/kg. Alternatively, if we know the gravitational constant G, and radius of the Earth and have measured g we would then know the mass of the Earth.

The downward force on m can also be written

$$\mathbf{F} = -mg\hat{\mathbf{z}}.$$

6.3 Newton's Third law

The force from mass m_1 exerted on m_2 is equal and opposite to that from m_2 exerted on m_1. That forces are equal and oppositely applied is known as Newton's third law.

Let's look at the momenta of the two particles $\mathbf{p}_1$ and $\mathbf{p}_2$. Because forces are equal and opposite

$$\mathbf{F}_{12} = -\mathbf{F}_{21}.$$

Recall that $d\mathbf{p}/dt = \mathbf{F}$.

$$\begin{aligned} \frac{d\mathbf{p}_1}{dt} &= -\frac{d\mathbf{p}_2}{dt} \\ \frac{d}{dt}(\mathbf{p}_1 + \mathbf{p}_2) &= 0 \\ \mathbf{p}_1 + \mathbf{p}_2 &= \text{constant}. \end{aligned}$$

This means that the sum of momenta is **conserved**.

We have been talking about gravitational forces, but Newton's third law and conservation of momenta applies for all types of pair-wise forces on point masses.

What happens for relativistic particles. Our gravitational force is instantaneously applied. So how does this information travel across space instantaneously from one particle to the other one? This would be faster than the speed of light! A better theory of gravity would propagate the force with gravitons. A relativistic and quantum mechanical theory for the electromagnetic force exists (and it is known as Quantum electrodynamics). However, we lack a tested theory for gravity that is both consistent with General Relativity and Quantum mechanics.

6.4 Circular orbits

We consider an object of mass m in orbit about a mass M with distance between them R. With $m \ll M$ we can take M fixed at the origin and m in a circular orbit about M. The gravitational attractive force balances mass times centripetal acceleration

$$-\frac{GmM}{R^2}\hat{\mathbf{r}} = -m\frac{v^2}{R}\hat{\mathbf{r}}.$$

We solve for the tangential velocity

$$v = \sqrt{\frac{GM}{R}}$$

Using $v = R\Omega$ we find the angular rotation

$$\Omega = \sqrt{\frac{GM}{R^3}}.$$

From the angular rotation rate we find the rotation period

$$P = \frac{2\pi}{\Omega} = 2\pi\sqrt{\frac{R^3}{GM}}.$$

If we square this and rearrange it

$$P^2 = \frac{R^3 4\pi^2}{GM}.$$

This is essentially Kepler's third law but with radius instead of the more general semi-major axis a. If the orbit is not a perfect circle but an ellipse the relation is also true but with R replaced by a.

6.5 Kepler's laws

1. The orbit of a planet is an ellipse with the Sun at one of the two foci.
2. A line segment joining a planet and the Sun sweeps out equal areas during equal intervals of time.
3. The square of the orbital period of a planet is directly proportional to the cube of the semi-major axis of its orbit.

Kepler's second law follows from conservation of angular momentum.

The Earth goes around the Sun in 1 year. The distance between Earth and Sun is 1 AU (astronomical unit).

It is convenient to write

$$\left(\frac{P}{1 \text{ yr}}\right) = \left(\frac{a}{1 \text{ AU}}\right)^{\frac{3}{2}} \left(\frac{1\ M_\odot}{M_*}\right)^{\frac{1}{2}} \tag{8}$$

with $M_\odot$ the mass of the Sun and a the orbit semi-major axis.

What is the orbital period for a planet at 1 AU that is orbiting a star that is $1/2\ M_\odot$?

Equation 8 lets us swiftly answer this question: the answer is $\sqrt{2} \approx 1.4$ years.

What is the orbital period for a planet at 100 AU that is orbiting a solar mass star? $100^{\frac{3}{2}} = 10^3 = 1000$. The answer is 1000 years.

7 Summary

- Motion in 3 dimensions under constant acceleration.

$$\mathbf{v}(t) = \mathbf{a}t + \mathbf{v}_0$$
$$\mathbf{x}(t) = \mathbf{a}\frac{t^2}{2} + \mathbf{v}_0 t + \mathbf{x}_0$$

- Principles of relativity and momentum.

  1) The natural force free state, is to keep moving at a constant velocity.

  2) Physics laws should be independent of the velocity of an observer.

  3) The momentum principle.

- How momentum depends on rest mass and velocity even at relativistic velocities. $\mathbf{p} = \gamma m \mathbf{v}$ with $\gamma = (1 - v^2/c^2)^{-\frac{1}{2}}$.

- How applied forces are related to changes in momentum, velocity and position. $\mathbf{F} = \frac{d\mathbf{p}}{dt}$.

- Velocity and acceleration in polar coordinates.

- Uniform circular motion. The angular rotation rate, $\dot{\theta}$ and rotation period. Tangential velocity $v_\theta = r\dot{\theta}$. Acceleration $\mathbf{a} = -\frac{v_\theta^2}{r}\hat{\mathbf{r}}$. Rotation period $P = \frac{2\pi}{\dot{\theta}}$.

- The gravitational force law.

$$\mathbf{F} = \frac{G m_1 m_2}{r^2}\hat{\mathbf{r}}$$

- Reciprocity. Forces are applied equally and oppositely. This implies momentum conservation.
- The shell theorem: The gravitational force from a uniform density sphere at positions outside it is the same as that from a point mass of the same total mass.

8 Props

Doppler ball.