PHY141 Lectures 14,15 notes

Alice Quillen

October 25, 2022

Contents

1 Collisions of two objects

We send two particles toward each other. When they are in contact, we say a collision is taking place. The force between them is strong during their contact phase. The change in momentum is $\Delta p = \int F dt$. Sometimes $J = \int F dt$ is called the *collision impulse* and it is in units of momentum.

The result of the collision is a large change in the momentum of the colliding particles. The change in momentum takes place during a short time. Before and after the strong interaction phase we assume the particles are ballistic. That means their energy is kinetic only (if non-relativistic) and γmc^2 only if relativistic.

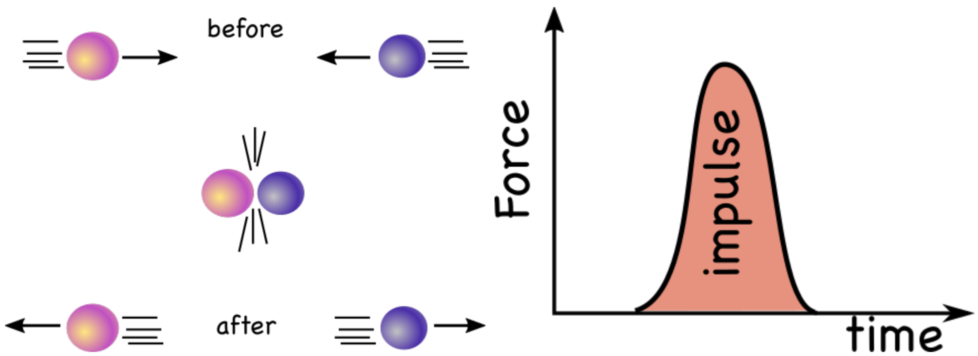

Figure 1: Collisions and impulse during contact.

Consider how long it takes a projectile to stop. The total momentum must go to zero during this time. If the stopping time is short then the force is high. If you increase the stopping time, then you would decrease the force. Air bags, crushable foam and metal, and padding are designed to increase the time of momentum changes during impacts.

Collisions involve interactions between two systems of particles. The forces can be considered internal so the total momentum must be conserved.

Energy need not be conserved. If energy is lost then the collision is called *inelastic*. If energy is conserved then the collision is *elastic*. For inelastic collisions energy is lost into heat, deformation, vibrations, particle ejection or radiation.

Figure 2: Strobe light illumination of a ball bouncing on a flat surface. This image is by Michael Maggs with Edits by Richard Bartz and from Bouncing ball strobe edit.jpg under the license Creative Commons Attribution-Share Alike 3.0 Unported license.

An example is a ball bouncing on a flat surface. The ball is pushing on something much more massive than it. Momentum is conserved, but the Earth barely moves. At impact the vertical component of velocity is reversed and multiplied by the coefficient of restitution COR $v_z \to -COR \times v_z$. If the coefficient of restitution is 1 then kinetic energy is conserved. Otherwise kinetic energy is reduced by COR^2 during each impact.

2 Non-relativistic elastic collisions in 1 dimension

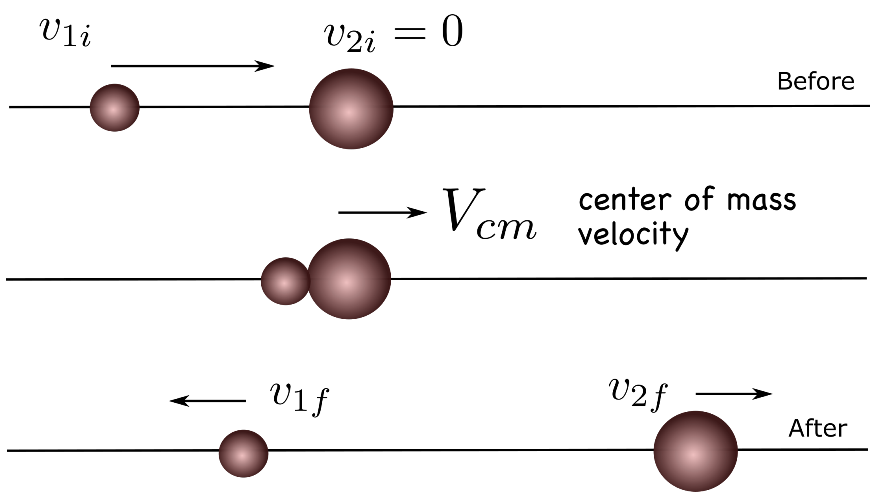

Figure 3: A collision in 1 dimension. In the lab frame.

Consider two point masses m_1, m_2 undergoing a collision in one dimension. The initial and final velocities of m_1 are v_{1i}, v_{1f}. The initial and final velocities of m_2 are v_{2i}, v_{2f}. We assume that the particle masses are not changed during the collisionl The total momentum is conserved.

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f}$$

The center of mass velocity is

$$V_{cm} = \frac{1}{m_1 + m_2}(m_1 v_{1i} + m_2 v_{2i}) = \frac{1}{m_1 + m_2}(m_1 v_{1f} + m_2 v_{2f})$$

Conservation of momentum and that the center of mass velocity remained unchanged are equivalent.

If the collision is elastic then the total kinetic energy remains unchanged.

$$K = \frac{1}{2} m_1 v_{1i}^2 + \frac{1}{2} m_2 v_{2i}^2 = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2$$

It can be helpful to write the kinetic energy as

$$K = \frac{1}{2}MV_{cm}^2 + \frac{1}{2}\mu(v_1 - v_2)^2$$

where the reduced mass is $\mu = m_1 m_2/M$ and total mass $M = m_1 + m_2$.

If we know the masses and initial velocities, conservation of momentum and conservation of kinetic energy give two equations in two unknowns (v_{1f}, v_{2f}). To solve for the final velocities in terms of the initial ones, is usually a tedious calculation. However there are two frames in which this calculation is faster.

2.1 In a lab-frame with one of the masses initially not moving

We set the initial velocity of the second mass to be zero; $v_{2i} = 0$. Conservation of momentum and kinetic energy become

$$\begin{aligned} m_1 v_{1i} &= m_1 v_{1f} + m_2 v_{2f} \\ \frac{1}{2}m_1 v_{1i}^2 &= \frac{1}{2}m_1 v_{1f}^2 + \frac{1}{2}m_2 v_{2f}^2. \end{aligned}$$

We solve for the final velocities in terms of the initial velocity of the first mass v_{1i}. We write the two equations as

$$\begin{aligned} m_1(v_{1i} - v_{1f}) &= m_2 v_{2f} \\ m_1(v_{1i}^2 - v_{1f}^2) &= m_2 v_{2f}^2 \\ m_1(v_{1i} - v_{1f})(v_{1i} + v_{1f}) &= m_2 v_{2f}^2 \\ (v_{1i} + v_{1f}) &= v_{2f} \end{aligned}$$

Insert this back into the first equation

$$\begin{aligned} m_1 v_{1i} &= m_1 v_{1f} + m_2(v_{1i} + v_{1f}) \\ v_{1f}(m_1 + m_2) &= (m_1 - m_2)v_{1i} \end{aligned}$$

Finally

$$\begin{aligned} v_{1f} &= \frac{m_1 - m_2}{m_1 + m_2} v_{1i} \\ v_{2f} &= \frac{m_1}{m_2}(v_{1i} - v_{1f}) = \frac{2m_1}{m_1 + m_2} v_{1i} \end{aligned} \tag{1}$$

Do these formulas make sense? Let's check by consider a limit, $m_2 \gg m_1$:

$$\begin{aligned} \lim_{\frac{m_2}{m_1} \to \infty} v_{1f} &= -v_{1i} \\ \lim_{\frac{m_2}{m_1} \to \infty} v_{2f} &= 0 \end{aligned}$$

This limit makes sense as m_2 does not move if it is massive.

Let's check when $m_1 = m_2$

$$\lim_{\frac{m_2}{m_1} \to 1} v_{1f} = 0$$

$$\lim_{\frac{m_2}{m_1} \to 1} v_{2f} = v_{1i}$$

This limit also makes sense as it is similar to the setting with Newton's cradle.

2.2 Newton's cradle

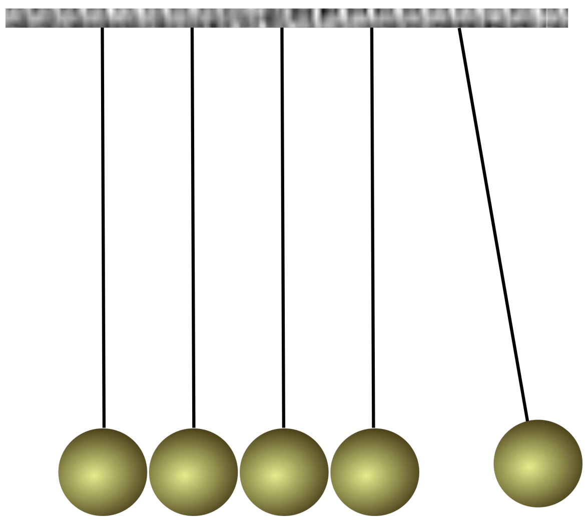

Figure 4: An illustration of Newton's cradle. Note that there is a small space between each ball.

Consider Newton's cradle, shown in Figure 4. All the balls have the same mass. The ball on the right strikes the one second from right. If this ball is not touching any other balls and the collision is elastic, then the second ball recoils with the same velocity and the first one is left at rest. This reaction propagates through the system, eventually ejecting the leftmost ball on the left at the same speed as the rightmost one had initially.

However if the left 4 balls are touching, then the collision is different. We can model the system as two masses $m_1 = m$ and $m_2 = 4m$. We use equations 1 to compute the final velocities

$$v_{1f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1i} = \frac{1-4}{1+4} v_{1i} = -\frac{3}{5} v_{1i}$$
$$v_{2f} = \frac{2m_1}{m_1 + m_2} v_{1i} = \frac{2}{1+4} v_{1i} = \frac{2}{5} v_{1i}$$

The result is different than you expect with Newton's cradle. Newton's cradle is designed with small gaps between balls so that the collisions between each pair of balls are separated in time.

2.3 In the center of mass frame

We go back to the problem of an elastic collision between two masses m_1, m_2 with initial and final velocities $v_{1i}, v_{2i}, v_{1f}, v_{2f}$ in the lab frame. In the center of mass frame we write the velocities as $u_{1i}, u_{2i}, u_{1f}, u_{2f}$.

In the center of mass frame

$$m_1 u_{1i} + m_2 u_{2i} = m_1 u_{1f} + m_2 u_{2f} = 0$$

and kinetic energy

$$K = \frac{1}{2}\mu(u_{1i} - u_{2i})^2) = \frac{1}{2}\mu(u_{1f} - u_{2f})^2$$

where the reduced mass $\mu = \frac{m_1 m_2}{m_1 + m_2}$.

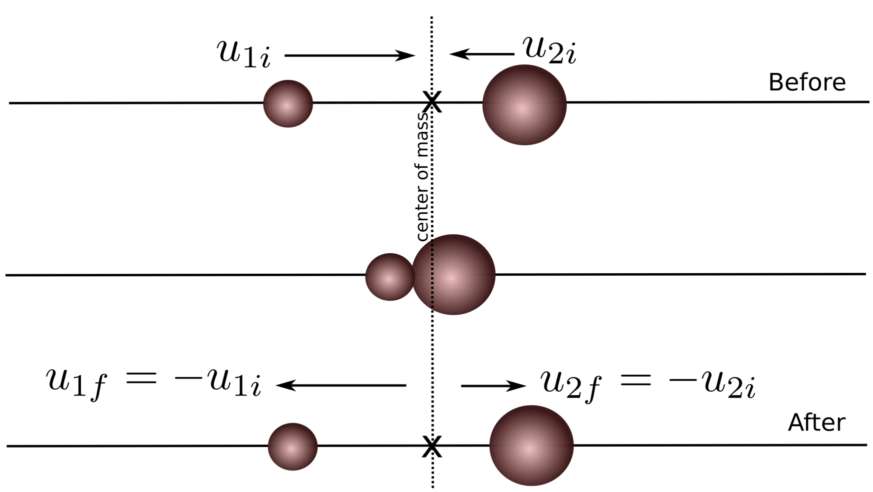

Figure 5: A collision in 1 dimension in the center of mass frame.

Since the total momentum is zero

$$u_{2i} = -\frac{m_1}{m_2}u_{1i}$$
$$u_{2f} = -\frac{m_1}{m_2}u_{1f}$$

Inserting these into our expression for kinetic energy

$$\left(u_{1i} + \frac{m_1}{m_2}u_{1i}\right)^2 = \left(u_{1f} + \frac{m_1}{m_2}u_{1f}\right)^2$$
$$\left(1 + \frac{m_1}{m_2}\right)^2 u_{1i}^2 = \left(1 + \frac{m_1}{m_2}\right)^2 u_{1f}^2$$
$$|u_{1i}| = |u_{1f}|$$

Likewise by inserting expressions for u_{2i}, u_{2f} instead of u_{1i}, u_{1f} into the kinetic energy expression we can show that

$$|u_{2i}| = |u_{2f}|$$

The solution is

$$u_{1i} = -u_{1f}$$
$$u_{2i} = -u_{2f}. \tag{2}$$

The velocities rebound perfectly in the center of mass frame.

We can show that the results of our calculation in lab frame and center of mass frame are equivalent. In the previous example the velocity of the center of mass

$$V_{cm} = \frac{1}{m_1 + m_2}(m_1 v_{1i} + m_2 v_{2i}) = \frac{m_1 v_{1i}}{m_1 + m_2}$$

because the second mass is initially at rest. Lets go into the center of mass frame

$$\begin{aligned} u_{1i} &= v_{1i} - V_{cm} \\ &= v_{1i}\frac{m_1 + m_2}{m_1 + m_2} - v_{1i}\frac{m_1}{m_1 + m_2} \\ &= \frac{m_2 v_{1i}}{m_1 + m_2} \end{aligned}$$

After the collision (using equation 2)

$$u_{1f} = -u_{1i} = -\frac{m_2 v_{1i}}{m_1 + m_2}$$

Now we go back into the lab frame

$$\begin{aligned} v_{1f} &= u_{1f} + V_{cm} \\ &= -\frac{m_2 v_{1i}}{m_1 + m_2} + \frac{m_1 v_{1i}}{m_1 + m_2} \\ &= \frac{m_1 - m_2}{m_1 + m_2} v_{1i} \end{aligned}$$

and this is identical to the expression we found in a previous section (equation 1) where we worked in the lab frame rather than center of mass frame.

2.4 Qualitatively what do we mean by the center of mass frame?

Think about sitting in a train at night. You look out your window and see an experiment taking place inside another train. The lights are on in the other train and you can see inside it. Because it is dark outside you can't tell how fast you are moving with respect to the underlying scenery.

Suppose the train carrying the experiment has a collision as we described in the "lab frame". If your train is moving at the same speed as the other train then you would see the same pre- and post-collision velocities as measured inside the train carrying the experiment.

However, if your train is moving with the center of mass of the two colliding objects, then then you would see the collision as viewed in the "center of mass frame".

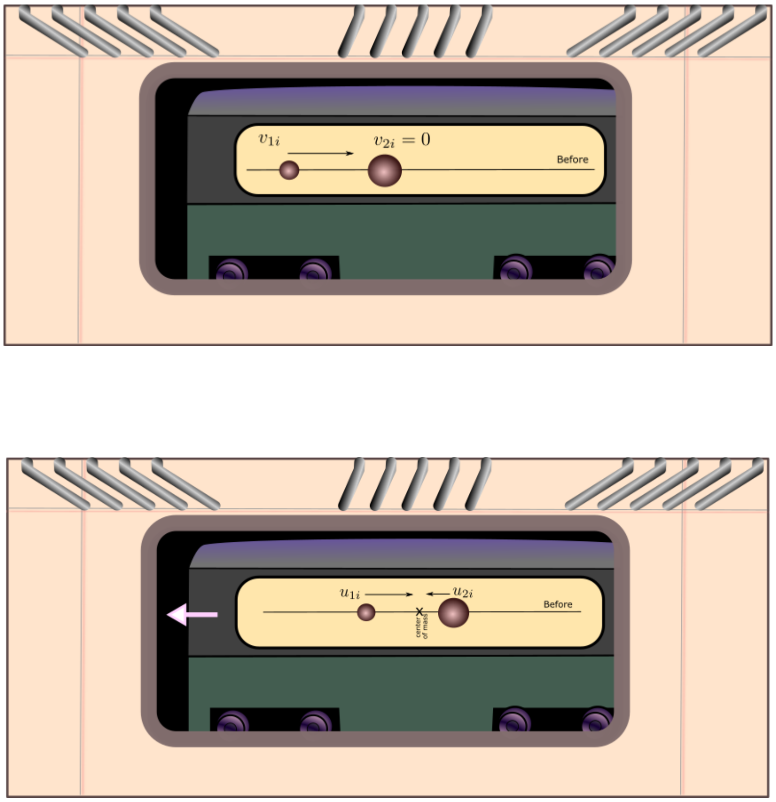

Figure 6: A collision in 1 dimension takes place in a train at night. A viewer is in a different train and looks at the collision through the windows of both trains. The velocities of the objects seen by the viewer depends on the object velocities with respect to the experiment's train and the relative velocity of the two trains. On the top, both trains are moving at the same speed and the viewed velocities are v_{1i}, v_{2i}, the same as viewed in the experiment's train. On the bottom the experiment's train appears to be moving to the left and the velocities seen by the viewer are u_{1i}, u_{2i}. The viewer's train is moving with the collision's center of mass.

3 Non-relativistic anelastic collisions in 1 dimension where the particles stick together

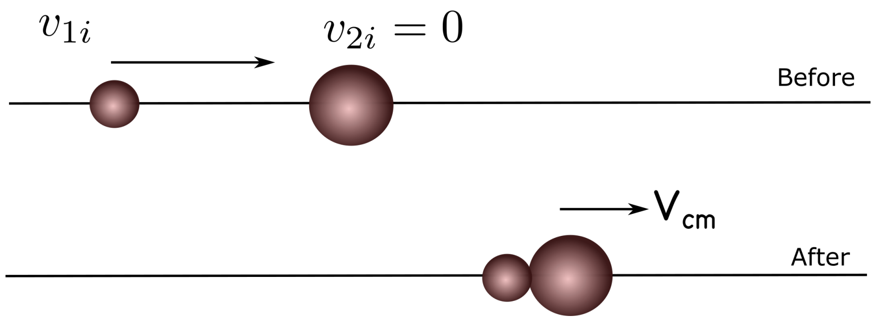

Figure 7: An anelastic collision in 1 dimension. The particles stick together during the collision.

An extremely inelastic collision is one where the two particles stick together. We consider two masses m_1, m_2. The first mass has initial velocity v_{1i}. The second has $v_{2i} = 0$. Afterwards they have the same velocity. This gives us an additional constraint that lets us solve for the final velocity.

Conservation of linear momentum implies that the velocity after the collision is equal to the center of mass velocity.

$$v_f = V_{cm} = \frac{m_1 v_{1i}}{m_1 + m_2}$$

The kinetic energy prior to the collision is

$$K_i = \frac{1}{2} m_1 v_{1i}^2$$

The kinetic energy after the collision is

$$\begin{aligned} K_f &= \frac{1}{2}(m_1 + m_2) V_{cm}^2 \\ &= \frac{1}{2}(m_1 + m_2) \left(\frac{m_1 v_{1i}}{m_1 + m_2} \right)^2 \\ &= \frac{1}{2} \frac{m_1^2}{m_1 + m_2} v_{1i}^2 \end{aligned}$$

The kinetic energy afterwards is less than that initially.

Let's compute the change in kinetic energy

$$\begin{aligned}\Delta K = K_i - K_f &= \frac{1}{2} m_1 v_{1i}^2 \left(1 - \frac{m_1}{m_1 + m_2}\right) \\ &= \frac{1}{2} m_1 v_{1i}^2 \left(\frac{m_2 + m_1 - m_1}{m_1 + m_2}\right) \\ &= \frac{1}{2} v_{1i}^2 \frac{m_1 m_2}{m_1 + m_2}\end{aligned}$$

We recognize the reduced mass. The velocity v_{1i} is also the relative velocity. The kinetic energy lost is equal to the relative kinetic energy of the two body system. Prior to the collision we can describe the total kinetic energy as a sum of translational and relative kinetic energies. When the two particles stick, the relative kinetic energy is lost, leaving only the translational kinetic energy.

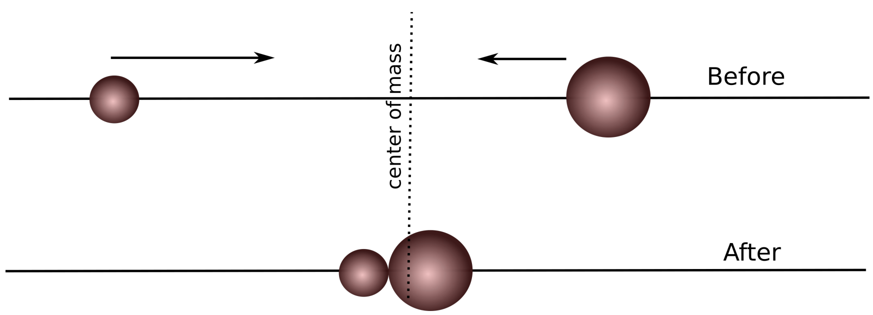

Figure 8: The same anelastic collision in 1 dimension but in the center of mass frame. The particles stick together during the collision.

4 Non-relativistic collisions in three dimensions

Collisions in three-dimensions are similar to those in 1-dimension. Each component (x,y,z) of the total linear momentum is conserved. If the collision is elastic, then the kinetic energy is also conserved.

4.1 Disintegration in 2D

Instead of two particles sticking we consider the opposite processes, a single particle that disintegrates or explodes.

We consider the disintegration shown in Figure 9. M splits into three pieces M_1, M_2, M_3. The trajectories lie in the xy plane. The initial velocity of M is V and is along the x axis.

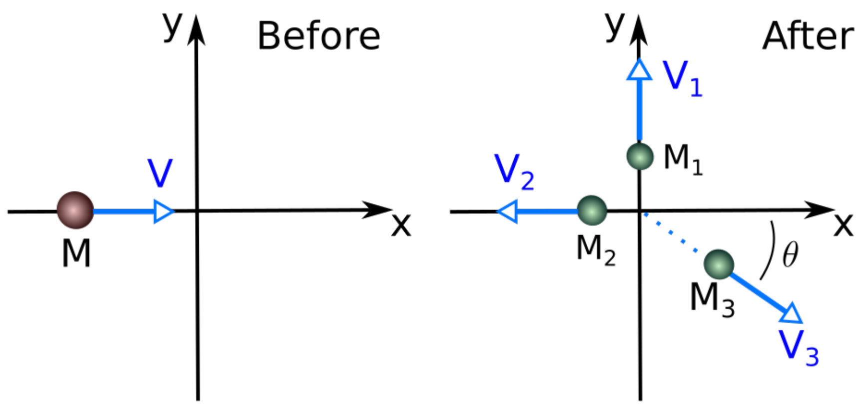

Figure 9: A mass M with velocity V disintegrates into three pieces M_1, M_2, M_3.

M_2 rebounds directly along M's inward trajectory on the x axis. M_1's velocity is perpendicular to M_2's and along the y-axis. The angle between M_3's velocity and the x-axis is θ.

Suppose we measure M, V, M_1, $\mathbf{v}_1$, M_2, $\mathbf{v}_2$. We assume that $M_3 = M - M_1 - M_2$ so no mass is lost. We don't know the magnitude or direction of the velocity of the third mass, $\mathbf{v}_3$.

What is the magnitude of the velocity $\mathbf{v}_3$ of M_3?

The component of momentum in the x direction is conserved

$$MV = M_3 v_3 \cos\theta - M_2 v_2$$

The component of momentum in the y direction is conserved

$$0 = M_1 v_1 - M_3 v_3 \sin\theta$$

We rewrite these as

$$\begin{aligned} M_3 v_3 \cos\theta &= M_2 v_2 + MV \\ M_3 v_3 \sin\theta &= M_1 v_1 \end{aligned} \tag{3}$$

Summing the square of these two equations

$$\begin{aligned} (M_3 v_3)^2 &= (M_2 v_2 + MV)^2 + (M_1 v_1)^2 \\ v_3 &= \frac{1}{M_3}\sqrt{(M_2 v_2 + MV)^2 + (M_1 v_1)^2} \end{aligned} \tag{4}$$

We have found the velocity v_3!

Using equations 3 again we can also solve for the angle θ

$$\theta = \mathrm{atan2}(M_1 v_1, M_2 v_2 + MV). \tag{5}$$

We have found the velocity of a particle that might not have been easy to observe using the velocities and masses of the other particles involved in a decay.

5 Relativistic collisions or reactions

The sum of four-vectors of all particles going into and out of a collision or reaction is conserved. The total four vector

$$(E, \mathbf{p})_{total} = \sum_i (E_i, \mathbf{p}_i)$$

The energy of each particle is the relativistic version

$$E_i = \gamma_i m_i c^2$$

and the momentum is also the relativistic version

$$\mathbf{p}_i = \gamma_i m_i \mathbf{v}_i$$

If a particle is massless, like a photon then $p_i = E_i/c$.

In other words before and after sums are the same.

$$\sum_i E_{i,initial} = \sum_j E_{j,final} \tag{6}$$

$$\sum_i \mathbf{p}_{i,initial} = \sum_j \mathbf{p}_{j,final} \tag{7}$$

The total number of particles or rest masses of the particles need not be the same before and after the collision or reaction.

Lorenz transformations are used to transfer between reference frames.

5.1 Absorption of a photon

A massive particle of rest mass m and initially at rest, absorbs a photon of energy E_γ. Afterwards, the particle has increased in mass and it gains a velocity.

What is the particle's momentum, p_f and energy E_f afterwards?

The initial total energy is the sum of the particle and photon's energy

$$E = E_\gamma + mc^2 = E_f. \tag{8}$$

This is equal to the particle's energy afterwards because energy is conserved.

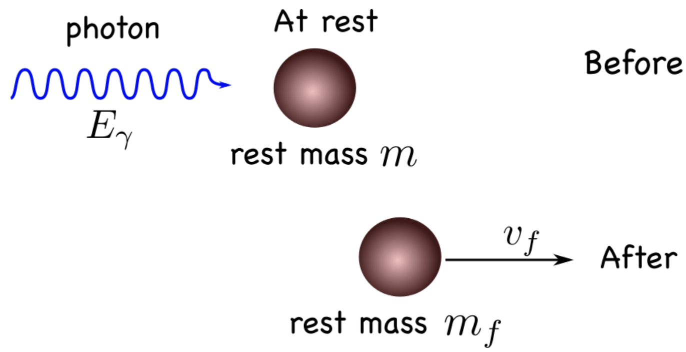

Figure 10: A particle of rest mass m absorbs a photon of energy E_γ. The particle is initially at rest. Afterwards the particle has rest mass m_f and is moving at velocity v_f.

The total momentum is that of the photon alone because initially the particle is at rest and has zero momentum.

$$P = \frac{E_\gamma}{c} = p_f. \tag{9}$$

This is equal to the particle's momentum afterwards because momentum is conserved.

What are the particle's final rest mass m_f and velocity, v_f?

The energy and momentum of the particle afterwards

$$\begin{aligned} E_f &= \gamma_f m_f c^2 \\ p_f &= \gamma_f m_f v_f \end{aligned}$$

where γ_f is the Lorenz factor of the particle. Taking the ratio of these and using equations 8 and 9 we find

$$v_f = \frac{p_f c^2}{E_f} = c \frac{E_\gamma}{E_\gamma + mc^2}.$$

We can use this to compute γ_f and then from this compute m_f.

Another way to do the computation is use the fact that $E^2 - P^2c^2 = (m_f c^2)^2$. Equations

8 and 9 give

$$\begin{aligned}
(m_f c^2)^2 &= E^2 - P^2c^2 \\
&= (E_\gamma + mc^2)^2 - \left(\frac{E_\gamma}{c}\right)^2 c^2 \\
&= E_\gamma^2 + 2E_\gamma mc^2 + m^2c^4 - E_\gamma^2 \\
&= mc^2(2E_\gamma + mc^2) \\
&= m^2c^4\left(\frac{2E_\gamma}{mc^2} + 1\right) \\
m_f &= m\sqrt{\frac{2E_\gamma}{mc^2} + 1}.
\end{aligned}$$

This example was taken from one contributed by Timon Idema to LibreTexts.

5.2 Electron positron annihilation

Electron positron annihilation is when an electron and positron combine to produce two photons. When they combine, they annihilate (that means they both disappear).

$$e^- + e^+ \to \gamma + \gamma$$

The energy of the photons is at minimum equal to the rest mass energy ($m_e c^2$) of an electron and is 511 keV (and a gamma ray).

Why can't an electron and positron annihilate to produce a single photon?

Momentum must be conserved. We can always transfer to the center of mass frame for the electron and positron and in that frame the total momentum must be zero. However a photon always has momentum equal to its energy divided by the speed of light $p_\lambda = E_\lambda/c$. The momentum of a single photon cannot be zero, but that of two photons can be zero as they can propagate in opposite directions.

5.3 The Geiger-Marsdon experiments, also known as the Rutherford gold foil experiment

Much of nuclear and particle physics consists of the study of collisions.

α particles (essentially Helium nuclei) that hit a gold foil can scatter backwards elastically. Most pass through the foil without momentum change. The gold foil was interpreted to be comprised of a few small dense and heavy atomic nuclei. See Figure 11. The set of experiments that showed this are known as the Geiger-Marsdon experiments or the Rutherford gold foil experiment.

With more energy in the collision, nuclei exhibit what is known as **deep inelastic scattering**. In that setting the nucleus can deform, rotate and be put in excited states. Eventually the extra energy in the nucleus decays via emission of radiation or/and particles.

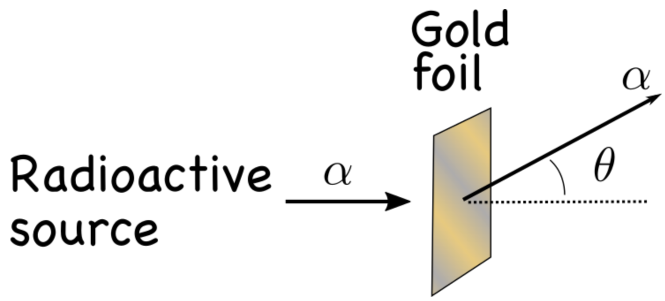

Figure 11: The Geiger-Marsdon experiments are also called the Rutherford gold foil experiment. Most of the α particles pass right through the gold foil without changing angle. A few of them are strongly scattered and are deflected to large angle θ. The gold foil must be comprised of a large volume of empty space and a small volume of dense atomic nuclei.

5.4 Beta decay

An example of beta decay is the following reaction

$$^{14}_{6}\mathrm{C} \rightarrow {}^{14}_{7}\mathrm{N} + e^- + \bar{\nu}_e$$

The half-life the carbon 14 atom is about 5,730 years. Here the electron is the emitted (and easier to detect) beta particle.

A neutron in the nucleus of the carbon atom decays into a proton. Emitted is an electron and an anti-neutrino. An intermediate virtual W- boson mediates the weak interaction. Beta decay is a consequence of the weak force, which is characterized by relatively lengthy decay times.

The study of beta decay provided the first physical evidence for the existence of the neutrino. The kinetic energy distribution of beta particles measured by Lise Meitner and Otto Hahn in 1911 and by Jean Danysz in 1913 showed multiple lines on a diffuse background. The distribution of beta particle energies was in apparent contradiction to the law of conservation of energy. Wolfgang Pauli attempted to resolve the beta-particle energy conundrum by suggesting that, in addition to electrons and protons, atomic nuclei also contained an extremely light neutral particle, which he called the neutron. A particle called the neutron was discovered in 1932 by J. Chadwick but was too massive to account for beta decay. Enrico Fermi renamed Pauli's "neutron" the *neutrino*. The neutrino interaction with matter was so weak that detecting it was an experimental challenge. Further indirect evidence of the existence of the neutrino was obtained by observing the recoil of nuclei that emitted such a particle after absorbing an electron.

6 Summary

- Final velocities following the non-relativistic elastic collision in 1d of two masses in a lab frame where one mass is initially at rest.

$$\begin{aligned} v_{1f} &= \frac{m_1 - m_2}{m_1 + m_2} v_{1i} \\ v_{2f} &= \frac{m_1}{m_2}(v_{1i} - v_{1f}) = \frac{2m_1}{m_1 + m_2} v_{1i} \end{aligned} \tag{10}$$

- Pre and post velocities of the elastic collision of two masses in a center of mass frame where the center of mass is at rest. The velocities flip sign; $u_{1i} = -u_{1f}$ and $u_{2i} = -u_{2f}$.
- How to go back and forth between center of mass and lab frames.
- How to compute the velocity of an anelastic non-relativistic collision where two particles stick together. Using conservation of momentum, the final velocity is equal to the center of mass velocity.
- For relativistic collisions the total momentum and total energy are both conserved. The total energy is the sum of the particle energies and the total momentum is the sum of the particle momentums.