PHY141 Lectures 16,17,18 notes

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November 10, 2022

Contents

1 Rigid body rotation

Our goal is to go beyond point masses. A rigid body has mass that is extended, however its dynamics is still specified by only a few quantities, such as the total mass, the center of mass position and velocity, the body orientation, a spin axis, and a spin angular rotation rate. Also important might some parameters describing the mass distribution of our rigid body.

Before we explore rigid body dynamics we first consider a single point mass particle of mass m undergoing uniform rotation. The distance between particle and center of rotation is R.

Variables used to describe the rotational motion of this single particle are:

The angular position θ.

The angular rotation rate $\omega = d\theta/dt$.

The angular acceleration $\alpha = d\omega/dt = d^2\theta/dt^2$.

The tangential velocity $v_\theta = R\omega$.

From one of our earlier lectures we showed that the acceleration vector in polar coordinates

$$\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\boldsymbol{\theta}}$$

where unit vector $\hat{\mathbf{r}} = \mathbf{r}/r$ and unit vector $\hat{\boldsymbol{\theta}}$ is perpendicular to $\hat{\mathbf{r}}$ and in the xy plane (and pointing in the direction of counter-clockwise rotation). With constant radius $r = R$

$$\begin{aligned}\mathbf{a} &= -R\dot{\theta}^2\hat{\mathbf{r}} + R\ddot{\theta}\hat{\boldsymbol{\theta}} \\ &= R\omega^2\hat{\mathbf{r}} + R\dot{\omega}\end{aligned} \tag{1}$$

Note the second term involves the angular acceleration!

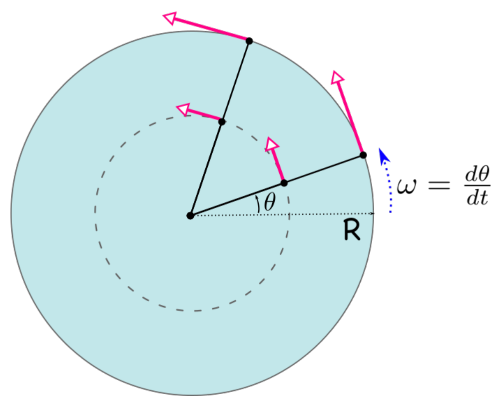


Figure 1: Rotation of a rigid disk about its center of mass.

We now go beyond a single particle. We approximate a rigid body as a series of point masses that do not move with respect to each other, however the whole body can rotate. We choose an axis that remains fixed and the body rotates about this axis.

1.1 The moment of inertia

Consider a uniform disk rotating about its center with angular rotation rate ω. Each particle with mass m_i moves with tangential velocity that depends on its radius r_i from the center. The tangential velocity $v_i = r_i\omega$ increases linearly with radius. This means the kinetic energy of each piece of the disk increases with the square of the radius.

For a rotating rigid body that is comprised of point masses m_i, the kinetic energy

$$K = \sum_i \frac{1}{2}m_i v_i^2 = \sum_i \frac{1}{2}m_i r_i^2\omega^2 = \frac{1}{2}\left(\sum_i m_i r_i^2\right)\omega^2. \tag{2}$$

Note r_i are the **distances from the axis of rotation** of each point mass.

We define the *moment of inertia* of a series of point masses

$$I \equiv \sum_i m_i r_i^2. \tag{3}$$

Then the kinetic energy of a rigid body rotating at ω

$$K = \frac{1}{2} I \omega^2. \tag{4}$$

The moment of inertia is in units of mass times length^2 or kg m^2.

The moment of inertia depends on the location and orientation of the rotation axis.

The moment of inertia depends on the mass distribution.

Rather than consider our rigid body as a series of point masses, we can describe it with a mass or density distribution. In this case the integral over the volume

$$I = \int r^2 dm$$

where dm is the mass in a small volume element. Again r is the distance to the rotation axis, not a three dimensional radial distance to any point. We could also write

$$dm = \rho dV$$

where ρ is the density of the volume element and $dV = dx\ dy\ dz$ is its volume. Equivalently

$$I = \int r^2 \rho dV.$$

If we are computing the moment of inertia about the z axis then

$$r^2 = x^2 + y^2$$

and

$$I = \int dx \int dy \int dz\ (x^2 + y^2) \rho(x, y, z)$$

This is in Cartesian coordinates. In cylindrical coordinates $dV = rdr\ d\phi\ dz$ and

$$I = \int d\phi \int dz \int dr\ r^3 dr\ \rho(r, \phi, z)$$

for the moment of inertia about the z axis.

The moment of inertia is a measure of how extended the mass distribution is. If all the mass is concentrated near the axis of rotation then when the body spins it has little rotational kinetic energy. However if most of the mass is concentrated at large radius, like a bicycle wheel where much of the mass is in the rim, then a small spin corresponds to a large rotational kinetic energy.

The state of a rigid body is described via the position of its center of mass and its body orientation. The kinetic energy of a rigid body is described by the kinetic energy of the center of mass, and associated with translation, and the kinetic energy associated with body rotation about the center of mass. The rotational kinetic energy is present instead of the vibrational kinetic energy that we discussed in some previous examples.

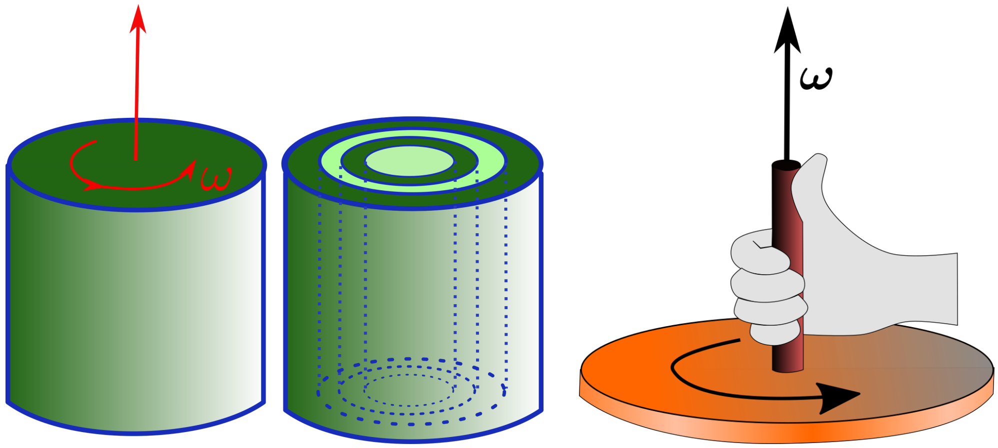


Figure 2: Left: A solid cylinder rotating about its axis of symmetry. The moment of inertia is a sum over layers. The contribute of each layer depends on its distance from the axis of rotation. Right: the axis of rotation of a rotating disk using the right hand rule.

1.2 Calculating the moment of inertia

1.2.1 The moment of inertia of a rod about its midpoint

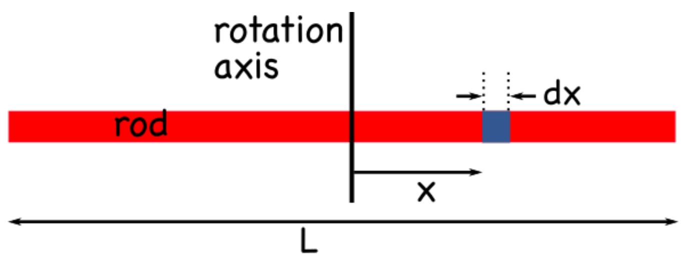


Figure 3: A rod rotating about its midpoint.

Consider a rod that has length L and mass m. What is the momentum of inertia about its center of mass? If the rod lies in the xy plane we let the axis of rotation be in the z direction. We can describe the rod with a uniform linear mass density λ

$$\lambda = \frac{\text{mass}}{\text{length}}$$

such that

$$L\lambda = m.$$

We orient the rod along the x axis and take the origin to be at the midpoint of the rod. The origin is the rotation axis. Each parcel of the rod that is dx long has mass $dm = \lambda dx$. A parcel that is located at x has distance from the rotation axis or origin $|x|$.

$$
\begin{aligned}
I &= \int_{-L/2}^{L/2} \lambda x^2 \; dx \\
&= \lambda \frac{x^3}{3} \Bigg]_{-L/2}^{L/2} \\
&= \lambda \left(\frac{L^3}{8 \times 3} + \frac{L^3}{8 \times 3} \right) \\
&= \lambda \frac{L^3}{12} = \frac{mL^2}{12}
\end{aligned}
$$

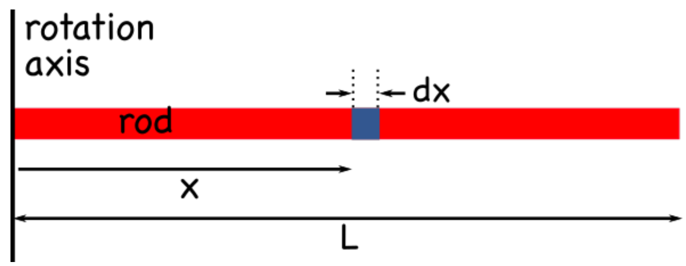


Figure 4: A rod rotating about its endpoint.

1.2.2 The moment of inertia of a rod about its endpoint

What is the moment of inertia about a rod endpoint rather than about the midpoint?

$$
\begin{aligned}
I &= \int_{0}^{L} \lambda x^2 \; dx \\
&= \lambda \frac{x^3}{3} \Bigg]_{0}^{L} \\
&= \lambda \frac{L^3}{3} = \frac{mL^2}{3}
\end{aligned}
$$

The moment of inertia is larger for the rod rotating about its end than for the same rod rotating about its center of mass.

1.2.3 The moment of inertia of a solid cylinder about its axis of symmetry

To compute the moment of inertia of a uniform density solid cylinder about is axis of symmetry (see Figure 2), it is convenient to work in cylindrical coordinates r, ϕ, z. We take a cylinder with radius R, length h and density ρ. Its total mass is

$$\begin{aligned} M &= \int_0^R rdr \int_0^{2\pi} d\phi \int_0^h dz\rho \\ &= 2\pi h\rho \int_0^R rdr = 2\pi h\rho \frac{R^2}{2} \\ &= \pi h\rho R^2 \end{aligned}$$

Note the volume element in cylindrical coordinates is $dV = rdrd\phi dz$. The moment of inertia is

$$\begin{aligned} I &= \int_0^R r^3 dr \int_0^{2\pi} d\phi \int_0^h dz\rho \\ &= 2\pi h\rho \int_0^R r^3 dr \\ &= 2\pi h\rho R^4/4 \\ &= \frac{\pi h\rho R^4}{2} \\ I &= \frac{1}{2} M R^2 \end{aligned} \tag{5}$$

1.2.4 The moment of inertia of a diatonic molecule

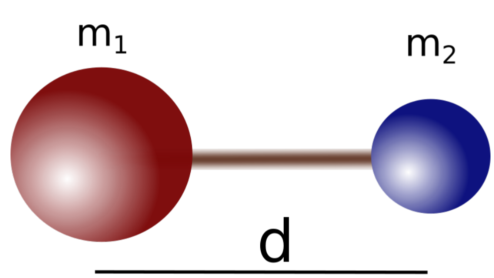


Figure 5: A diatonic molecule.

We consider a molecule comprised of two atoms, separated by distance d and with masses m_1, m_2. We assume that d is fixed, so that the molecule is rigid. We compute the moment of inertia about the center of mass and with rotation axis perpendicular to the line containing the two masses. We orient the coordinate system with center of mass at the origin and the atoms on the x axis. The axis of rotation can be the z axis. In the center of mass coordinate system

$$\begin{aligned} m_1x_1 + m_2x_2 &= 0 \\ m_1x_1 + m_2x_2 - m_2x_1 + m_2x_1 &= 0 \\ x_1(m_1 + m_2) + m_2(x_2 - x_1) &= 0 \end{aligned}$$

$$x_1 = -d\frac{m_2}{M}, \tag{6}$$

with $d = x_2 - x_1$ and $M = m_1 + m_2$. Likewise

$$x_2 = d\frac{m_1}{M}. \tag{7}$$

The moment of inertia about the center of mass is

$$\begin{aligned} I &= m_1x_1^2 + m_2x^2 \\ &= \frac{d^2}{M^2}(m_1m_2^2 + m_2m_1^2) = \frac{d^2}{M^2}m_1m_2(m_1 + m_2) \\ &= \frac{m_1m_2}{M}d^2 \\ I &= \mu d^2 \end{aligned} \tag{8}$$

where $\mu = \frac{m_1m_2}{m_1+m_2}$ is the reduced mass.

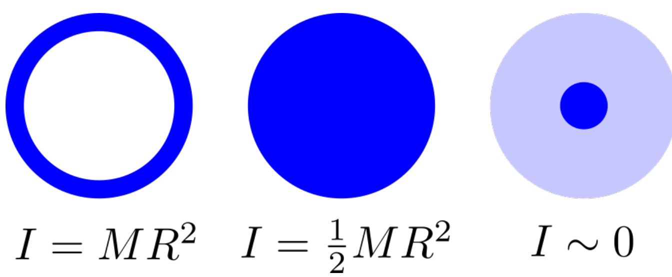


Figure 6: Three disks with the same total mass M and the same radius R. The moment of inertia about the center of mass is highest for the ring on the left. The moment of inertia is intermediate for the uniform density disk in the middle. The moment of inertia is lowest for the disk that is low density except near its center, as see on the right.

1.3 Mass concentration and the moment of inertia

Suppose you have two circular flat objects with the same mass M and both have radius R. We consider the moment of inertia about an axis going through the center of these circular objects which is also their center of mass.

One of them has a high moment of inertia. The other has a lower one. What can you say about their mass distributions?

A ring has a moment of inertia MR^2. This is largest possible value for a disk shaped object of mass M and radius R. In this case all the mass is concentrated at the rim.

A uniform disk has moment of inertial $MR^2/2$. The moment of inertia is lower if the mass is more centrally concentrated.

If all the mass were concentrated at the center (a heavy mass surrounded by foam), the moment of inertia would be nearly zero.

A measurement of the moment of inertia can give you some information about internal mass distribution. An example where this can be useful is in planets, where measurements of the moment of inertia, derived from shape, spin or gravity field, can place constraints on models of their internal density distribution.

1.4 The parallel axis theorem

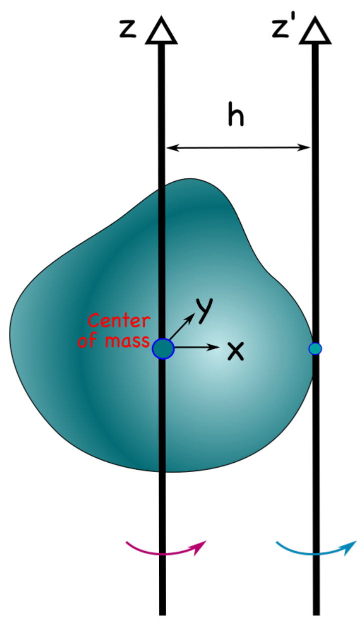


Figure 7: The moment of inertia of a body around an axis can be determined from the moment of inertia around a parallel axis that goes through the centre of mass.

If you know the moment of inertia about an axis $\hat{\mathbf{z}}$ that goes through the center of mass, I_{cm}, then you can use the **parallel axis theorem** to compute it about another rotation axis $\hat{\mathbf{z}}'$ that is parallel to the one that goes through the center of mass $\hat{\mathbf{z}}$. The parallel axis theorem

$$I_{new} = I_{cm} + Mh^2 \tag{9}$$

where h is the distance between the two rotation axes, and M is the mass.

Let's show that this is true.

Let us choose coordinate system with origin at the center of mass. The rotation axis goes through the center of mass and lies along the z axis. We orient the coordinate system so that the new rotation axis is at $x = h, y = 0$.

We first compute the moment of inertia about the origin and center of mass and then compute the moment of inertia about an axis that goes through $x = h, y = 0$. Both rotation axes are parallel to the z axis.

$$I_{cm} = \int r^2 dm = \int dx\ dy\ dz \rho\ (x^2 + y^2)$$

$$\begin{aligned}
I_{new} &= \int dx\ dy\ dz\ \rho\ ((x-h)^2 + y^2) \\
&= \int dx\ dy\ dz\ \rho\ (x^2 + y^2 + h^2 - 2xh) \\
&= \int dx\ dy\ dz\ \rho\ (x^2 + y^2) + h^2 \int dx\ dy\ dz\ \rho - 2h \int dx\ dy\ dz\ \rho\ x \\
&= I_{cm} + Mh^2
\end{aligned}$$

where the rightmost term integrating x drops out because we have taken the origin to be at the center of mass.

Let's check to see that the parallel axis theorem is consistent with our previous calculations on the rod. The moment of inertia about its center of mass is $I_{cm} = ML^2/12$. We shift the axis of rotation by $h = L/2$.

$$I_{new} = \frac{1}{12} ML^2 + M\frac{L^2}{4} = ML^2 \left(\frac{1}{12} + \frac{3}{12} \right) = \frac{1}{3} ML^2$$

This is consistent with our previous calculation for the moment of inertia of the rod but rotating about an endpoint rather than the midpoint.

2 Rolling without slipping

We consider round or cylindrical objects that are rolling without slipping on a flat surface. Take a cylinder with radius R. How far does it need to go horizontally to rotate one full revolution?

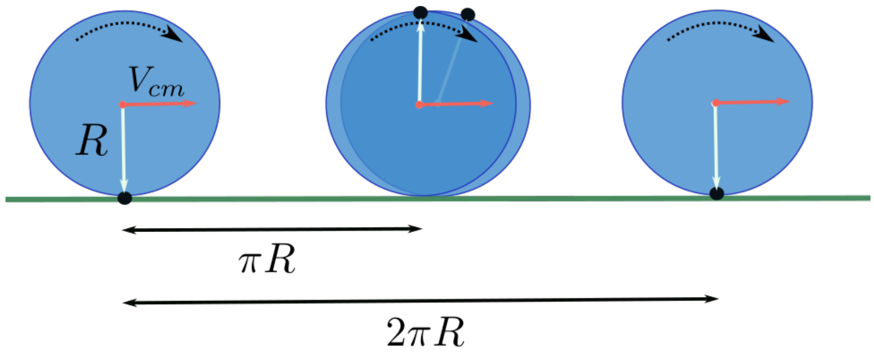


Figure 8: A cylinder of radius R rolls without slipping. The angular rotation rate $\omega = V_{cm}/R$ in terms of the center of mass velocity.

To undergo one full revolution, the cylinder must travel $2\pi R$. Suppose this takes time T. This is a full rotation period so in terms of the angular rotation rate $T = 2\pi/\omega$. The distance the center of mass travels in this time is $2\pi R$ and so its velocity must be $V_{cm} = 2\pi R/T$. Combining these we find that the center of mass velocity

$$V_{cm} = \frac{2\pi R}{T} = \frac{2\pi\omega R}{2\pi} = \omega R$$

Alternatively we can write the angular rotation rate in terms of the center of mass velocity

$$\omega = \frac{V_{cm}}{R}. \tag{10}$$

These two expressions are valid when the cylinder is *rolling without slipping.*

2.1 Translational and Rotational kinetic energy

In a previous lecture we showed that kinetic energy of a multi-particle system could be written as a sum

$$K = \frac{1}{2}MV_{cm}^2 + \frac{1}{2}\sum_i m_i(\mathbf{v}_i - \mathbf{V}_{cm})^2$$

where $m_i, \mathbf{v}_i$ are the masses and velocities of the particles. Here M and $\mathbf{V}_{cm}$ are the total mass and center of mass velocity. Taking the rotation axis to be the center of mass we recognize that

$$\frac{1}{2}\sum_i m_i(\mathbf{v}_i - \mathbf{V}_{cm})^2 = \frac{1}{2}I\omega^2$$

where I is computed for the axis of rotation that goes through the center of mass.

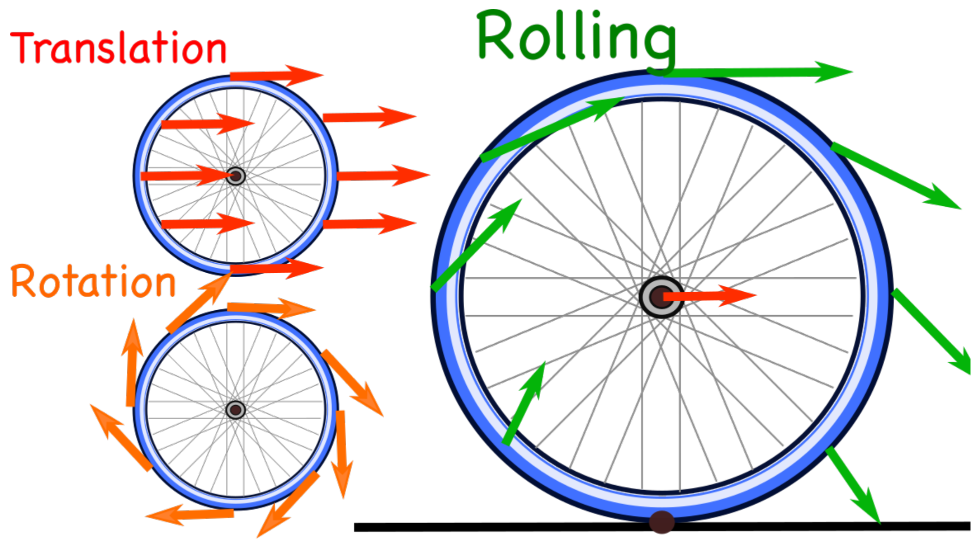


Figure 9: A wheel rolls without slipping. The rolling motion is a sum of rotation and translation.

The total kinetic energy of a rigid body is

$$K = \frac{1}{2}MV_{cm}^2 + \frac{1}{2}I\omega^2 \tag{11}$$

where I is computed for the axis of rotation that goes through the center of mass.

2.2 Rolling down an inclined plane

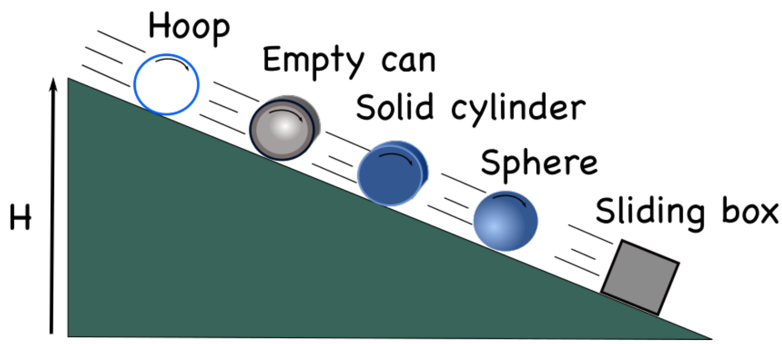


Figure 10: Objects with the same mass and radius released at rest from the top of an inclined plane. The object that slides without friction and does not roll reaches the bottom first. The more centrally concentrated and rolling objects reach the bottom before the more extended objects.

Objects of the same mass and radius but different moments of inertia, are released from rest from the top of an inclined plane. Which ones reach the bottom first? Here H is the height of the inclined plane.

Energy is the sum of potential energy, center of mass kinetic energy and rotational kinetic energy. We take the initial energy to be equal to the potential energy or

$$E_0 = MgH.$$

The final energy

$$E = \frac{1}{2} M V_{cm}^2 + \frac{1}{2} I \omega^2$$

If the objects are round and rolling without slipping then $\omega = V_{cm}/R$. We use this to remove ω.

$$E = \frac{1}{2} M V_{cm}^2 \left(1 + \frac{I}{MR^2} \right)$$

As energy is conserved, this should be equal to the initial energy

$$\frac{E}{M} = gH = \frac{1}{2} V_{cm}^2 \left(1 + \frac{I}{MR^2} \right) \tag{12}$$

The smaller the moment of inertia, the larger V_{cm} becomes.

The quantity $M(1 + \frac{I}{MR^2})$ could be considered an *effective* mass.

If the object does not roll at all and the surface is frictionless then

$$\frac{E}{M} = gH = \frac{1}{2} V_{cm}^2$$

and V_{cm} is even larger. This is equivalent to equation 12 in the limit of $I \to 0$ as there is no contribution to the kinetic energy from rotation.

Question: Why is no energy lost from friction?

When rolling without slipping, the point of contact on the rolling object is not moving with respect to the surface. There is no friction. However friction does act to keep the rolling object rolling without slipping. The rolling without slipping condition is a **constraint** that does no work.

The constraint is in the form $f(\mathbf{x}, \theta) =$ constant and these are called *holonomic* constraints.

2.3 The roller coaster loop

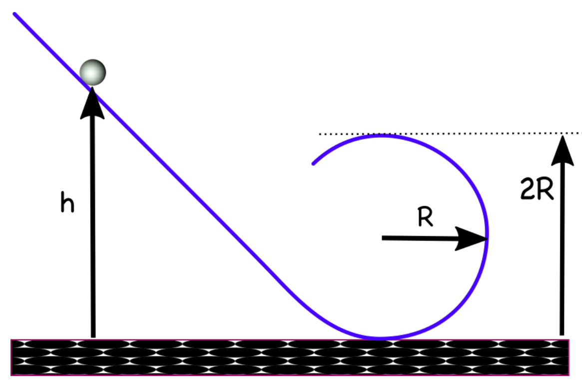


Figure 11: What height h is required for an object initially at rest to make it to the top of the loop?

What height h is required for an object of mass m initially at rest to make it to the top of the loop? The loop has radius R.

We compute the energy. Initially the energy is all potential energy

$$E_0 = mgh$$

At the top of the loop the object is moving with speed v. For the object to not fall when the object is at the top of the loop, the centrifugal acceleration must exceed gravity

$$v^2/R > g$$

At the top of the loop we set v to be the critical value so

$$v^2 = gR$$

If there is no rolling and the surface is frictionless, then the total energy is

$$E = mg2R + \frac{1}{2}mv^2 = mg2R + \frac{1}{2}mRg.$$

We equate the two expressions for energy finding

$$\begin{aligned} mgh &= mg2R + \frac{1}{2}mRg \\ h &= 2R + \frac{1}{2}R = \frac{5}{2}R \end{aligned}$$

If there is no rolling then the roller coast height must be $h > \frac{5}{2}R$.

What happens if the object is a uniform density **sphere** that rolls? Then the total energy includes a rotational term. The moment of inertia for a sphere is $I = \frac{2}{5}mr^2$, where r is the radius of the sphere. Note that the radius of the sphere $r \neq R$ the radius of the loop.

With rolling without slipping the angular rotation rate $\omega = v/r$. The rotational kinetic energy at the top of the loop is

$$\begin{aligned} K_{rot} &= \frac{1}{2}I\omega^2 \\ &= \frac{1}{2}\frac{2}{5}mr^2\frac{v^2}{r^2} \\ &= \frac{1}{5}mv^2 \\ &= \frac{1}{5}mRg \end{aligned}$$

The total energy at the top of the loop is

$$\begin{aligned} E &= mg2R + \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 \\ &= mg2R + \frac{1}{2}mRg + \frac{1}{5}mRg \\ &= mg2R + \frac{7}{10}mRg \end{aligned}$$

We set this equal to the initial energy and solve for h

$$h = 2R + \frac{7}{10}R = \frac{27}{10}R$$

If a uniform density sphere rolls without slipping then the roller coast height must be $h > \frac{27}{10}R$ in order for the sphere to make it over the top of the loop.

This is somewhat higher than the $\frac{5}{2}R = \frac{25}{10}R$ required for the object that does not roll!

3 Angular momentum

We define the angular momentum of a particle with momentum $\mathbf{p}$. We chose an origin. The particle position is given with respect to this position $\mathbf{r}$. The angular momentum is defined with a cross product

$$\mathbf{L} \equiv \mathbf{r} \times \mathbf{p}. \tag{13}$$

The value of the angular momentum vector $\mathbf{L}$ depends on our choice for the origin.

The angular momentum vector $\mathbf{L}$ is perpendicular to both $\mathbf{r}$ and $\mathbf{p}$. In the non-relativistic limit $\mathbf{L} \equiv m\mathbf{r} \times \mathbf{v}$.

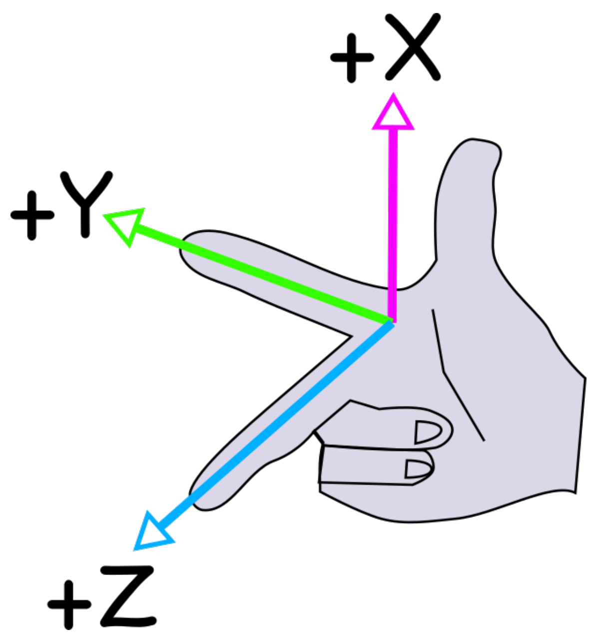


Figure 12: A right handed coordinate sytem. The thumb is $\hat{\mathbf{x}}$, the index finger is $\hat{\mathbf{y}}$, the 3rd finger is $\hat{\mathbf{z}}$.

3.1 Angular momentum for a particle undergoing uniform circular motion

Let's look at a mass undergoing counter clockwise circular motion in the xy plane and about the origin. We compute the angular momentum using the center of rotation as the

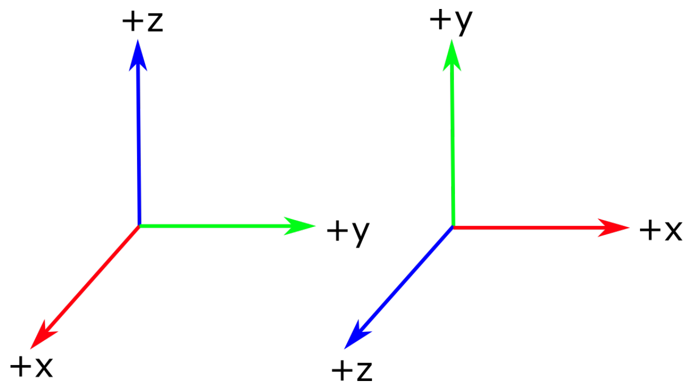


Figure 13: Right handed coordinate systems.

origin. The position and velocity vectors

$$\begin{aligned}\mathbf{r} &= r\cos(\omega t)\hat{\mathbf{x}} + r\sin(\omega t)\hat{\mathbf{y}}\\ \mathbf{v} &= -r\omega\sin(\omega t)\hat{\mathbf{x}} + r\omega\cos(\omega t)\hat{\mathbf{y}}\end{aligned}$$

where $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$ are unit vectors in the +x,+y,+z directions, respectively.

It may help to write

$$\begin{aligned}&\hat{\mathbf{x}}\times\hat{\mathbf{y}} = \hat{\mathbf{z}} \qquad \hat{\mathbf{y}}\times\hat{\mathbf{x}} = -\hat{\mathbf{z}}\\ &\hat{\mathbf{y}}\times\hat{\mathbf{z}} = \hat{\mathbf{x}} \qquad \hat{\mathbf{z}}\times\hat{\mathbf{y}} = -\hat{\mathbf{x}}\\ &\hat{\mathbf{z}}\times\hat{\mathbf{x}} = \hat{\mathbf{y}} \qquad \hat{\mathbf{x}}\times\hat{\mathbf{z}} = -\hat{\mathbf{y}}\\ &\hat{\mathbf{x}}\times\hat{\mathbf{x}} = \hat{\mathbf{y}}\times\hat{\mathbf{y}} = \hat{\mathbf{z}}\times\hat{\mathbf{z}} = 0\end{aligned}$$

We compute the angular momentum vector

$$\begin{aligned}\mathbf{L} &= m\mathbf{r}\times\mathbf{v}\\ &= mr^2\omega\left[(\cos(\omega t)\hat{\mathbf{x}} + \sin(\omega t)\hat{\mathbf{y}})\times(-\sin(\omega t)\hat{\mathbf{x}} + \cos(\omega t)\hat{\mathbf{y}})\right]\\ &= mr^2\omega(\cos^2(\omega t)\hat{\mathbf{x}}\times\hat{\mathbf{y}} - \sin^2(\omega t)\hat{\mathbf{y}}\times\hat{\mathbf{x}})\\ &= mr^2\omega(\cos^2(\omega t) + \sin^2(\omega t))\hat{\mathbf{x}}\times\hat{\mathbf{y}}\\ &= mr^2\omega\hat{\mathbf{z}}\end{aligned}$$

We could also write this as $L = mrv_\theta$ as $r\omega$ is equal to the tangential velocity for an object undergoing circular motion.

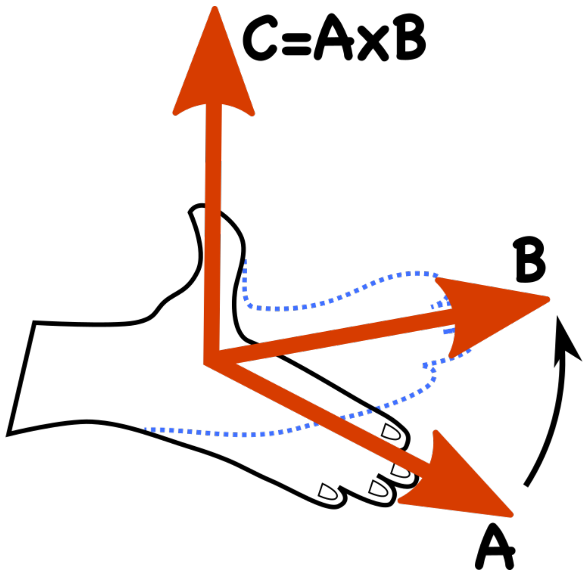


Figure 14: The direction of $\mathbf{A} \times \mathbf{B}$ using the right hand rule.

3.2 Angular momentum for a particle in orbit

In polar coordinates the position of a particle

$$\mathbf{r} = r\hat{\mathbf{r}}$$

and the velocity of this particle

$$\mathbf{v} = r\dot{\theta}\hat{\boldsymbol{\theta}} + \dot{r}\hat{\mathbf{r}}$$

We recognize velocity components $v_\theta = r\dot{\theta}$ and $v_r = \dot{r}$.

We compute the angular momentum about the origin,

$$\begin{aligned}
\mathbf{L} &= \mathbf{r} \times \mathbf{p} \\
&= r\hat{\mathbf{r}} \times m(v_r\hat{\mathbf{r}} + v_\theta\hat{\boldsymbol{\theta}}) \\
&= mrv_\theta\hat{\mathbf{z}} = mr^2\dot{\theta}\hat{\mathbf{z}}
\end{aligned}$$

The angular momentum of this particle only depends on tangential velocity component and is perpendicular to the orbital plane.

Another way to compute the angular momentum is in polar coordinates. The radial vector to the particle is $\hat{\mathbf{r}}$. The velocity has velocity vector $\mathbf{v} = r\omega\hat{\boldsymbol{\theta}}$. Because the velocity is tangential, it is perpendicular to the radial vector. It is convenient to compute $\hat{\mathbf{r}} \times \hat{\boldsymbol{\theta}} = \hat{\mathbf{z}}$. We can see this is correct if we start with $\hat{\mathbf{r}}$ in the $\hat{\mathbf{x}}$ direction then $\hat{\boldsymbol{\theta}}$ is in the positive $\hat{\mathbf{y}}$ direction. Then since $\hat{\mathbf{x}} \times \hat{\mathbf{y}} = \hat{\mathbf{z}}$, it must follow that $\hat{\mathbf{r}} \times \hat{\boldsymbol{\theta}} = \hat{\mathbf{z}}$.

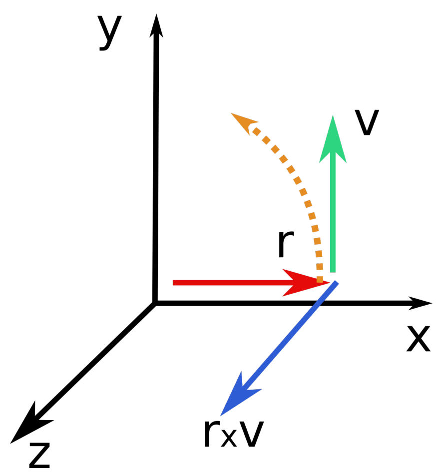


Figure 15: The angular momentum of an object undergoing circular motion in the xy-plane and in the counter-clockwise direction has angular momentum in the +z direction.

The angular momentum vector

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = m\mathbf{r} \times \mathbf{v} = mr^2\omega \ \hat{\mathbf{r}} \times \hat{\boldsymbol{\theta}} = mr^2\omega \ \hat{\mathbf{z}}$$

3.3 Kepler's second law

Kepler's second law is: *A line segment joining a planet and the Sun sweeps out equal areas during equal intervals of time.*

Consider a particle of mass m with radius r from the origin in orbit about the Sun. The rate that area is swept per unit time is

$$\frac{dA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt} = \frac{1}{2}r^2\dot{\theta} = \frac{1}{2}r^2\omega$$

and this is true even if radius r is varying. The rate that area is swept per unit time is proportional to the angular momentum per unit mass.

$$\frac{dA}{dt} = \frac{1}{2}r^2\omega = \frac{L_z}{2m}$$

Thus constant $\frac{dA}{dt}$ follows from conservation of orbital angular momentum.

Why is angular momentum conserved?

$$\begin{aligned}
\mathbf{L} &= \mathbf{r} \times m\mathbf{v} \\
\frac{d\mathbf{L}}{dt} &= \frac{d\mathbf{r}}{dt} \times m\mathbf{v} + \mathbf{r} \times m\frac{d\mathbf{v}}{dt} \\
&= \mathbf{v} \times m\mathbf{v} + \mathbf{r} \times \mathbf{F} \\
&= 0
\end{aligned}$$

The left term is zero because it is a cross product of two parallel vectors. The right term is zero because gravity is a radial force and so this term is also a cross product of two parallel vectors.

3.4 Angular momentum for linear motion

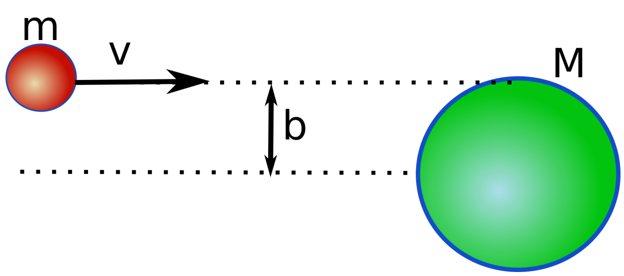


Figure 16: A non-normal or grazing impact can change the target's spin.

We consider a mass m that is moving at velocity v and impacts a mass M that is initially at rest. If the impact is grazing, then the impact can cause the larger body to spin. We compute the angular momentum from the center of the larger mass. The distance b between the trajectory of m and the center of mass of M is known as the **impact parameter**. The angular momentum is

$$L = mbv$$

The angular momentum in the figure causes the object to spin in the clockwise direction. If $\mathbf{v}$ is in the $\hat{\mathbf{x}}$ direction and the vector $\mathbf{b}$ in the $\hat{\mathbf{y}}$ direction then $\mathbf{L}$ is in the $-\hat{\mathbf{z}}$ direction or into the page.

3.5 Angular momentum for a rigid body

We have so far considered angular momentum and torque on a single particle. We now define them for rigid bodies.

The total angular momentum of a rigid body is the sum of the angular momenta of each mass component.

All transformations of a rigid body can be described in terms of a rotation and a translation. If the body is comprised of more than three non-coplanar masses, then the only transformations that preserve all distances between the masses, and their chirality (so excluding reflections) would be rotations and translations. These transformations form a group known as the *special Euclidean group*.

Motions are time derivatives of these transformations. This means that the motion of a rigid body can always be described in terms of a translation velocity and a time dependent rotation about an axis.

The angular momentum can similarly be decomposed into that of the total mass and that associated with rotation.

It is often convenient to use the center of mass position and velocity and a rotation about the center of mass.

An angular rotation rate about an axis is near identity rotation that happens in a small length of time. Angular velocities live in the space of infinitesimal rotations. The space of possible orientations of a rigid body is described by the continous (Lie) group of rotations in 3D. Angular rotations are infinitesimal generators of this continuous group.

Let's integrate the angular momentum over a mass distribution for a rigid body that is rotating

$$\begin{aligned}
\mathbf{L} &= \int \mathbf{r} \times \mathbf{p} \; dV \\
&= \int \mathbf{r} \times \rho \mathbf{v} \; dV \\
&= \int \mathbf{r} \times \mathbf{v} \; dm
\end{aligned}$$

We need three coordinate systems for points in the body.

- $\mathbf{x}$. Coordinates in inertial frame.
- $\mathbf{x}'$. Coordinates with respect to the center of mass. The body points rotate in this frame.
- $\mathbf{x}_i$. Coordinates with respect to the body itself $\mathbf{x}_i$ and with origin the center of mass. If the body rotates, this frame moves with the body.

To compute this angular momentum integral we chose rotation axis going through the center of mass. We let rotation only take place about the z axis. The body orientation is then specified by rotation angle θ with angular rotation rate $\dot{\theta} = \omega$. The center of mass position and velocity $\mathbf{X}_{cm}, \mathbf{V}_{cm}$. We describe the body's orientation with a rotation matrix $\mathbf{R}(\theta)$.

$$\mathbf{R}(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The rotation matrix gives us

$$\mathbf{x}' = R(\theta)\mathbf{x}_i.$$

Here $\mathbf{x}_i$ are coordinates inside the body in a frame that moves with the body and with origin center of mass and $\mathbf{x}'$ are coordinates with respect to the center of mass.

Inertial coordinates are

$$\mathbf{x} = \mathbf{X}_{cm} + R(\theta)\mathbf{x}_i = \mathbf{X}_{cm} + \mathbf{x}'$$

Let's write this out

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} X_{cm} \\ Y_{cm} \\ Z_{cm} \end{pmatrix} + \begin{pmatrix} \cos\theta x_i - \sin\theta y_i \\ \sin\theta x_i + \cos\theta y_i \\ z_i \end{pmatrix}$$
$$= \begin{pmatrix} X_{cm} \\ Y_{cm} \\ Z_{cm} \end{pmatrix} + \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

The velocity of a particle inside the body at this position

$$\mathbf{v} = \mathbf{V}_{cm} + \mathbf{v}'.$$

We compute the velocity $\mathbf{v}'$

$$\mathbf{v}' = \frac{d\mathbf{x}'}{dt} = \frac{d}{dt}\begin{pmatrix} \cos\theta x_i - \sin\theta y_i \\ \sin\theta x_i + \cos\theta y_i \\ z_i \end{pmatrix}$$
$$= \begin{pmatrix} -\sin\theta x_i - \cos\theta y_i \\ \cos\theta x_i - \sin\theta y_i \\ 0 \end{pmatrix} \dot{\theta}$$

The velocity in the inertial frame

$$\begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = \begin{pmatrix} V_{x,cm} \\ V_{y,cm} \\ V_{z,cm} \end{pmatrix} + \begin{pmatrix} (-\sin\theta x_i - \cos\theta y_i)\dot{\theta} \\ (\cos\theta x_i - \sin\theta y_i)\dot{\theta} \\ 0 \end{pmatrix}$$
$$= \begin{pmatrix} V_{x,cm} \\ V_{y,cm} \\ V_{z,cm} \end{pmatrix} + \omega \begin{pmatrix} -y' \\ x' \\ 0 \end{pmatrix}$$

where $\dot{\theta} = \omega$. The angular momentum per unit mass at a single point is

$$\begin{aligned} l_{\mathbf{x}} &= \mathbf{x} \times \mathbf{v} = (\mathbf{X}_{cm} + \mathbf{x}') \times (\mathbf{V}_{cm} + \mathbf{v}') \\ &= \mathbf{X}_{cm} \times \mathbf{V}_{cm} + \mathbf{X}_{cm} \times (-y', x', 0)\omega + \mathbf{x}' \times \mathbf{V}_{cm} + \mathbf{x}' \times \mathbf{v}' \end{aligned}$$

To find the total angular momentum we integrate over the mass distribution

$$\mathbf{L} = \int \rho dV \left[\mathbf{X}_{cm} \times \mathbf{V}_{cm} + \mathbf{X}_{cm}\omega \times (-y', x', 0) + (x', y', z') \times \mathbf{V}_{cm} + \mathbf{x}' \times \mathbf{v}'\right]$$

The first term is equal to $M\mathbf{X}_{cm} \times \mathbf{V}_{cm}$. This is the angular momentum of the center of mass The two middle terms are zero because the $\mathbf{x}'$ coordinate system is centered at the center of mass. The last term is the angular momentum with respect to the center of mass which we can call the *spin angular momentum*.

$$\mathbf{L} = \mathbf{L}_{cm} + \mathbf{L}_{spin} \tag{14}$$

where

$$\begin{aligned}
\mathbf{L}_{cm} &= M\mathbf{X}_{cm} \times \mathbf{V}_{cm} \\
\mathbf{L}_{spin} &= \int_{body} \rho dV \mathbf{x}' \times \mathbf{v}' \\
&= \int_{body} \rho dV (\mathbf{x} - \mathbf{X}_{cm}) \times (\mathbf{v} - \mathbf{V}_{cm})
\end{aligned}$$

Let's now use our expressions for coordinates with respect to the center of mass.

$$\begin{aligned}
\mathbf{L}_{spin} &= \int_{body} \rho dV \mathbf{x}' \times \mathbf{v}' \\
&= \int \rho dV (x', y', z') \times (-y', x', 0)\omega \\
&= \int \rho dV (-x'z', -y'z', x'^2 + y'^2)\omega
\end{aligned}$$

If the body is nicely aligned then the last term only contributes a component in the z direction.[1] Because we are only allowing rotations about the z axis $x'^2 + y'^2 = x^2 + y^2$. Simplifying we find that

$$\begin{aligned}
\mathbf{L}_{spin} &= \int \rho(x^2 + y^2) dV \omega \hat{\mathbf{z}} \\
&= I\omega\hat{\mathbf{z}}.
\end{aligned}$$

We recognize the right hand term as $I\omega$ but with direction that is along the axis of rotation.

We define a **spin vector** $\boldsymbol{\omega}$ that has direction aligned along the rotation axis and magnitude that is equal to the angular rotation rate about this axis. Using the spin vector, the angular momentum

$$\mathbf{L} = \mathbf{L}_{cm} + I\boldsymbol{\omega}. \tag{15}$$

A more general formulation treats the moment of inertia as a matrix (or tensor), and

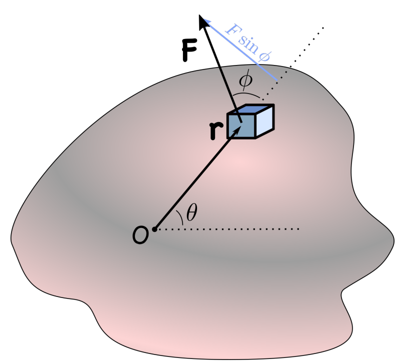
$$\mathbf{L}_{spin} = \mathbf{I}\boldsymbol{\omega}. \tag{16}$$

In this case the angular momentum vector is not necessarily aligned with the spin vector.

[1] This is true if the body is spinning about a principal axis. Otherwise the moment of inertia is a matrix and the spin is not aligned with the angular momentum vector.

Above we considered a rigid body that was moving and rotating. Another setting might be a compound pendulum where the rigid body is allowed to rotate about an axis that does not the center of mass or a top that has a fixed pivot point that is not the center of mass. In this case the angular momentum $L = I\omega$ is also true, but moment of inertia I and angular rotation rate ω are computed about that axis or pivot point.

4 Torque



Consider an external force exerted on a series of point masses that makes up a rigid body. Alternatively we can consider an extended rigid body as a distributed mass where the mass distribution is in the body's frame does not change. We don't only allow the body to rotate about a point 'O' and we keep that point fixed.

We apply a force $\mathbf{F}$ to a small volume with mass m that is inside our mass distribution.

The position vector $\mathbf{r}$ is that from the origin to the small volume.

The angle between $\mathbf{F}$ and the vector from the origin is ϕ.

The angle θ describes body orientation.

The force $\mathbf{F}$ can be decomposed into two components:

A radial component directed along the direction of the position vector $\mathbf{r}$ from the origin. The magnitude of this component is $F \cos\phi$. This component will not produce any motion if the origin is fixed.

We now look at the tangential force component, perpendicular to the direction of the position vector r. The magnitude of this component is $F \sin\phi$. This component causes rotation.

If there is a mass m at the point where the force is applied, but the length between it and the origin (rotation axis) is fixed, and the rest of the object's mass is negligible, then the acceleration on m is $\frac{F}{m} \sin\phi$. The component directed along $\mathbf{r}$ is cancelled by the tension in the body. Because the body is rigid it cannot elongate along $\mathbf{r}$.

The acceleration on m has magnitude $a = \frac{F}{m} \sin\phi$ and it is in the direction of the perpendicular component of force. As the body can only rotate this gives an angular acceleration

$$\alpha = \ddot{\theta} = \frac{d^2\theta}{dt^2}$$

with

$$a = r\ddot{\theta}$$

The angular acceleration

$$\alpha = \frac{F}{mr}\sin\phi = \frac{rF}{mr^2}\sin\phi \tag{17}$$

Note that $mr^2 = I$ is the moment of inertia of m about the origin (neglecting all the rest of the extended mass except m). Our equation 17 is equivalent to the statement: the rate of change of angular momentum is equal to the torque;

$$\frac{d}{dt}\mathbf{L} = \frac{d}{dt}I\boldsymbol{\omega} = \mathbf{r}\times\mathbf{F} = \boldsymbol{\tau}$$

Why is this statement equivalent? Equation 17 can be rewritten as $I\alpha = rF\sin\phi$ using $I = mr^2$. Note that $rF\sin\phi = |\mathbf{r}\times\mathbf{F}|$ as the cross product removes the radial component.

We are now going to justify this statement in more detail!

We define **torque** on a point mass as a vector

$$\boldsymbol{\tau} \equiv \mathbf{r}\times\mathbf{F}. \tag{18}$$

The cross product gives us the tangential force component. As was true for angular momentum, $\mathbf{r}$ is defined with respect to a particular point for the origin.

Recall our definition for angular momentum

$$\mathbf{L} \equiv \mathbf{r}\times\mathbf{p}$$

Take the time derivative of both sides

$$\begin{aligned}
\frac{d\mathbf{L}}{dt} &= \frac{d\mathbf{r}}{dt}\times\mathbf{p} + \mathbf{r}\times\frac{d\mathbf{p}}{dt} \\
&= \mathbf{v}\times\mathbf{p} + \mathbf{r}\times\frac{d\mathbf{p}}{dt} \\
&= \mathbf{r}\times\frac{d\mathbf{p}}{dt} \\
&= \mathbf{r}\times\mathbf{F} = \boldsymbol{\tau}
\end{aligned}$$

The term $\mathbf{v}\times\mathbf{p}$ is zero because $\mathbf{v}$ and $\mathbf{p}$ are in the same direction.

Our definitions for angular momentum and torque are consistent with the rate of change in angular momentum is equal to the torque.

$$\frac{d\mathbf{L}}{dt} = \mathbf{r}\times\mathbf{F} = \boldsymbol{\tau}$$

We now extend our definition for torque on a point mass to cover a rigid body which is rotating about a fixed axis. The angular momentum

$$\mathbf{L} = I\boldsymbol{\omega}.$$

We can take the time derivative of the angular momentum to find the torque

$$\boldsymbol{\tau} = \frac{d\mathbf{L}}{dt} = I\dot{\boldsymbol{\omega}} = I\boldsymbol{\alpha} \tag{19}$$

where $\boldsymbol{\alpha}$ is an angular acceleration.

Note that $d\boldsymbol{\omega}/dt$ can involve changes in the direction of the rotation axis as well as changes in the angular rotation rate.

This is for a rigid body with a constant moment of inertia I.

4.1 A string wrapped around a disk

Consider a uniform disk with mass M and radius R. The disk is mounted on a fixed axle. A block with mass m hangs from a light cord that is wrapped around the rim of the disk.

Find the acceleration of the falling block, a, the angular acceleration of the disk α, and the tension of the cord, T.

The moment of inertia of the disk is

$$I = MR^2/2$$

We adopt a coordinate system with y increasing upward and x increasing to the right. This gives positive $\dot{\theta}$ with positive L_z and counter clockwise rotation.

The force on the mass m is

$$T - mg = ma$$

This gives tension

$$T = m(g + a) \tag{20}$$

We expect $|a| < g$.

The torque on the disk is radius times force with force equal to the tension.

$$\tau = -TR = -m(g + a)R$$

The minus sign is because the tension force points down. This is a torque that turns the disk clockwise and that is consistent with the negative sign for the torque. A negative torque causes clockwise rotation.

$$\tau = I\alpha = \frac{1}{2}MR^2\alpha$$

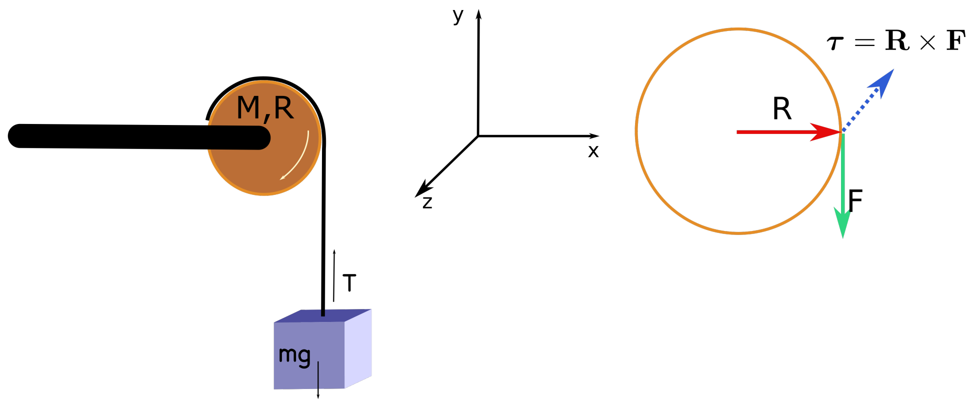


Figure 17: A block with mass m hangs from a light cord that is wrapped around the rim of the disk. The disk has mass M and radius R and can turn without friction about an axle that is fixed. The torque on the disk by the cord is in the -z direction if we place the disk in the xy plane. This causes the disk to rotate in the clockwise direction.

Together

$$-m(g+a)R = \frac{1}{2}MR^2\alpha \tag{21}$$

We are expecting $\alpha < 0$ as the disk should rotate clockwise.

Because the string is wrapped about the disk, there is a relation between the velocity of m and the angular rotation rate of the disk ω. Similar to the case of rolling without slipping, $v_m = \omega R$. This implies that

$$a = \alpha R$$

Both a and α are negative if m drops.

Known quantities are m, R, M, g, I. Unknown quantities are a, α, T.

We replace α with a/R in the torque equation (equation 21) giving

$$\begin{aligned}
-m(g+a)R &= \frac{1}{2}MR^2\frac{a}{R} \\
-mg &= \frac{1}{2}Ma + ma \\
a &= -g\frac{m}{M/2+m} = -g\frac{2m}{M+2m}
\end{aligned}$$

The angular acceleration follows

$$\alpha = \frac{a}{R} = -\frac{g}{R}\frac{2m}{M+2m}$$

The tension also follows from equation 20

$$T = m(g+a) = mg\left(1 - \frac{2m}{M+2m}\right) = mg\frac{M}{M+2m}$$

Notice that tension T and acceleration a do not depend on disk radius R. They do depend on the moment of inertia or the factor I/MR^2.

4.2 Units for angular momentum and torque

Angular momentum is in units of mass distance^2 / time. This is the same as momentum times distance.

Torque is in units of mass distance^2 / time^2 which is the same as energy.

4.3 The compound pendulum

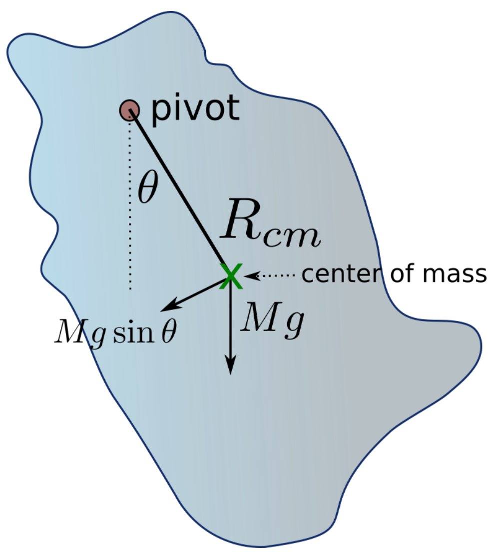


Figure 18: A rigid body of mass M swinging about a pivot point. This is called a compound pendulum.

A swinging rigid (and extended) rigid body that is free to rotate about a fixed horizontal axis is called a **compound pendulum**. Relevant is the moment of inertia about the pivot point I and the radius of the center of mass from the pivot point R_{cm}. The pendulum has total mass M.

If the center of mass position is at an angle θ from the vertical the torque from gravity is

$$\tau = -MgR_{cm}\sin\theta.$$

We set this equal to the rate of change of angular momentum

$$\tau = I\ddot{\theta}.$$

For small oscillations $\sin\theta \sim \theta$ and

$$\ddot{\theta} = -\frac{MgR_{cm}}{I}\theta.$$

This gives angular frequency of oscillations

$$\omega = \sqrt{\frac{MgR_{cm}}{I}}$$

and period of oscillations

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{MgR_{cm}}}.$$

This formula can be used to find the moment of inertia of an object. Alternatively, an object of known moment of inertia and center of mass can be used to measure g.

Note that the moment of inertia I is that about about the pivot point, not that about the center of mass, I_{cm}. How do you relate I with I_{cm}?

Answer: The parallel axis theorem.

4.4 Tugged spool

A spool of string is gently tugged by its string, as shown in Figure 19. The spool is assumed to roll without slipping.

One way to understand this is to turn the spool upside down and think about it as if it were a compound pendulum. Compute the torque around the contact point. For the spool the roll-without slip condition means that instantaneously we can think about the spool as a compound pendulum pivoting about the contact point (see Figure 20).

If you extend a line along the force vector it intersects the ceiling either on the right hand side or left hand side of the pivot point. It only intersects the pivot point if the applied force is radial and in that case it is exactly balanced by the constraint on the pivot. Which side this line intersects the pivot determines the direction of the tangential component of force and so the sign of the torque.

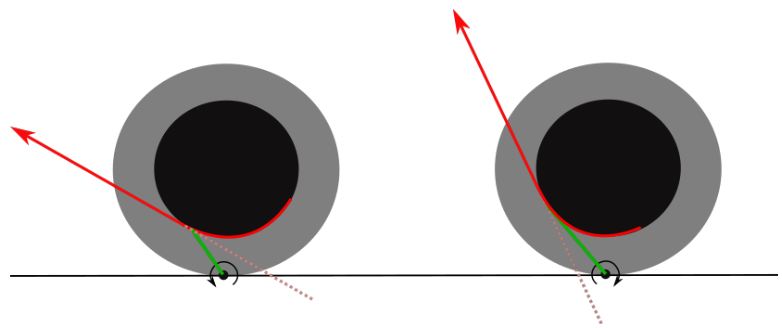


Figure 19: A spool of string is gently tugged by its string. The direction that the spool rolls depends on the angle of the string. In the figure on the left, the spool rolls left, reeling the string in. In the figure on the right, the spool rolls right, reeling the string out. You can find the direction of the torque by considering the direction of $\mathbf{r} \times \mathbf{F}$ where $\mathbf{r}$ is the vector between contact point and tangent point and $\mathbf{F}$ is the tension which is exerted at the point where string stops curving.

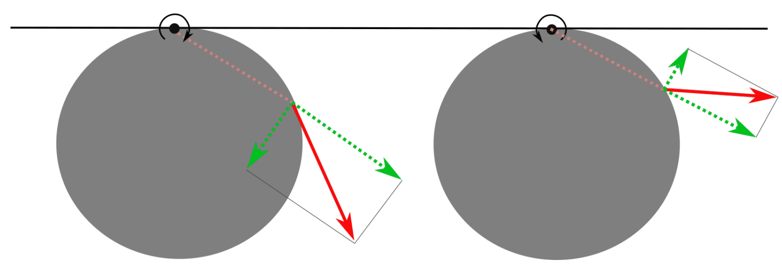


Figure 20: A compound pendulum with black dot as pivot point. The red vectors are applied forces. The direction of motion depends on the tangential component of the applied force, computed with origin at the pivot point.

5 Stability

5.1 Tightrope walkers

A tight-rope or wire-walker may use a horizontal pole for balance. This technique distributes mass away from the pivot point where the feet touch the wire, thereby increasing the moment of inertia. A greater torque is required to tip the walker. Alternatively, at a tip angle θ, torque due to gravity would give a smaller angular acceleration rate because the moment of inertia is larger. The walker can also tip the pole to help adjust his or her balance.

For a colorful local historical example, see https://en.wikipedia.org/wiki/Tightrope_walking#/media/File:Maria_Spelterini_at_Suspension_Bridge.jpg Maria Spelterini

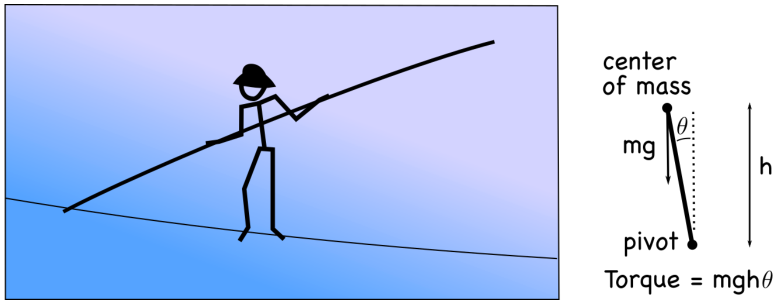


Figure 21: A wire walker using a pole to help balance.

walking across a tightrope across the Niagara Gorge in 1876.

If the tightrope walker's center of mass is a distance h from the wire and tilted by small angle θ from vertical, the torque on the walker with mass m is $\tau \sim mgh\theta$ where g is the gravitational acceleration. The equation of motion for θ is

$$\tau \sim mgh\theta = I\ddot{\theta}.$$

Notice the sign of this equation. It resembles a harmonic oscillator but with the opposite sign. Solutions are exponentials instead of sines and cosines. A solution is

$$\theta(t) = \theta_0 e^{at}$$

where θ_0 is the value of θ at $t = 0$. Here a is a growth rate

$$a = \sqrt{\frac{mgh}{I}}.$$

The larger the moment of inertia, the slower the growth rate. If a is small, the the tightrope walker has a longer time to adjust his or her balance before falling.

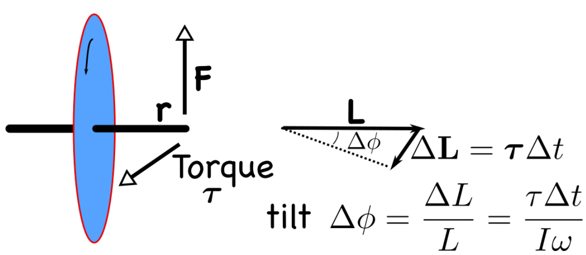


Figure 22: A torque is exerted on a spinning wheel or disk. Force is applied in a direction perpendicular to the axle.

5.2 Tilting a spinning wheel or disk

Consider a wheel that is spinning. A force is exerted perpendicular to the spin axis, as shown in Figure 22 giving a torque that is perpendicular to the spin axis. Because the torque is perpendicular to the spin axis, the spin rate ω does not change, only the axis of rotation changes. Over a time Δt, the spin axis shifts by an angle

$$\Delta\phi = \frac{\Delta L}{L} = \frac{\tau \Delta t}{I\omega},$$

where in the last step we have used the fact that $\tau = \Delta L/\Delta t$. The smaller the initial spin ω, the larger the angle change or tilt. This estimate shows why it is harder to tilt a more rapidly spinning object and why gyroscopes are used to stabilize space craft and scooters.

6 Conservation of Angular momentum

Suppose we have N point masses m_i with positions $\mathbf{x}_i$. They interact with forces $\mathbf{F}_{ij}$. The force from particle i on particle j is equal and opposite to that from j onto particle i. The total force on particle i is that from all other particles

$$\frac{d\mathbf{p}_i}{dt} = \sum_{j\neq i} \mathbf{F}_{ji}$$

The total angular momentum

$$\mathbf{L} = \sum_i \mathbf{x}_i \times \mathbf{p}_i$$

$$\begin{aligned} \frac{d\mathbf{L}}{dt} &= \sum_i \mathbf{x}_i \times \frac{d\mathbf{p}_i}{dt} \\ &= \sum_i \mathbf{x}_i \times \sum_{j\neq i} \mathbf{F}_{ji} \end{aligned}$$

Consider a single pair of particles i, j. There are two terms in the sum

$$\mathbf{x}_i \times \mathbf{F}_{ji} + \mathbf{x}_j \times \mathbf{F}_{ij}$$

However $\mathbf{F}_{ji} = -\mathbf{F}_{ij}$ so we can write the two terms as

$$(\mathbf{x}_i - \mathbf{x}_j) \times \mathbf{F}_{ji}$$

If the force vector is along the direction connecting the two particles then the cross product is zero.

$$(\mathbf{x}_i - \mathbf{x}_j) \times \mathbf{F}_{ji} = 0$$

And this would be true for the sum of the terms, giving

$$\frac{d\mathbf{L}}{dt} = \sum_i \mathbf{x}_i \times \sum_{j \neq i} \mathbf{F}_{ji} = 0$$

As long as all forces are internal forces, then the total angular momentum is conserved. By internal I mean the forces between particle pairs are oriented in the direction connecting the particle pairs. This situation arises naturally for distance dependent conservative force laws, but may not be obeyed by some numerical algorithms.

6.1 An effective potential for Keplerian motion

We have shown that the orbital angular momentum is conserved. More specifically, we showed that the component of angular momentum perpendicular to the orbital plane is conserved.

We consider a small mass of mass m in orbit about a much larger mass M which we assume is at the origin. The energy per unit mass is

$$\frac{E}{m} = \frac{1}{2}v^2 - \frac{GM}{r}$$

We can write the velocity in terms of polar coordinates

$$v^2 = v_r^2 + v_\theta^2$$

The tangential velocity component is related to the angular momentum (from the origin). The orbital angular momentum per unit mass

$$l \equiv \frac{L}{m} = rv_\theta = r^2\dot{\theta}$$

so

$$v_\theta^2 = \frac{l^2}{r^2}$$

giving

$$\begin{aligned}
\frac{E}{m} &= \frac{1}{m}(K+U) \\
&= \left(\frac{1}{2}v_r^2 + \frac{1}{2}v_\theta^2 - \frac{GM}{r}\right) \\
&= \frac{1}{2}v_r^2 + \frac{l^2}{2r^2} - \frac{GM}{r}. \qquad (22)
\end{aligned}$$

We can treat the function

$$U_{\rm eff}(r) = \frac{l^2}{2r^2} - \frac{GM}{r} \qquad (23)$$

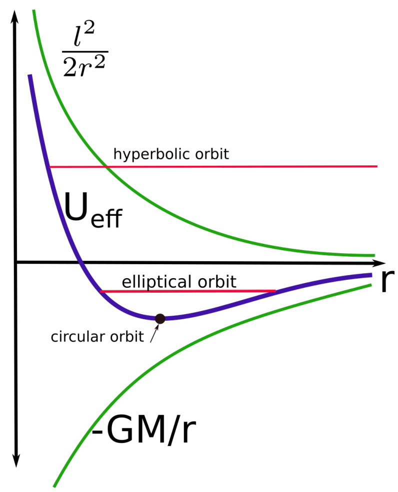


Figure 23: The effective potential $U_{\rm eff}(r) = \frac{l^2}{2r^2} - \frac{GM}{r}$ for a Keplerian orbit.

as an **effective potential energy** per unit mass. This effective potential is that for the 1 dimensional problem for radial motion alone.

In Figure 23 we show the effective potential per unit mass (equation 23) for a given angular momentum value. The total energy, which is the sum of radial component of kinetic energy and the effective potential, is conserved. Once energy E and L are chosen, it is possible to determine what range of radii are available for the orbit. Figure 23 illustrates elliptical, hyperbolic and circular orbits.

6.2 Kepler's first law: elliptical orbits

Kepler's first law is: "The orbit of a planet is an ellipse with the Sun at one of the two foci."

The radial equation of motion for small mass in orbit about M is

$$\ddot{r} = -\frac{dU_{\rm eff}}{dr} = \frac{l^2}{r^3} - \frac{GM}{r^2} \tag{24}$$

where r is the distance from m to M and l is the angular momentum per unit mass.

It is convenient to use a variable that is the inverse radius $u = 1/r$ with

$$\frac{du}{dt} = \dot{u} = -\frac{\dot{r}}{r^2} \tag{25}$$

$$= \frac{du}{d\theta}\frac{d\theta}{dt} \tag{26}$$

using $l = r^2\dot{\theta}$ which is conserved

$$\frac{du}{dt} = \frac{du}{d\theta}\frac{l}{r^2} \tag{27}$$

Putting this together with our previous expression for $\dot{u}$ (equation 25)

$$-\dot{r} = \frac{du}{d\theta} l. \tag{28}$$

Taking the time derivative of this

$$\begin{aligned} -\ddot{r} &= \frac{d}{dt}\frac{du}{d\theta} l = \dot{\theta}\frac{d}{d\theta}\frac{du}{d\theta} l \\ &= \frac{d^2u}{d\theta^2} l^2 u^2 \end{aligned} \tag{29}$$

The equations of motion are

$$\ddot{r} = \frac{l^2}{r^3} - \frac{GM}{r^2} = l^2u^3 - GMu^2. \tag{30}$$

Putting this together with equation 29

$$\frac{d^2u}{d\theta^2} l^2 u^2 \quad = \quad -l^2u^3 + GMu^2$$

This can be simplified to

$$\frac{d^2u}{d\theta^2} + u = \frac{GM}{l^2}. \tag{31}$$

This has a general solution for inverse radius in the form

$$u(\theta) = (1 + e\cos\theta)p^{-1} \tag{32}$$

with constant

$$p = \frac{l^2}{GM}$$

and free parameter e known as the **eccentricity**. Inverting this for radius

$$r(f) = \frac{p}{1 + e\cos f} \tag{33}$$

and we have replaced θ with an equivalent angle $\theta = f$ which is called the **true anomaly**.

Equation 33 is a formula for a conic section, in other words, it describes an ellipse, parabola or hyperbola. If $e = 1$ you get a parabola, if $e > 1$ you get a hyperbola and if $e < 1$ you get an ellipse. Newtonian gravity gives Kepler's first law.

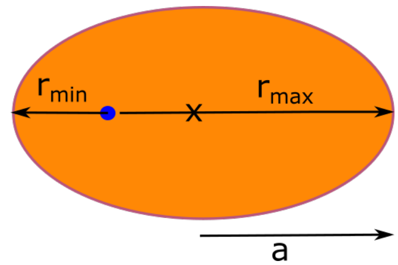


Figure 24: The sum of the maximum and minimum orbital radii of an ellipse are equal to twice the semi-major axis.

For true anomaly $f = 0$ the orbit is at pericenter. The minimum and maximum radius are $r_{min} = p/(1+e)$ and $r_{max} = p/(1-e)$ giving a semi-major axis a

$$2a = \frac{p}{1+e} + \frac{p}{1-e} = \frac{2p}{1-e^2}$$

so that

$$p = a(1-e^2)$$

The orbit is then

$$r(f) = \frac{a(1-e^2)}{1+e\cos f}. \tag{34}$$

We have found the orbit as a function of true anomaly f. It is much hard to find $r(t)$, or radius as a function of time or $f(t)$ true anomaly as a function of time.

With some manipulation it is possible to show that energy per unit mass and angular momentum per unit mass are

$$\begin{aligned} \frac{E}{m} &= -\frac{GM}{2a} \\ l &= \sqrt{GMa(1-e^2)}. \end{aligned}$$

These are appropriate for elliptical orbits. With some generalization a similar description covers parabolic and hyperbolic orbits (e.g., hyperbolic orbits have e greater than 1, $a < 0$).

It turns out that for the Keplerian setting the rotation period is independent of eccentricity. It might be nice to show this here.

6.3 The beetle on the lazy Susan

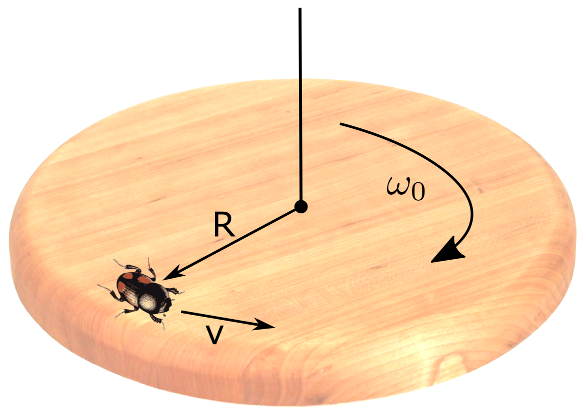


Figure 25: The beetle on the lazy Susan. A little beetle of mass m runs counterclockwise on the rim of a lazy Susan. The lazy Susan is a disk of mass M, radius R, and moment of inertia I that can rotate about its center. Initially the beetle's speed with respect to an inertial frame is v and it is at a radius R from the center of the lazy Susan. The lazy Susan is rotating clockwise with angular speed ω_0.

The beetle abruptly stops.
a) What is the angular rotation rate of the lazy Susan after the beetle stops moving?
b) Is mechanical energy conserved?
Initially the angular momentum

$$L = mvR - I\omega_0$$

Here L is the z component of the angular momentum vector. The signs of the terms are set by the direction of motion with counter clockwise giving a positive value. With this choice, the lazy Susan rotation (which is clockwise) gives a negative angular momentum. Note I am ignoring the spin angular momentum of the beetle – this is valid as the beetle is much smaller than the lazy Susan.

Afterward the beetle stops moving and the angular momentum

$$L = mR^2\omega + I\omega.$$

The forces that bring the beetle to a stop are internal to the system, so angular momentum is conserved. Angular momentum is conserved so we set these expressions to be equal and solve for ω

$$\omega = \frac{mvR - I\omega_0}{mR^2 + I}$$

This is the angular rotation rate of the lazy Susan after the beetle stops moving. Note $\omega < 0$ for clockwise rotation and $\omega > 0$ for counterclockwise rotation.

Now we check the rotational energy. Initially

$$E_0 = \frac{1}{2} I\omega_0^2 + \frac{1}{2} mv^2$$

Afterwards

$$E_1 = \frac{1}{2} I\omega^2 + \frac{1}{2} m(R\omega)^2$$

We compute the difference

$$\Delta E = -\frac{1}{2}\frac{mI}{I + mR^2}(v + R\omega_0)^2 < 0$$

Bringing the beetle to a stop involved friction and likely went into heat. Energy was lost from the system.

6.4 Grazing sticky impacts with a sphere

6.4.1 Sphere pivots about its center

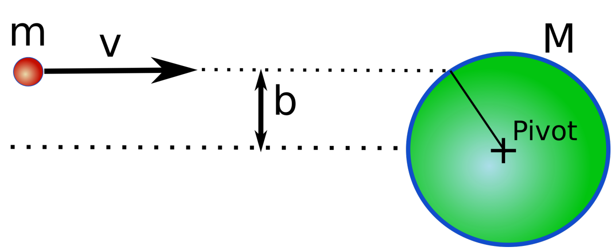


Figure 26: A non-normal or grazing impact. The mass m sticks to sphere M and M is held so it pivots about its center.

Let's consider a grazing impact with point mass m and sphere M. The sphere M is constrained so it pivots about its center and the pivot is fixed. Let's assume that M is initially not rotating and it is a homogeneous sphere of radius R and moment of inertia about its center of mass is $I_M = \frac{2}{5}MR^2$. We let m be a point mass. The impact parameter is b and m has initial velocity v.

We let m stick to M when it collides.

What is the angular rotation rate of the system after the collision?

Angular momentum is conserved. The moment of inertia of the combined mass about the center of mass of M is

$$I = \frac{2}{5}MR^2 + mR^2$$

We set

$$L = mbv = I\omega$$

and solve for

$$\omega = \frac{mbv}{\frac{2}{5}MR^2 + mR^2}$$

What direction is the rotation?

Putting the collision in the xy plane, with $\mathbf{v} = v\hat{\mathbf{x}}$ and vector b in the y direction

$$\mathbf{L} = mvb\hat{\mathbf{y}} \times v\hat{\mathbf{x}} = -mvb\hat{\mathbf{z}}$$

the angular momentum is in the -z direction. This gives clockwise rotation.

6.4.2 Sphere is free to move

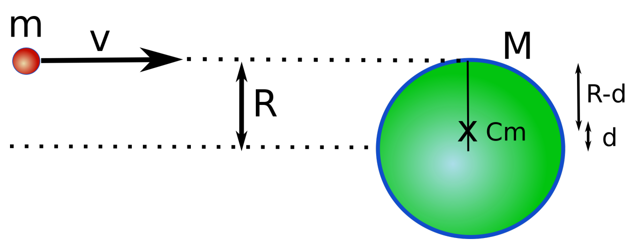


Figure 27: A non-normal or grazing impact. Here sphere M is free to move. The impact parameter $b = R$ and point mass m sticks to M at the moment of contact.

Again we consider an impact between point mass m and uniform density sphere M. We let sphere M be free to move and adjust the impact parameter to be equal to $b = R$. Again m sticks to M at the moment of impact.

What is the velocity and angular rotation rate of the system after the collision?

Momentum is conserved so the final translational velocity is to the right and equal to that of the center of mass initially.

$$V_{cm} = \frac{m}{M+m}v$$

Rotation takes place about the center of mass of the combined system. The center of mass when the two collide is a distance

$$d = \frac{m}{m+M}R$$

from the center of mass of M. The moment of inertia about the center of mass for the combined system is

$$I = \frac{2}{5}MR^2 + Md^2 + m(R-d)^2$$

where I have used the parallel axis theorem.

Angular momentum is conserved. We take origin to be the center of mass

$$L = m(R-d)v = I\omega$$

and solve for

$$\omega = \frac{m(R-d)v}{\frac{2}{5}MR^2 + Md^2 + m(R-d)^2}$$

Once again rotation is clockwise.

6.5 Elroy's Beanie

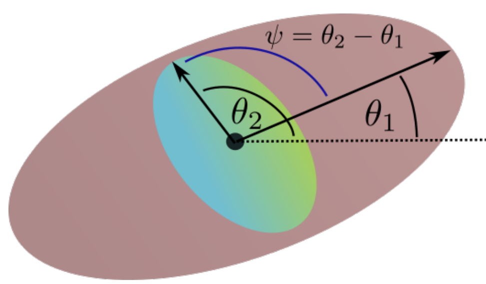


Figure 28: Elroy's Beanie is a nice example illustrating how internal rotations can affect body orientation. Assuming angular momentum conservation and that the object initially has angular momentum of zero, a full rotation in ψ of the inner oval with respect to the outer one, gives a non-zero change in $\Delta\theta_1$, the orientation of the outer oval.

We consider two oval solid bodies connected with a pivot point at their center of mass, see Figure 28. The angle of the outer oval body, with moment of inertia I_1 is oriented with respect to the inertial frame with angle θ_1. The inner oval body, with moment of inertia

I_2, is oriented with respect to the inertial frame with angle θ_2. The angle between the two ovals is

$$\psi = \theta_2 - \theta_1.$$

The total configuration space is described by two angles. The two bodies with respect to one another can be described by ψ alone. So we can think of the internal freedom as described by angle ψ.

Changes in body can be described as a torque from one oval by the other. The torque exerted by the inner one on the outer one would be equal and opposite to that exerted on the outer one by the inner one.

We describe the orientation of the body with respect to the outside with $\theta = \theta_1$. The total angular momentum

$$L = I_1\dot{\theta}_1 + I_2\dot{\theta}_2 = I_1\dot{\theta} + I_2(\dot{\psi} + \dot{\theta})$$

Assume that angular momentum L is initially zero and is conserved, then

$$I_1 d\theta + I_2(d\psi + d\theta) = 0,$$

giving

$$d\theta = -\frac{I_1}{I_1 + I_2} d\psi$$

If we move around a loop in ψ

$$\Delta\theta = \int_0^{2\pi} -\frac{I_1}{I_1 + I_2} d\psi = -\frac{2\pi I_1}{I_1 + I_2} \tag{35}$$

An internal rotation of the smaller oval gives a partial rotation of the entire mechanism.

A closed loop in body configuration space gives a translation in the body orientation space.

Internal deformations of the body can cause body rotation. An example is a falling cat who can reorient him/her self by twisting his/her body.

Many locomotion systems can be described as loops in body configuration space that give translations in real space.

7 Precession of a gyroscope or top

We consider a spinning top that is tilted at an angle with respect to the vertical direction, as shown in Figure 29. The angle θ is the angle between the spin vector (or axis of rotation) and the vertical direction. R_{cm} is the distance between the top tip (where it touches the table) to its center of mass. Let

$$\mathbf{R}_{cm} = R_{cm}(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

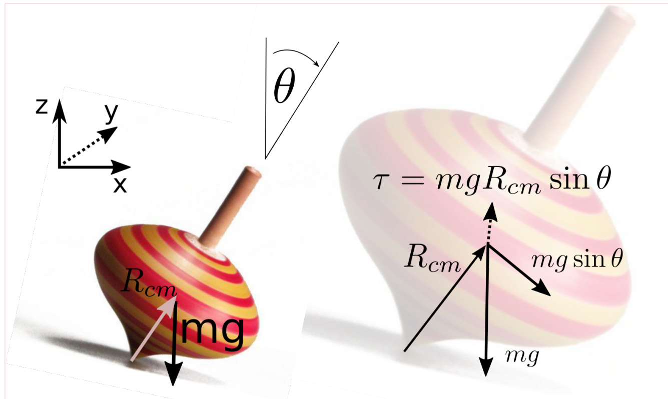


Figure 29: A top! The torque is perpendicular to the spin vector and the downward gravity vector. On this figure the torque due to gravity is toward the viewer (-y direction). This causes the top to precess.

where ϕ gives the orientation of the top spin axis projected onto the xy plane and θ gives how far the top is tilted over.

Gravity exerts a torque on the top. We compute the torque

$$\begin{aligned}\boldsymbol{\tau} &= \mathbf{R}_{cm} \times -mg\hat{\mathbf{z}} \\ &= R_{cm}mg\sin\theta(\cos\phi, \sin\phi, 0) \times (0, 0, -1) \\ &= R_{cm}mg\sin\theta(-\sin\phi, \cos\phi, 0)\end{aligned}$$

The torque is perpendicular to the angular momentum, so the angular momentum changes direction without changing its magnitude. The top does not speed up or slow down but the angular momentum vector which is the spin axis moves in a circle. The direction of the torque is horizontal so θ remains fixed.

Let us describe $\mathbf{L}$ as a vector

$$\mathbf{L} = L(\cos\phi\sin\theta, \sin\phi\sin\theta, \cos\theta) \tag{36}$$

It is in the same direction as $\mathbf{R}_{cm}$. Because the torque is in the horizontal direction, ϕ varies. We set $\phi = \Omega t$ where we call Ω the **precession** rate.

Taking the time derivative of the angular momentum vector (equation 36 with θ constant but ϕ depending upon time) we find

$$\frac{d\mathbf{L}}{dt} = L\Omega\sin\theta(-\sin\phi, \cos\phi, 0)$$

We set this to be equal to the torque

$$L\Omega\sin\theta = mgR_{cm}\sin\theta$$

and we can solve for the precession rate Ω

$$\Omega = \frac{mgR_{cm}}{L} = \frac{mgR_{cm}}{I\omega} \tag{37}$$

The precession rate increases with a lower spin or ω value! As the top slows down, it precesses more quickly.

8 Quantization of angular momentum

The units of angular momentum are kg m^2/s. If we multiply by a frequency then we have units of energy. Energy in quantum mechanical systems are quantized in units of $\hbar\omega$. This means that angular momentum has the same units as $\hbar$.

Consider a particle with momentum p that is confined to radius r. The de Brogelie wavelength

$$\lambda = \frac{h}{p}$$

Let's see if we can fit one wavelength λ into the circumference of a circle or $2\pi r$. This gives

$$\frac{h}{p} = 2\pi r$$

$$pr = \frac{h}{2\pi} = \hbar$$

We recognize pr as angular momentum.

We could fit more wavelengths into a circle.

This is not at all rigorous, but the main point is that angular momentum is quantized, and its quantum values tend to be in units of $\hbar$.

We can describe a planet as having both orbital and spin angular momentum. Likewise particles can be described as having both spin and orbital angular momentum. When they are alone they only have spin angular momentum and it has to be in units of $1/2\ \hbar$. The behavior of half integer spin particles is different than those with integer spin values.

Electrons by themselves have spin angular momenta of $\pm\hbar/2$ in a particular component. When in the hydrogen atom, the orbital angular momentum contributes a quantum number that specifies a eigenstate solution to Shrodinger's equation.

9 Summary

Angular momentum

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}$$

Torque

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} = \frac{d\mathbf{L}}{dt}$$

For a rigid body, the moment of inertia

$$I = \int r^2 dm$$

where r is the distance from the axis of rotation and dm are mass elements.

The parallel axis theorem

$$I_{new} = I_{cm} + Md^2$$

The moment of inertia about an axis can be computed from the moment of inertia about the center of mass and the distance d between center of mass and the new rotation axis.

The total angular momentum of a rigid body is a sum of translational and spin angular momentum.

$$\mathbf{L}_{tot} = \mathbf{L}_{cm} + \mathbf{L}_{spin}$$

The spin angular momentum of a rigid body depends on the mass distribution, rotation axis and rotation rate

$$\mathbf{L}_{spin} = I\boldsymbol{\omega}$$

where I is computed about the axis of rotation, the angular rotation rate is the length of a spin vector $\boldsymbol{\omega}$ that is aligned with the rotation axis, $\dot{\theta} = |\boldsymbol{\omega}|$. Here we are assuming principal axis rotation so that $\mathbf{L}$ is in the same direction as $\boldsymbol{\omega}$. The torque on the rigid body is

$$\boldsymbol{\tau} = \frac{d\mathbf{L}}{dt}$$

If the torque is applied along (is parallel to) the axis of rotation then $\tau = I\dot{\omega} = I\ddot{\theta}$.

The rolling without slipping constraint is $V = R\omega$ where V is the velocity of the center of mass of a wheel with radius R and angular rotation rate ω.

When there are no external forces, the total angular momentum is conserved.

The total kinetic energy of a moving rigid body is the sum of the kinetic energy of the center of mass and the rotational kinetic energy.

$$K = \frac{1}{2}MV_{cm}^2 + \frac{1}{2}I\omega^2$$

Here the moment of inertia I is computed about an axis that contains the center of mass and $\omega = \dot{\theta}$ is the angular rotation rate about this axis.