PHY141 Lectures 19 notes

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Contents

1 Force Balance at Equilibrium (Statics)

If a body is not moving then its center of mass is fixed. This means that the sum of forces on it is zero.

$$\sum_i \mathbf{F}_i = 0$$

If a body is not rotating then its angular momentum is fixed. This means that the sum of torques on it is zero.

$$\sum_i \boldsymbol{\tau}_i = 0$$

Both of these equations are vector equations.

The torques can be computed about any origin!

All forces or torques on the body must be included in the sums!

1.1 Torque on a rigid body due to gravity computed at the center of mass position

For an extended body, why is it that we compute the torque using the position vector, $\mathbf{X}_{cm}$ of the center of mass?

We consider the force element on a small mass $dm = \rho dV$ inside the body, where ρ is the density and dV is the volume element. The force on the mass element is

$$\mathbf{F} = dm\ \mathbf{g} = \rho dV\ \mathbf{g}$$

where $\mathbf{g}$ is a vector with length equal to the gravitational acceleration and direction pointing downward. We recall that the center of mass position

$$\mathbf{X}_{cm} = \frac{1}{M} \int dV\ \rho\ \mathbf{r}$$

where the total mass $M = \int dV\ \rho$.

To find the total torque we integrate over the body

$$\begin{aligned} \boldsymbol{\tau} &= \int \mathbf{r} \times \mathbf{F} \\ &= \int dV\ \rho\ \mathbf{r} \times \mathbf{g} \\ &= \left(\int dV\ \rho\ \mathbf{r} \right) \times \mathbf{g} \\ \boldsymbol{\tau} &= M\mathbf{X}_{cm} \times \mathbf{g}. \end{aligned}$$

The torque only depends on the center of mass position.

1.2 A horizontal beam with a block sitting on it

We consider a horizontal beam of length L and mass m, as show in Figure **??**. Each end of the beam rests on a scale. A block of mass M rests on the beam 1/4 of the way from the beam's left end.

What do the scales read?

We take F_l, F_r to be the forces on the left and right scales.

The convention on a Figure like Figure 1 is to draw vectors with arrows pointing in their expected direction. However the signs of the vectors are given with respect to a fixed coordinate system. The coordinate system we will adopt is shown on the top left side of Figure 1. I use a coordinate system with horizontal axis x and $x = 0$ at the left end of the beam. The vertical y axis is positive upward. Even though the force Mg vector is pointing down, we would take its sign to be negative with $\mathbf{F} = -Mg\hat{\mathbf{y}}$. I am using a xy coordinate system on Figure 1 so as to be careful about the signs and directions of torques.

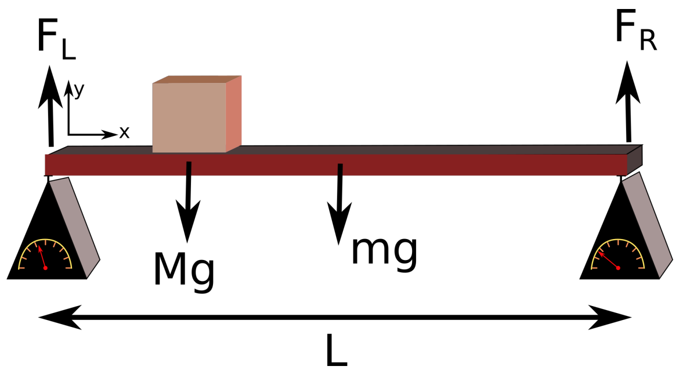

Figure 1: A beam of mass m and length L. Both ends of the beam are resting on scales. A block of mass M rests on the beam. The block is a distance $L/4$ from the beam's left end. What are the weights shown on each scale?

This problem only has vertical forces. Vertical force balance (in the $\hat{\mathbf{y}}$ direction) gives

$$F_l + F_r - mg - Mg = 0. \tag{1}$$

The torque computed with origin at the left end is

$$\begin{aligned}\boldsymbol{\tau}_l &= \frac{L}{4}\hat{\mathbf{x}} \times (-Mg)\hat{\mathbf{y}} + \frac{L}{2}\hat{\mathbf{x}} \times (-mg)\hat{\mathbf{y}} + L\hat{\mathbf{x}} \times F_r\hat{\mathbf{y}} \\ &= (-\frac{MgL}{4} - \frac{mgL}{2} + F_r L)(\hat{\mathbf{x}} \times \hat{\mathbf{y}}) \\ &= (-\frac{MgL}{4} - \frac{mgL}{2} + F_r L)\ \hat{\mathbf{z}}.\end{aligned}$$

Torque balance gives

$$\tau_l = -Mg\frac{L}{4} - mg\frac{L}{2} + F_r L = 0. \tag{2}$$

All position vectors are in the positive x direction. The first two terms are negative because the forces are downwards. Recall that $\hat{\mathbf{x}} \times \hat{\mathbf{y}} = \hat{\mathbf{z}}$.

We use τ_l (equation 2) to solve for the force from the right scale F_r

$$F_r = \frac{Mg}{4} + \frac{mg}{2}$$

The torque computed with orgin at the right end is

$$\begin{aligned}\boldsymbol{\tau}_r &= -\frac{3L}{4}\hat{\mathbf{x}} \times (-Mg)\hat{\mathbf{y}} + -\frac{L}{2}\hat{\mathbf{x}} \times (-mg)\hat{\mathbf{y}} - L\hat{\mathbf{x}} \times F_l\hat{\mathbf{y}} \\ &= \left(Mg\frac{3L}{4} + mg\frac{L}{2} - F_l L\right)\hat{\mathbf{x}} \times \hat{\mathbf{y}}.\end{aligned}$$

Torque balance gives

$$\tau_r = Mg\frac{3L}{4} + mg\frac{L}{2} - F_l L = 0$$

We use τ_r to solve for F_l

$$F_l = \frac{3Mg}{4} + \frac{mg}{2}$$

We check to find that the sum of $F_r + F_l = mg + Mg$ as expected! (This is equation 1 for vertical force balance).

We could have used the condition of vertical force balance instead of $\tau_r = 0$ to solve for F_l once we had already found F_r.

1.3 Ladder against a wall with a person on the ladder

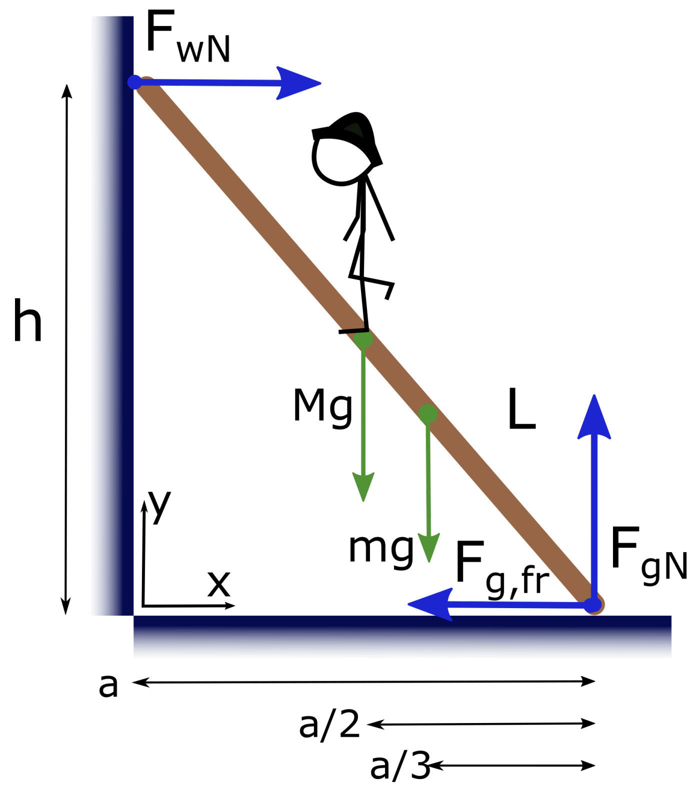

Figure 2: What is the force exerted on the ladder by the wall and by the ground?

A ladder with length L and mass m rests against a wall. Its upper end is a distance h above the ground. The center of gravity of the ladder is one-third of the way up the ladder. A person with mass M climbs halfway up the ladder. Assume that the wall, but not the ground, is frictionless.

What is the force exerted on the ladder by the wall and by the ground?

The wall exerts a normal force which is horizontal, F_{wN}. The wall does not exert a friction force, so there is no vertical component to this force.

The ground exerts a normal force that is vertical and is F_{gN}. The ground also exerts a friction force that is horizontal, $F_{g,fr}$.

We use a coordinate system with x increasing to the right and y increasing upward. The directions of the arrows show the directions we think the forces are in, but their vector components might be negative.

The total ground force is

$$\mathbf{F}_g = F_{gN}\hat{\mathbf{y}} + F_{g,fr}\hat{\mathbf{x}}. \tag{3}$$

The horizontal distance between wall and end of ladder is

$$a = \sqrt{L^2 - h^2}.$$

Vertical force balance for the sum of y-components of all forces on the ladder

$$F_y = -mg - Mg + F_{gN} = 0.$$

This gives us the component of the ground force that is normal

$$F_{gN} = (m + M)g.$$

Horizontal force balance for the sum of x-components of all forces on the ladder

$$F_x = F_{wN} + F_{g,fr} = 0. \tag{4}$$

Torque balance from the end of the ladder that is on the ground (the sum of torques on the ladder)

$$\begin{aligned}\boldsymbol{\tau} &= \left(-\frac{a}{2}\hat{\mathbf{x}}\right) \times (-Mg\hat{\mathbf{y}}) + \left(-\frac{a}{3}\hat{\mathbf{x}}\right) \times (-mg\hat{\mathbf{y}}) + (h\hat{\mathbf{y}}) \times (F_{wN}\hat{\mathbf{x}}) \\ &= \left(\frac{aMg}{2} + \frac{amg}{3} - hF_{wN}\right)\hat{\mathbf{z}} = 0.\end{aligned}$$

This lets us solve for the wall force

$$\mathbf{F}_{wN} = \left(\frac{M}{2} + \frac{m}{3}\right)\frac{ga}{h}\hat{\mathbf{x}}. \tag{5}$$

We can use our condition for horizontal force balance (equation 4 and equation 5) to find the ground friction force,

$$F_{g,fr} = -\left(\frac{M}{2} + \frac{m}{3}\right)\frac{ga}{h}.$$

Taking both components the ground force is

$$\begin{aligned}\mathbf{F}_g &= F_{g,fr}\hat{\mathbf{x}} + F_{gN}\hat{\mathbf{y}} \\ &= -\left(\frac{M}{2} + \frac{m}{3}\right)\frac{ga}{h}\hat{\mathbf{x}} + (m+M)g\hat{\mathbf{y}}.\end{aligned}$$

The friction force has a maximum magnitude of μF_{gN} where μ is the coefficient of static friction. If a is too large (or h too small), and the ladder tilted over too far, the ladder will slip.

The condition to keep the ladder from slipping depends on the ratio of friction to normal force

$$\left|\frac{F_{g,fr}}{F_{gN}}\right| = \frac{\left(\frac{M}{2} + \frac{m}{3}\right)\frac{ga}{h}}{(m+M)g} = \frac{\left(\frac{M}{2} + \frac{m}{3}\right)}{(m+M)}\frac{\sqrt{L^2 - h^2}}{h} < \mu. \tag{6}$$

We notice that if h is smaller, then a/h is larger and the friction force is larger. This is consistent with our intuition. If the ladder is nearly horizontal then it is likely to slip and vice versa, a nearly vertical ladder is more likely to be stable.

1.4 When does the ladder start to slip?

We consider a ladder with center of mass at its mid-point and a person walking up the ladder. The normal force from the ground must balance the gravitational force of ladder and person. However the torque from the person's gravitational force sets the size of the friction force. As the person walks up the ladder the size of the torque increases and so must the friction force. However, the normal force from the ground does not increase. At some point stability requires a friction force that is larger than μF_{gN} and the ladder starts to slip.

Let's revise our previous calculation. This time the center of mass of the ladder is in its center. The fraction of the way up the ladder that the person has climbed is α. If $\alpha = 0$ he or she is at the bottom and if $\alpha = 1$ he or she is at the top. Equation 6 becomes

$$\left|\frac{F_{g,fr}}{F_{gN}}\right| = \frac{\left(\alpha M + \frac{m}{2}\right)\frac{ga}{h}}{(m+M)g} < \mu. \tag{7}$$

As the person gets higher on the ladder, α increases. To keep from slipping, the required friction force is larger.

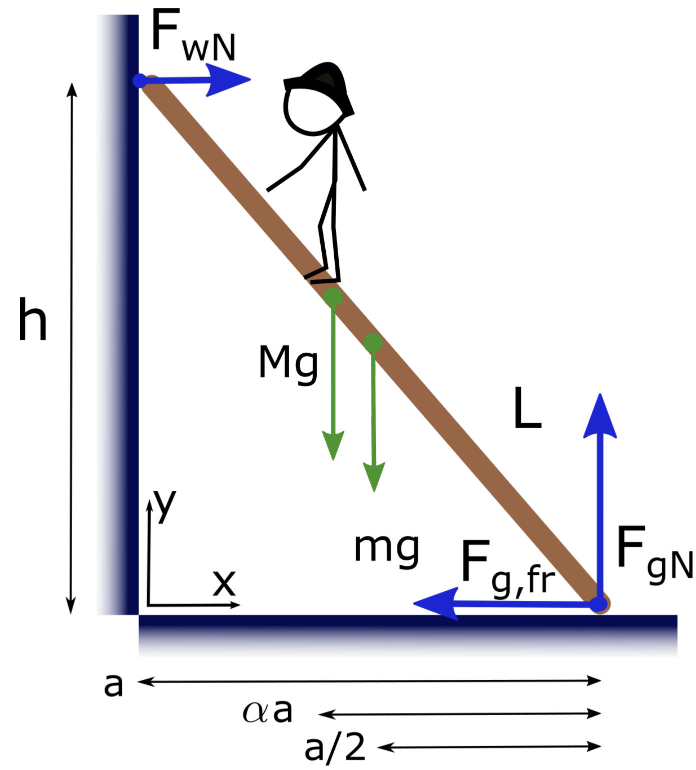

Figure 3: As the person walks up the ladder the friction force needed to keep the ladder from slipping must be larger. The person is a fraction α up the ladder.

1.5 Center of gravity and rotational stability

We consider an extended object that can rotate about a specific pivot axis.

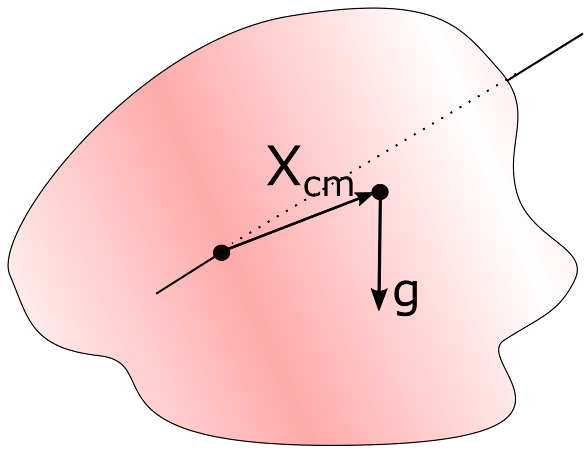

Figure 4: What is the condition for rotational equilibrium?

What is the condition for rotational equilibrium?

The total gravitational force

$$F_z = -\int dV \rho g \hat{\mathbf{z}}$$

Is exactly balanced by the upward force on the pivot.

We locate the origin of our coordinate system so the rotation axis contains it. However the torque about the origin may not be zero

$$\begin{aligned} \boldsymbol{\tau} &= -\int dV \mathbf{r} \times \hat{\mathbf{z}} \rho g \\ &= -g \left(\int dV \mathbf{r} \rho \right) \times \hat{\mathbf{z}}. \end{aligned} \tag{8}$$

The center of mass position is

$$\mathbf{X}_{cm} = \frac{1}{M} \int dV \mathbf{r} \rho.$$

We insert this into our torque expression (equation 8)

$$\boldsymbol{\tau} = -Mg\mathbf{X}_{cm} \times \hat{\mathbf{z}}.$$

For equilibrium we require that the torque is zero or

$$-Mg\mathbf{X}_{cm} \times \hat{\mathbf{z}} = 0.$$

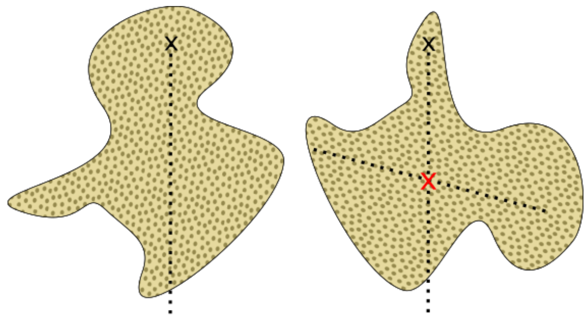

Figure 5: On the left, a rigid body is suspended from a pivot point. After it reaches an equilibrium state, points directly below the pivot are marked with a plumb line (a weight on a string). The rigid body is then suspended from a different pivot point, on the right. Again points below the pivot are marked with a plumb line. The center of mass is where the two plumb lines cross.

It is some times convenient to define a vector $\mathbf{g} = -g\hat{\mathbf{z}}$. Our torque condition for equilibrium becomes

$$\mathbf{X}_{cm} \times \mathbf{g} = 0.$$

What positions for $\mathbf{X}_{cm}$ satisfy this equilibrium condition? The length of $\mathbf{X}_{cm}$ could be zero. If the center of mass is at the pivot point, then gravity does not exert a torque on it. If $|\mathbf{X}_{cm}| > 0$ then for the cross product to be zero, $\mathbf{X}_{cm}$ must be parallel to $\mathbf{g}$. For equilibrium, the center of mass position must be directly above or below the rotation axis. For stability it must be below the rotation axis. An equilibrium point can be unstable if trajectories nearby diverge away from the equilibrium point rather than oscillating about it.

Note: the *center of gravity* is essentially the same thing as the *center of mass*. There can be a difference if deviations from uniform acceleration are taken into account. In this case, the force from gravity is often decomposed into a sum of the force onto the center of mass and a tidal force which can also be described as a quadrupolar force.

We can find the center of mass of a rigid body using a plumb line and hanging it from different pivot points. A drop or plumb line is a string with a weight on it. First hang it from a pivot point and use a vertical drop line to mark points below it, as shown in Figure 5. Then hang it from a different pivot points and use a vertical drop line to mark points below it. Where the two sets of marked points cross, is the center of mass.

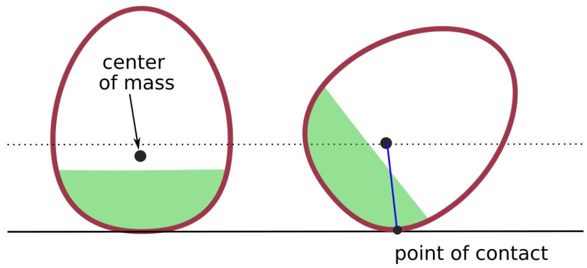

Figure 6: The roly-poly toy has an uneven internal mass distribution. Typically most of the weight is near the bottom, and here it is shown in green. When the toy is tilted, the center of mass rises, as shown on the right. Since the contact point is to the right of the center of mass, gravity exerts a torque that pushes the toy back into the equilibrium position.

1.6 The Roly-Poly Toy

A cartoon of a roly-poly toy is shown in Figure 6. Typically the inside is mostly empty and most of the mass is near the bottom. These toys do not have a homogenous mass distribution. The density is not a constant inside the body.

Vladimir Arnold in 1995 conjectured that there might be a class of homogeneous (uniform density) and convex objects that have only one stable and one unstable point of equilibrium when resting on a flat surface. In 2006 Gábor Domokos and Péter Várkonyi constructed both mathematical and actual real physical examples of such objects. The type of object is called a Gömböc. See https://en.wikipedia.org/wiki/G%C3%B6mb%C3%B6c . The shape of a Gömböc is pretty weird!

The definition of a convex set https://en.wikipedia.org/wiki/Convex_set in Euclidean space: Given any two points in the set, the set contains the whole line segment that joins them.

1.7 A block on another block

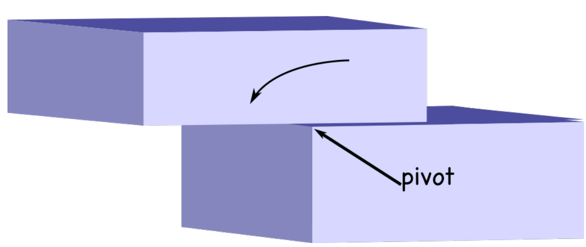

Figure 7: When does the top block fall over?

Consider two identical wood blocks, as shown in Figure 7. We compute the torque about the pivot point which is the left top corner of the lower block.

If the center of mass is directly above the pivot point, then both the radial vector to the pivot point and the gravity vector are vertical. The torque is zero.

If the center of mass is the right of the pivot point, then the torque can be balanced by an opposing force from the lower block. This can be estimated by considering two little triangles under the top block, one under the pivot point, the other under the right lower end of the top block, similar to our previous example of a plank on top of two supports. The torque on the top block can be opposed by an upward normal force from the left triangle.

If the center of mass of the top block is to the left of the pivot point then it must fall over.

1.8 A stack of blocks

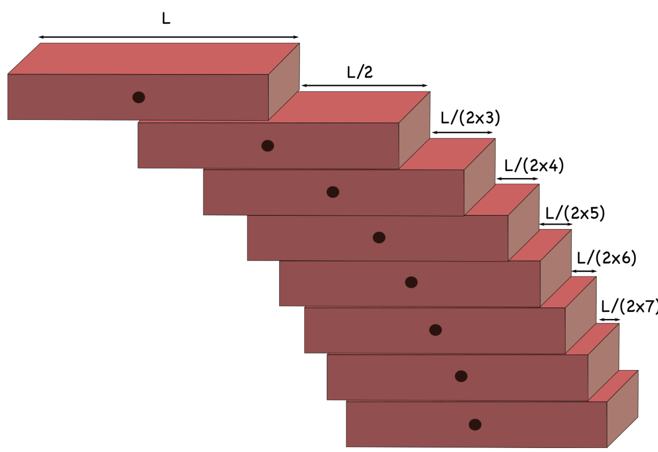

Figure 8: Black dots show the center of mass of each block. For each block, the center of mass of all blocks above it must lie to the right of its left end.

Consider a stack of blocks as shown in Figure 8. We assume uniform density blocks and they all have the same mass and length.

What is a criterion for stability of a stack of blocks?

Consider the center of mass of all blocks above the i-th block and the pivot point that is the left top corner of the i-th block.

If the center of mass of all blocks above block i is to the left of the pivot point of the i-th block (its top left corner), then the stack falls over.

There is no limit to the possible distance spanned.

My notes on the pattern: I compute the center of mass of j blocks (from the top) w.r.t to the center of mass position of the j-th block from the top.

First block's cm is 1/2 from the center of mass of the second. Here I take the length of each block to be $L = 1$ so I am describing distances in units of L.

Blocks	distance of center of mass to	cm of	overlap
1	0	Block 1	$L/2$ on Block 2
1,2	$L\left(\frac{1}{2}+0\right)\frac{1}{2}=\frac{L}{4}$	Block 2	$\frac{L}{4}-\frac{L}{2}=\frac{L}{4}$ on Block 3
1–3	$L\left[\left(\frac{1}{2}+\frac{1}{4}\right)+\left(\frac{1}{4}\right)+0\right]\frac{1}{3}=\frac{L}{3}$	Block 3	$\frac{L}{2}-\frac{L}{3}=\frac{L}{6}$ on Block 4
1–4	$L\left[\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}\right)+\left(\frac{1}{4}+\frac{1}{6}\right)+\left(\frac{1}{6}\right)+0\right]\frac{1}{4}=\frac{3L}{8}$	Block 4	$\frac{L}{2}-\frac{3L}{8}=\frac{L}{8}$ on Block 5

The first + second block's cm is (1/2 + 0)/2 from the cm of the second block. This gives an overlap of 1/2-1/4 = 1/4. The overlap is between second and third block and follows from requiring that the center of mass of the first and second blocks lies to the right of the third block's left top corner.

The first + second + third blocks cm is ([1/2 + 1/4] + 1/4+0)/3 = 1/3 from the cm of the third. The overlap is 1/2 - 1/3 = 1/6. The overlap is between third and fourth block and follows from requiring that the center of mass of the first, second and third blocks lies to the right of the fourth block's left top corner.

The first+second+third+fourth blocks cm is ([1/2 + 1/4 + 1/6] + [1/4 + 1/6] + 1/6 + 0)/4 = 3/8 from the cm of the fourth. The overlap is 1/2 - 3/8 = 1/8.

The fifth to first block's cm is ([1/2 + 1/4 + 1/6 + 1/8] + [1/4 + 1/6 + 1/8] + [1/6 + 1/8] + 1/8 + 0)/5 from the cm of the fifth block. The overlap is 1/2 - 2/5 = 1/10.

We can see a pattern!

What is the distance spanned by the bridge? It is a sum of the overlap distances.

$$d_{spanned} = \frac{L}{2} + \frac{L}{4} + \frac{L}{6} + \frac{L}{8} + \frac{L}{10} \ldots\ldots$$
$$d_{spanned} = \frac{L}{2}\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} \ldots\right)$$

The sum does not converge!

The sum can become arbitrarily large.

This means that any distance can be spanned with a finite number of blocks.

Question: As long as the mass and length of each block is the same and they have uniform linear density, does it matter what their height is?

Answer: no!

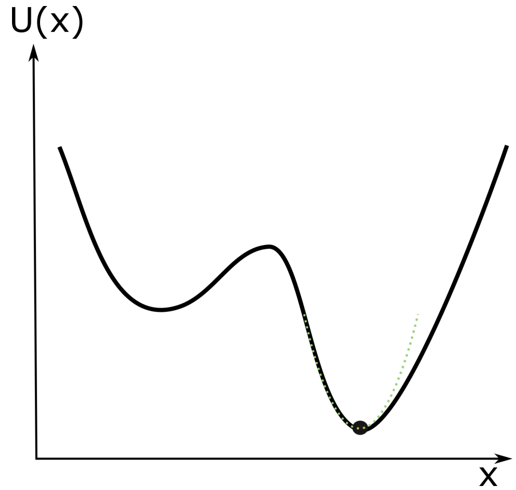

Figure 9: A stable equilibrium point in a potential well.

2 Harmonic motion near a stable equilibrium point

2.1 Harmonic motion at the bottom of a potential well

We consider a potential energy function in 1 dimension $U(x)$ that has a minimum at x_0. At x_0, because $dU/dx = 0$, the equation of motion has a fixed solution, $x(t) = x_0$ at all times. It is an equilibrium point.

Near x_0 we expand the potential energy U using a Taylor series

$$U(x) \approx U(x_0) + \left.\frac{dU}{dx}\right|_{x_0} (x - x_0) + \left.\frac{d^2U}{dx^2}\right|_{x_0} \frac{1}{2}(x - x_0)^2$$

Because x_0 is an extremum

$$U(x) \approx U(x_0) + \frac{1}{2}k(x - x_0)^2 + ...$$

where

$$k = \left.\frac{d^2U}{dx^2}\right|_{x_0}$$

To second order in $x - x_0$, we recognize that the potential energy is the same quadratic form as that of a spring, giving the equation of motion typical of a harmonic oscillator. We can define a variable that is shifted $y = x - x_0$. The time derivative $\ddot{y} = \ddot{x}$. The equation of motion near x_0 for a mass m is

$$m\ddot{y} = m\frac{d^2y}{dt^2} = -ky.$$

Near the bottom of the potential well, the mass would oscillate back and forth at angular frequency

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{1}{m}\frac{d^2U}{dx^2}\bigg|_{x_0}}. \tag{9}$$

The period of oscillations is $T = \frac{2\pi}{\omega}$ as we expect. Solutions near the equilibrium point would be similar to those of a harmonic oscillator

$$x(t) = x_0 + A\cos(\omega t) + B\sin(\omega t)$$

with constants A, B dependent on initial conditions. The frequency in equation 9 and associated period of oscillation are only accurate for **small** oscillations about the equilibrium point because the Taylor expansion is only accurate near the equilibrium point.

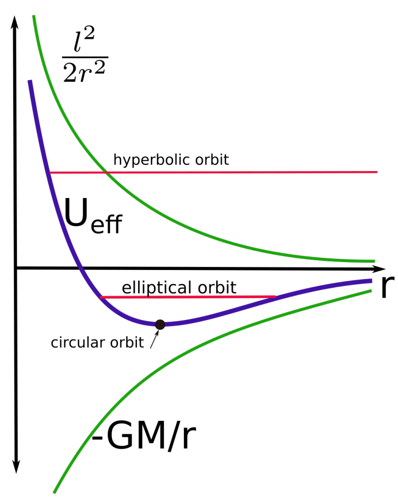

Figure 10: The effective potential $U_{eff}(r) = \frac{L^2}{2r^2} - \frac{GM}{r}$ for a Keplerian orbit.

2.2 Radial epicyclic motion

The effective potential per unit mass for radial motion in the Keplerian orbit problem

$$U_{eff} = \frac{L^2}{2r^2} - \frac{GM}{r}$$

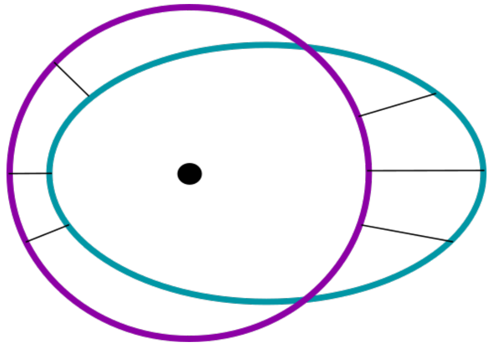

Figure 11: An elliptical orbit can be considered as the sum of a circular orbit plus a small radial oscillation. The radial oscillation is called the epicycle.

where L is the orbital angular momentum per unit mass and M is the mass of the central body like the Sun. At the extremum

$$\frac{dU_{eff}}{dr} = -\frac{L^2}{r^3} + \frac{GM}{r^2} = 0$$

We solve this for the radius r_* of a circular orbit with angular momentum per unit mass L, $L^2 = GMr_*$ or

$$r_* = \frac{L^2}{GM}.$$

Using the radius of the circular orbit we approximate the bottom of the potential well with the second derivative

$$\frac{d^2U_{eff}}{dr^2} = \frac{3L^2}{r^4} - \frac{2GM}{r^3}$$

We evaluate this at r_* to find the epicyclic frequency

$$\kappa^2 = \frac{3L^2}{r_*^4} - \frac{2GM}{r_*^3}$$

The energy per unit mass near the bottom of the potential well or for nearly circular orbits

$$E = \frac{1}{2}v_r^2 + \frac{1}{2}\kappa^2(r - r_*)^2.$$

where r_* is the radius for the circular orbit. The frequency κ is called the epicyclic frequency and it is the frequency of radial oscillations about a circular orbit.

For a Keplerian circular orbit of radius r_* the angular rotation rate

$$\Omega = \sqrt{\frac{GM}{r_*}^3}$$

so we can write

$$L^2 = r_*^4 \Omega^2 = GMr_* \tag{10}$$

giving

$$\kappa^2 = \Omega^2.$$

Not surprisingly the angular rotation rate is the same as the epicyclic frequency in the Keplerian setting. The frequency of epicyclic oscillations is the same as the orbital period. This is equivalent to saying that the orbit closes to form an ellipse.

More generally

$$U_{eff} = \frac{L^2}{2r^2} + V_0(r)$$

where $V_0(r)$ is derived from the radial distribution of mass. This gives epicyclic frequency

$$\kappa^2 = \frac{3L^2}{r^4} + \frac{dV_0^2}{dr^2}$$

For a flat rotation curve, typical of a disk galaxy, with potential $V_0(r) = v_c^2 \ln r$, the epicyclic frequency $\kappa = \sqrt{2}\Omega$. If the potential energy is not inversely proportional to r then orbits are usually not closed. The orbit is more like a rosette. In the nearly Keplerian setting, like the Solar system, where planets also perturb each other, the orbits are nearly but not exactly closed. The angle of perihelion slowly changes or precesses with a period of tens of thousands of years.

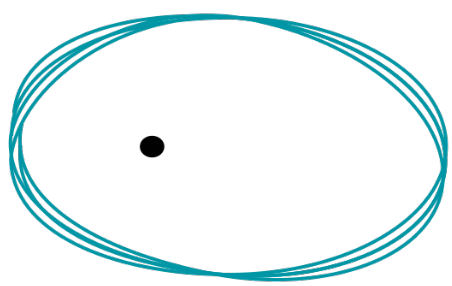

Figure 12: An nearly Keplerian orbit slowly precesses. The angle of perihelion slowly changes due to the average of perturbations from planets. In this setting the epicyclic frequency is approximately the same as the angular rotation rate. The difference between these two frequencies is the precession rate.

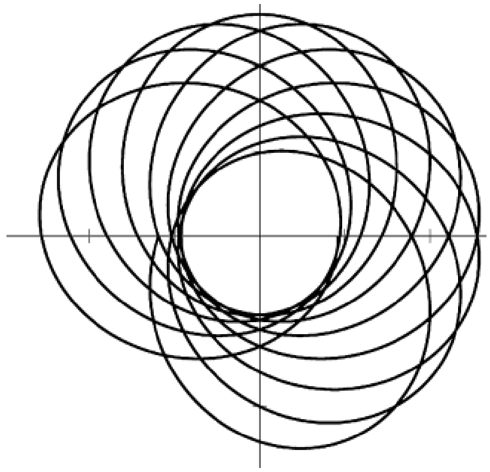

Figure 13: An orbit with a larger epicyclic frequency resembles a rosette. Orbits in the Milky Way's galactic disk look like this.

3 Summary

Forces on portions of a static structure can be computed from sums of forces and torques on objects in the system. These sums must total zero for the system to be in equilibrium.

$$\sum_i \mathbf{F}_i = 0 \qquad\qquad \sum_i \boldsymbol{\tau}_i = 0$$

Small oscillations of a mass m near a stable equilibrium point at the bottom of a potential well (described by potential energy function $U(x)$) have frequency dependent on the second derivative of the potential evaluated at the equilibrium point, x_*.

$$\omega = \sqrt{\frac{1}{m}\left.\frac{d^2U}{dx^2}\right|_{x_*}}$$

The period of the oscillations $T = \frac{2\pi}{\omega}$.