PHY141 Lectures 12,13 notes

Alice Quillen

October 20, 2022

Contents

1 The center of mass

We consider two masses, m_1 at $\mathbf{x}_1$ and m_2 at $\mathbf{x}_2$. The center of mass is at position

$$\mathbf{X}_{cm} = \frac{m_1\mathbf{x}_1 + m_2\mathbf{x}_2}{m_1 + m_2}. \tag{1}$$

The mass of each particle acts like a weight.

We can take the time derivative of this equation to find the center of mass's velocity.

$$\mathbf{V}_{cm} = \frac{m_1\mathbf{v}_1 + m_2\mathbf{v}_2}{m_1 + m_2}. \tag{2}$$

What is the distance of the center of mass from $\mathbf{x}_1$? This distance is $|\mathbf{X}_{cm} - \mathbf{x}_1|$.

$$\begin{aligned}
\mathbf{X}_{cm} - \mathbf{x}_1 &= \frac{m_1\mathbf{x}_1 + m_2\mathbf{x}_2}{m_1 + m_2} - \mathbf{x}_1 \\
&= \left((m_1 - m_1 - m_2)\mathbf{x}_1 + m_2\mathbf{x}_2\right)(m_1 + m_2)^{-1} \\
&= \frac{m_2}{m_1 + m_2}(\mathbf{x_2} - \mathbf{x}_1) \\
|\mathbf{X}_{cm} - \mathbf{x}_1| &= \frac{m_2}{m_1 + m_2}|\mathbf{x_2} - \mathbf{x}_1|. \qquad (3)
\end{aligned}$$

The distance between the center of mass and m_1 is $\frac{m_2}{m_1+m_2}$ times the distance between m_1 and m_2. This formula can be useful!

We can flip the particle ids to find

$$|\mathbf{X}_{cm} - \mathbf{x}_2| = \frac{m_1}{m_1 + m_2}|\mathbf{x_2} - \mathbf{x}_1|. \tag{4}$$

If we have N masses, each with mass m_i and position $\mathbf{x}_i$, the total mass is

$$M = \sum_{i=1}^{N} m_i.$$

The center of mass position is

$$\mathbf{X}_{cm} = \frac{\sum_{i=1}^{N} m_i\mathbf{x}_i}{\sum_{i=1}^{N} m_i} = \frac{\sum_{i=1}^{N} m_i\mathbf{x}_i}{M} \tag{5}$$

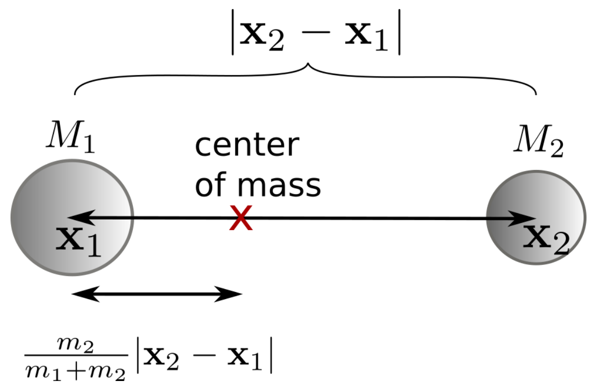

Figure 1: The position of the center of mass of two particles. The distance from m_1 to the center of mass is $\frac{m_2}{m_1+m_2}|\mathbf{x}_2 - \mathbf{x}_1|$ where $|\mathbf{x}_2 - \mathbf{x}_1|$ is the distance between the two particles.

If instead of a number of point masses, we have a continuum density distribution $\rho(x, y, z)$. The total mass

$$M = \int \rho(x, y, z)\ dx\ dy\ dz$$

where the integral is all mass within the surface. The position of the center of mass

$$\mathbf{X}_{cm} = \frac{\int dx\ dy\ dz\ \mathbf{x}\ \rho(x, y, z)}{\int dx\ dy\ dz\ \rho(x, y, z)} = \frac{1}{M} \int dx\ dy\ dz\ \mathbf{x}\ \rho(x, y, z). \tag{6}$$

This is equivalent to

$$\begin{aligned}
\mathbf{X}_{cm} &= (x_{cm}, y_{cm}, z_{cm}) \\
x_{cm} &= \frac{1}{M} \int dx\ dy\ dz\ x\rho(x, y, z) \\
y_{cm} &= \frac{1}{M} \int dx\ dy\ dz\ y\rho(x, y, z) \\
z_{cm} &= \frac{1}{M} \int dx\ dy\ dz\ z\rho(x, y, z)
\end{aligned}$$

1.0.1 Example

Suppose m_1 and m_2 are on a sliding table and their center of mass remains fixed. The distance between them is initially d but then shrinks to $d/2$.

Question: How far does m_1 move?

Answer: we use equation 3.

The position of m_1 initially has $|x_{1,init} - X_{cm}| = \frac{m_2}{m_1+m_2} d$.

The final position of m_1 is $|x_{1,final} - X_{cm}| = \frac{m_2}{m_1+m_2} d/2$.

The difference between initial and final positions for m_1 is

$$\frac{m_2}{m_1+m_2}(d-d/2) = \frac{m_2}{m_1+m_2}\frac{d}{2}.$$

1.1 A tilted cube of jello

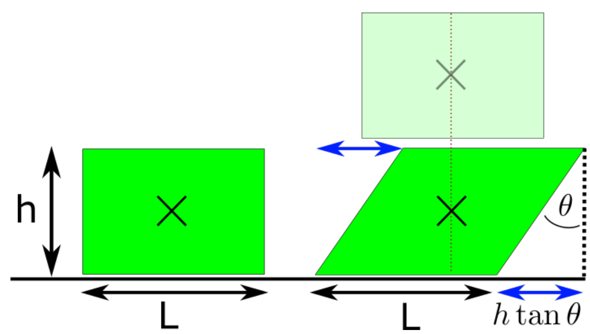

Figure 2: A tilting jello cube.

Question: A uniform density jello rectangle, as shown in Figure 2, is deformed so that it is tilted by angle θ. The height is h and the base has length L and both remain the same length. If the base is held fixed, how far does the center of mass move?

If l is the length of the right side after the jello is tilted, then the height of the jello $h = l\cos\theta$ and the base of the triangle in the overhang is $l\sin\theta = h\tan\theta$.

The top corner moves by a distance $h\tan\theta$.

Answer: The vertical coordinate of the center of mass stays fixed. The tilted block is symmetrical so its center of mass is half way between the two opposite corners. The center of mass moves by half of this distance or $\frac{h}{2}\tan\theta$.

Question: If the base is frictionless, how far does the lower left hand corner move as the jello tilts over by angle θ? Assume that the jello tilts via internal forces, not by being pushed externally.

Answer: The center of mass remains fixed. The lower left corner moves the same distance as in the previous question ($\frac{h}{2}\tan\theta$) but in the opposite direction as the top right corner.

1.2 Superposition

Suppose we have a jigsaw puzzle. We can compute the center of mass by integrating over the entire mass distribution

$$\mathbf{X}_{cm} = \frac{1}{M}\int dV\ \rho\ \mathbf{x}. \tag{7}$$

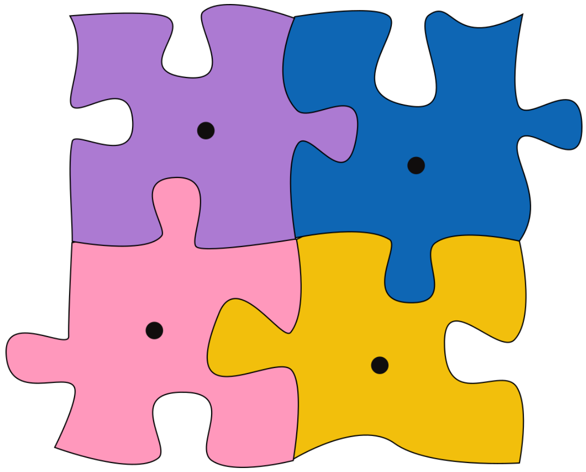

Figure 3: The center of mass of the entire puzzle can be computed using the center of mass positions of each piece and their masses.

Or we can compute the masses and center of mass of each puzzle piece, separately.

$$m_i = \int_{piece,i} dV \ \rho \tag{8}$$

$$\mathbf{x}_{i,cm} = \frac{1}{m_i} \int_{piece,i} dV \ \mathbf{x} \ \rho. \tag{9}$$

Then from the positions and masses of each piece (and with $M = \sum_i m_i$)

$$\begin{aligned}
\mathbf{X}_{cm} &= \frac{\sum_i \mathbf{x}_{i,cm} m_i}{\sum_j m_j} \\
&= \frac{1}{M} \sum_i \mathbf{x}_{i,cm} m_i \\
&= \frac{1}{M} \sum_i \left(\frac{1}{m_i} \int_{piece,i} dV \mathbf{x} \rho \right) m_i \\
&= \frac{1}{M} \int dV \mathbf{x} \rho.
\end{aligned} \tag{10}$$

The two descriptions are equivalent.

1.3 Center of mass of a uniform density disk with a hole in it

Looking at the left side of Figure 4 showing a disk with a hole in it, we ask, where is the center of mass?

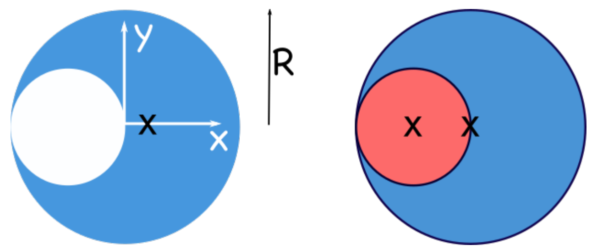

Figure 4: The center of mass of the disk with a hole in it can be computed using superposition. On the left we mark with an x the center of mass of a disk with a hole in it. On the right we mark the center of mass of the red plug and the filled in disk.

We call M_h, x_h the mass and center of mass position of a plug that would fit in the hole.

We call M_{dh}, x_{dh} the mass and center of mass position of the disk with the hole.

We call M_s, x_s the mass and center of mass position of the solid disk with radius R.

With a density of 1, the mass of the solid disk is $M_s = \pi R^2$.

The mass of the plug that would fit in the hole is $M_h = \pi(R/2)^2 = \pi R^2/4 = M_s/4$.

The mass of the disk with hole is $M_{dh} = \pi R^2(1 - 1/4) = \frac{3}{4}\pi R^2 = 3M_s/4$.

We define the x coordinate so that it is positive to the right and zero at the center of the solid disk.

The x coordinate of the center of mass of the plug for the hole (if filled in) is $x_h = -R/2$.

The x coordinate of the center of mass of the entire solid disk is $x_s = 0$.

We want to solve for the center of mass of the disk with hole in it or x_{dh}.

Via superposition, the center of mass of the solid disk is

$$x_s = 0 = \frac{M_h x_h + M_{dh} x_{dh}}{M_h + M_{dh}}.$$

Taking the numerator, we can solve for x_{dh}

$$x_{dh} = -\frac{M_h}{M_{dh}} x_h = \frac{1/4}{3/4} \frac{R}{2} = \frac{R}{6}$$

This (x_{dh}) is the distance of the center of mass from the center of the large disk. The center of mass is to the right of the center of the disk.

1.4 Coin vibrational motors

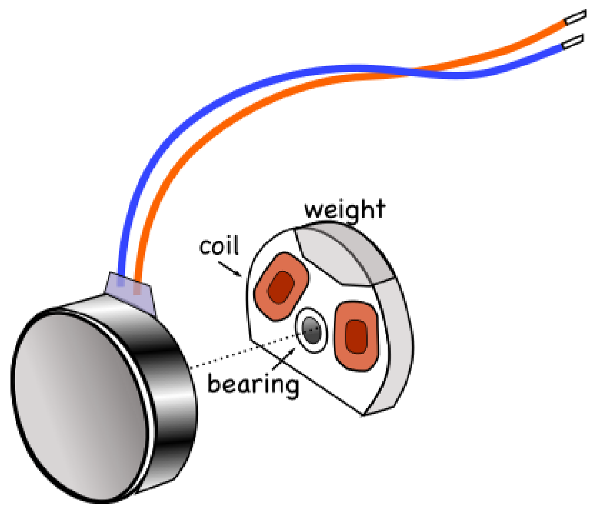

Figure 5: Inside a 1 cm coin vibrational motor which is used for haptic feedback in cellphones. The lopsided flywheel causes the center of mass to rotate at a few hundred Hz. This type of motor typically weighs less than a gram and is powered with a few volts DC. It can jump a few cm off a table top due to recoil.

2 Conservation of linear momentum in multiple particle systems

We consider N point masses, with masses m_i, that interact via forces. The forces act equally and oppositely on pairs of particles.

Recall that the velocity of the center of mass the system

$$\mathbf{V}_{cm} = \frac{1}{M} \sum_{i=1}^{N} m_i \mathbf{v}_i \tag{11}$$

where $M = \sum_{i=1}^{N} m_i$. This implies that

$$\sum_i m_i \mathbf{v}_i = M \mathbf{V}_{cm}. \tag{12}$$

In the non-relativistic limit the total momentum

$$\mathbf{P} = \sum_i \mathbf{p}_i = \sum_i m_i \mathbf{v}_i = M \mathbf{V}_{cm}. \tag{13}$$

For relativistic particles, with total rest mass M and the total momentum $\mathbf{P}$, we can find a velocity $\mathbf{V}_{cm}$ that gives

$$\mathbf{P} = \sum_i \mathbf{p}_i = \sum_i \gamma_i m_i \mathbf{v}_i = \gamma_{cm} M \mathbf{V}_{cm} \tag{14}$$

where Lorenz factor $\gamma_{cm} = \frac{1}{\sqrt{1-V_{cm}^2/c^2}}$. This center of mass velocity gives us a reference frame in which the center of mass has zero velocity. Here m_i and γ_i are the rest masses and Lorenz factors of each particle.

2.1 Internal forces alone — Total momentum is conserved

What happens if there are forces between pairs of particles? The total momentum

$$\mathbf{P} = \sum_i \mathbf{p}_i.$$

Consider a force between m_i and m_j. Over a short duration in time dt, each particle has a small change in momentum $d\mathbf{p}_i$.

$$d\mathbf{p}_i = -d\mathbf{p}_j$$

for equal and opposite forces during the time interval dt. The force between each pair of particles does not change the total momentum. Hence the total momentum $\mathbf{P}$ is **conserved**.

$$\mathbf{P} = \text{constant}.$$

A system with internal forces alone, **conserves total momentum**.

If the total momentum is conserved, then the center of mass velocity, $\mathbf{V}_{cm}$, **remains fixed**. The center of mass position **drifts at a constant velocity**. If $\mathbf{V}_{cm}$ is constant then so is $\frac{1}{2}MV_{cm}^2$ which is the kinetic energy of the center of mass. A non-relativistic system with internal forces alone has conserves the kinetic energy of its center of mass.

In a frame initially moving with the center of mass, the center of mass velocity is fixed at zero and remains so. This is true if particles exert forces on each other but not true in the presence of external forces.

An example of a system with only internal forces might be an N-body system of stars in an isolated star cluster. In this case the total momentum is conserved and the center of mass drifts at a constant velocity.

2.2 External forces

What if there is an external force operating on all masses? We consider changes in momentum over a short time interval caused by the external force.

$$d\mathbf{P} = \sum_i d\mathbf{p}_i$$

In the non-relativistic limit

$$d\mathbf{V}_{cm} = \frac{1}{M}\sum_i m_i d\mathbf{v}_i$$

$$dP = \sum_i m_i d\mathbf{v}_i = M d\mathbf{V}_{cm}$$

An external force changes the total momentum and the center of mass velocity.

The total work done on the particle system by an external force is

$$dW = \sum_i \mathbf{F}_i \cdot d\mathbf{x}_i = \sum_i \frac{d\mathbf{p}_i}{dt} \cdot d\mathbf{x}_i. \tag{15}$$

It is possible to decompose this into work done on the center of mass and work done internally by the external force. An example of internal work might be tidal deformation. An external gravitational field can be decomposed into that affecting the center of mass and that causing body deformation. Each particle may not necessarily move in the same direction so the work on the center of mass might not be the same as the total work done on the system. In section 4.2 we discuss this again.

2.3 Kinetic energy (non-relativistic) for multiple particle systems

The kinetic energy of all particles (in the non-relativistic limit)

$$K = \frac{1}{2} \sum_i m_i v_i^2$$

I am going to add and subtract some terms and then complete the square

$$\begin{aligned} K &= \frac{1}{2}\left[\sum_i m_i v_i^2 - 2\sum_i m_i \mathbf{v}_i \cdot \mathbf{V}_{cm} + 2\sum_i m_i \mathbf{v}_i \cdot \mathbf{V}_{cm} + \sum_i m_i V_{cm}^2 - \sum_i m_i V_{cm}^2\right] \\ &= \frac{1}{2}\left[\sum_i m_i (\mathbf{v}_i - \mathbf{V}_{cm})^2 + 2\sum_i m_i \mathbf{v}_i \cdot \mathbf{V}_{cm} - \sum_i m_i V_{cm}^2\right] \end{aligned}$$

It is useful to use $\sum m_i \mathbf{v}_i = M\mathbf{V}_{cm}$ (equation 12) and $\sum m_i = M$,

$$\begin{aligned} K &= \frac{1}{2}\left[\sum_i m_i (\mathbf{v}_i - \mathbf{V}_{cm})^2 + 2M\mathbf{V}_{cm} \cdot \mathbf{V}_{cm} - \sum_i m_i V_{cm}^2\right] \\ &= \frac{1}{2}\left[\sum_i m_i (\mathbf{v}_i - \mathbf{V}_{cm})^2 + 2MV_{cm}^2 - MV_{cm}^2\right] \\ &= \frac{1}{2}\sum_i m_i (\mathbf{v}_i - \mathbf{V}_{cm})^2 + \frac{1}{2}MV_{cm}^2 \end{aligned}$$

The total kinetic energy can be written as the kinetic energy of the center of mass plus the kinetic energy of all the particles with respect to the center of mass position.

Sometimes the kinetic energy of the center of mass motion is called the **translational kinetic energy**. The kinetic energy of particles with respect to the center of mass position is sometimes called the **relative kinetic energy**.

$$K = K_{relative} + K_{translational} \tag{16}$$
$$K_{relative} = \frac{1}{2}\sum_i m_i(\mathbf{v}_i - \mathbf{V}_{cm})^2 \tag{17}$$
$$K_{translational} = \frac{1}{2}MV_{cm}^2. \tag{18}$$

The relative kinetic energy can include vibrations and rotation.

The translational kinetic energy remains fixed if there are no external forces. This follows because when there are no external forces the total momentum is equal to the center of mass momentum and the velocity of the center of mass, $\mathbf{V}_{cm}$, is constant.

2.4 The reduced mass for two masses

Consider two particles with masses m_1, m_2 and velocities $\mathbf{v}_1, \mathbf{v}_2$. Let

$$\mathbf{u}_1 = \mathbf{v}_1 - \mathbf{V}_{cm} \qquad \text{and} \qquad \mathbf{u}_2 = \mathbf{v}_2 - \mathbf{V}_{cm}.$$

Here $\mathbf{u}_1, \mathbf{u}_2$ are velocities with respect to the center of mass frame. Let's consider the kinetic energy of particles w.r.t. to the center of mass. This is consistent with working in a center of mass frame. Because we are in the center of mass frame

$$m_1\mathbf{u}_1 + m_2\mathbf{u}_2 = 0$$

$$m_1(\mathbf{u}_1 - \mathbf{u}_2) + (m_1 + m_2)\mathbf{u}_2 = 0$$

$$\mathbf{u}_2 = -\frac{m_1}{m_1 + m_2}(\mathbf{u}_1 - \mathbf{u}_2)$$
$$\mathbf{u}_1 = \frac{m_2}{m_1 + m_2}(\mathbf{u}_1 - \mathbf{u}_2) \tag{19}$$

We define a velocity difference vector

$$\mathbf{u}_{diff} \equiv \mathbf{u}_1 - \mathbf{u}_2 = \mathbf{v}_1 - \mathbf{v}_2. \tag{20}$$

The kinetic energy in the center of mass frame

$$
\begin{aligned}
K &= \frac{1}{2}\left[m_1 u_1^2 + m_2 u_2^2\right] \\
&= \frac{1}{2}\left[m_1\left(\frac{m_2}{m_1+m_2}\right)^2 u_{diff}^2 + m_2\left(\frac{m_2}{m_1+m_2}\right)^2 u_{diff}^2\right] \\
&= \frac{1}{2}\left[\frac{m_1 m_2^2 + m_2 m_1^2}{m_1+m_2)^2} u_{diff}^2\right] \\
&= \frac{1}{2}\frac{m_1 m_2}{m_1+m_2} u_{diff}^2 \\
&= \frac{1}{2}\mu u_{diff}^2
\end{aligned} \tag{21}
$$

We have defined the reduced mass

$$
\mu \equiv \frac{m_1 m_2}{m_1+m_2}. \tag{22}
$$

We just computed the kinetic energy w.r.t. the center of mass. We now go back into the original frame by restoring the center of mass motion.

The total kinetic energy can be written

$$
K = \frac{1}{2}\mu u_{diff}^2 + \frac{1}{2} M V_{cm}^2. \tag{23}
$$

2.5 The two body problem

Consider the total energy of two point masses m_1, m_2 interacting via gravity. It is useful to compute

$$
m_1 m_2 = \mu(m_1+m_2) = \mu M
$$

and a position difference vector

$$
\mathbf{r}_1 - \mathbf{r}_2 = \mathbf{r}_{diff}
$$

where $\mathbf{r}_1, \mathbf{r}_2$ are the positions of the two masses. Here μ is the reduced mass and $M = m_1 + m_2$ is the total mass. As before the velocity difference $\mathbf{u}_{diff} = \dot{\mathbf{r}}_1 - \dot{\mathbf{r}}_2$. The total energy

$$
\begin{aligned}
E &= K + U \\
&= \frac{1}{2}\mu u_{diff}^2 + \frac{1}{2} M V_{cm}^2 - \frac{G m_1 m_2}{|\mathbf{r_1} - \mathbf{r_2}|} \\
&= \frac{1}{2}\mu u_{diff}^2 - \frac{G\mu M}{r_{diff}} + \frac{1}{2} M V_{cm}^2.
\end{aligned} \tag{24}
$$

The dynamics of 2 point masses of mass m_1, m_2 in orbit about each other is the same as a point mass of mass equal to the reduced mass μ about a very large mass M that has a fixed position. The center of mass motion is unaffected by the gravitational force between the two masses. This formalism is used to study binary stars!

2.6 Relativistic generalization

To shift reference frame, a Lorenz transformation must be applied to the energy momentum four vector of each particle $(E_i, \mathbf{p}_i)$. When there are internal forces alone $\sum_i \mathbf{p}_i$ is conserved and $\sum_i E_i$ is conserved. All four components of $(E, \mathbf{p})$ are conserved.

2.7 The break-up of a projectile

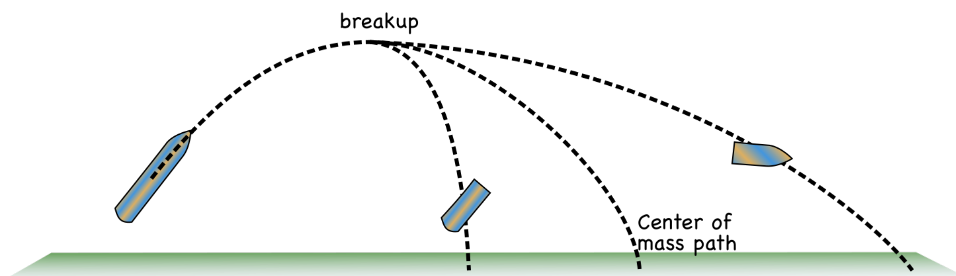

Figure 6: Internal forces are responsible for the breakup of a rocket into pieces. The center of mass position and velocity are not affected by the breakup.

Internal forces are responsible for the breakup of a rocket into pieces, so the center of mass position and velocity are not affected by the breakup.

After the breakup the gravitational acceleration on each piece is the same.

$$\begin{aligned}\mathbf{V}_{cm} &= \frac{1}{M}\sum_i m_i \mathbf{v}_i \\ \frac{d\mathbf{V}_{cm}}{dt} &= \frac{1}{M}\sum_i m_i \frac{d\mathbf{v}_i}{dt} \\ &= \frac{1}{M}\sum_i m_i(-g\hat{\mathbf{z}}) = -g\hat{\mathbf{z}}.\end{aligned}$$

The center of mass accelerates the same as each individual piece. This means that the center of mass of the system remains on the projectile's initial trajectory.

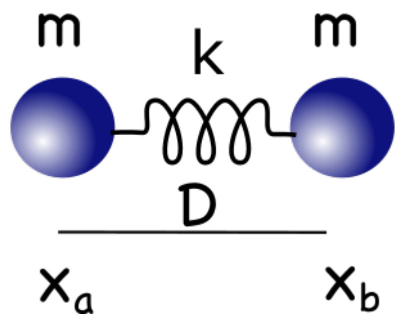

Figure 7: Two equal masses connected by a spring. The motion consists of a drifting center of mass coupled with harmonic motion. The position of the left mass is x_a and that of the right mass is x_b. The distance between them is D.

2.8 Two masses and a spring (a molecule)

We model a molecule as two equal masses m which we label with subscripts a and b, are connected by a spring with rest length L and spring constant k, as shown in Figure 7. We let the masses move in one dimension, along the x-axis. Their positions are $x_a(t), x_b(t)$. The two masses can oscillate back and forth while the entire molecule drifts through space. The equations of motion

$$m\frac{d^2x_a}{dt^2} = -k(x_a - x_b + L) \tag{25}$$

$$m\frac{d^2x_b}{dt^2} = k(x_a - x_b + L). \tag{26}$$

The spring forces are equal and opposite. The sign of the first term makes sense as when $x_b - x_a > L$, the force is positive giving an acceleration to the right. Here x_a refers to the mass on the left and x_b refers to the mass on the right.

The center of mass position X_{cm} and distance between the two masses D depend on the positions of each mass x_a, x_b

$$\begin{aligned} X_{cm} &= \frac{x_a + x_b}{2} \\ D &= x_b - x_a. \end{aligned} \tag{27}$$

The inverse transformation giving x_a, x_b from X_{cm}, D is

$$\begin{aligned} x_a &= X_{cm} - \frac{D}{2} \\ x_b &= X_{cm} + \frac{D}{2} \end{aligned} \tag{28}$$

We add the equations of motion (equation 26) together, and we subtract them to find

$$\begin{aligned}
\frac{d^2}{dt^2}(x_a + x_b) &= 0 \\
\frac{d^2}{dt^2}(x_b - x_a) &= -\frac{2k}{m}(x_b - x_a - L)
\end{aligned} \tag{29}$$

Equations 29 can be written

$$\frac{d^2 X_{cm}}{dt^2} = 0 \tag{30}$$

$$\frac{d^2 D}{dt^2} = -\omega^2(D - L) \tag{31}$$

with $\omega = \sqrt{k/\mu}$ and with μ the reduced mass. Each of these equations can be solved separately. The first one can be recognized as a drifting center of mass. The second one can be recognized as the equation for a harmonic oscillator with a constant added.

A general solution is

$$X_{cm}(t) = X_{cm,0} + V_{cm} t \tag{32}$$

$$D(t) = A\cos(\omega t) + B\sin(\omega t) + L \tag{33}$$

with constants $X_{cm,0}, V_{cm}, A, B$. Initial conditions could be used to set these constants.

For example, consider the initial condition for positions

$$x_a(t=0) = 0 \qquad x_b(t=0) = L$$

and velocities

$$v_a(t=0) = -V_0 \qquad v_b(t=0) = 0$$

where $v_a = \frac{dx_a}{dt}$ and similarly for v_b.

The center of mass position at $t = 0$ is

$$X_{cm,0} = \frac{mL}{m+m} = \frac{L}{2}.$$

The center of mass velocity is

$$V_{cm} = \frac{-mV_0}{m+m} = -\frac{V_0}{2}.$$

Using these constants (and equation 32), we find that the center of mass at later times

$$X_{cm}(t) = \frac{L}{2} - \frac{V_0 t}{2}. \tag{34}$$

The distance between the two masses at $t = 0$ is

$$D(t = 0) = x_b(t = 0) - x_a(t = 0) = L.$$

We evaluate equation 33 at $t = 0$

$$D(t = 0) = A + L = L$$

giving $A = 0$. The time derivative of separation D at $t = 0$ is

$$\frac{dD(t = 0)}{dt}(t = 0) = v_b(0) - v_a(0) = V_0$$

We take the time derivative of 33 at $t = 0$

$$\frac{dD(t = 0)}{dt} = B\omega = V_0$$

giving

$$B = \frac{V_0}{\omega}.$$

Now that we have values for A, B we can use equation 33 to find the separation between the two masses at later times

$$D(t) = \frac{V_0}{\omega} \sin \omega t + L. \quad (35)$$

Lastly we can use the inverse transform (equation 28), solutions for $X_{cm}(t)$ and $D(t)$ (equations 34 and 35) to write down expressions for positions $x_a(t), x_b(t)$ at later times.

$$x_a(t) = X_{cm}(t) - \frac{D(t)}{2} = \frac{L}{2} - \frac{V_0 t}{2} - \frac{V_0}{2\omega} \sin \omega t - \frac{L}{2}$$
$$x_b(t) = X_{cm}(t) + \frac{D(t)}{2} = \frac{L}{2} - \frac{V_0 t}{2} + \frac{V_0}{2\omega} \sin \omega t + \frac{L}{2}.$$

2.9 A mass spring jumper

We consider two blocks of masses m_1, m_2 that are connected by a massless spring of spring constant k. The bottom mass m_1 is resting on a table top. The top mass m_2 compresses the spring by d and is initially at rest. The rest spring length is L.

What condition allows m_1 to leave the table?

What is the center of mass position and velocity at the moment when m_1 leaves the table top?

We first solve the problem for m_1 fixed on the top of the table. We use this solution to find a condition for m_1 to leave the table surface. Assuming this condition is satisfied, we find the velocity and position of m_2 at the time when m_1 leaves the surface. These can be used as initial conditions for the solution for both masses after m_1 leaves the surface.

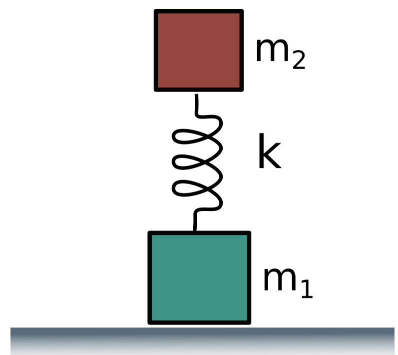

Figure 8: The top mass is pushed down, compressing the spring. When this force is removed, the system can jump off the table top.

2.9.1 Before m_1 leaves the table surface

Initially the bottom mass m_1 does not move. We take the vertical coordinates of the two masses to be z_1, z_2. Let $z_1 = 0$ when m_1 is at rest on the table top and let z_2 be measured with respect to that position. The equation of motion for m_2 (with m_1 fixed)

$$m_2\ddot{z}_2 = -k(z_2 - L) - m_2 g$$

The solution with with m_2 initially at rest and m_1 fixed

$$z_2 = L - \frac{m_2 g}{k} - d\cos\left(\sqrt{\frac{k}{m_2}}t\right) \tag{36}$$

The sign is because we start with compression by distance d. We chose the cosine because the spring is initially compressed and m_2 initially has zero velocity.

As long as the acceleration on m_1 is downward and it is resting on the table top, m_1 does not move. We want to know when the force of the spring overcomes gravity on m_1 so it can lift of the table. Ignoring the force from the table top, the equation of motion for m_1 is

$$m_1\ddot{z}_1 = k(z_2 - z_1 - L) - m_1 g.$$

When $z_1 = 0$ the force on m_1 and using equation 36 for z_2

$$\begin{aligned} m_1\ddot{z}_1 &= k(z_2 - L) - m_1 g \\ &= k(L - \frac{m_2 g}{k} + d\sin\left(\sqrt{\frac{k}{m_2}}t\right) - L) - m_1 g \\ &= -(m_1 + m_2)g - kd\cos\left(\sqrt{\frac{k}{m_2}}t\right) \end{aligned} \tag{37}$$

The cosine is at most ± 1. What condition allows m_1 to leave the table? If

$$kd > (m_1 + m_2)g, \tag{38}$$

then the acceleration on m_1 can exceed that from gravity and m_1 can leave the table top.

2.9.2 The system at the moment when m_1 leaves the table surface

When does m_1 leave the table top? The mass m_1 leaves the surface of the table at t_* when the right side of equation 37 is zero.

$$(m_1 + m_2)g = -kd\cos\left(\sqrt{\frac{k}{m_2}}t_*\right)$$

This gives us a z_2 value when m_1 leaves the table top.

$$z_{2*} = L - \frac{m_2 g}{k} + \frac{(m_1 + m_2)g}{k}$$

We can compute the center of mass position at this moment in time t_*

$$z_{cm,*} = \frac{1}{m_1 + m_2} m_2 z_{2*} = \frac{m_2}{m_1 + m_2}\left(L - \frac{m_2 g}{k} + \frac{(m_1 + m_2)g}{k}\right)$$

which is measured from the rest position of m_1 on the table top.

At that moment in time t_* what is the velocity of m_2? We differentiate equation 36.

$$\begin{aligned}
v_{2*} &= d\sqrt{\frac{k}{m_2}}\sin\left(\sqrt{\frac{k}{m_2}}t_*\right) \\
m_2 v_{2*}^2 &= kd^2\sin^2\left(\sqrt{\frac{k}{m_2}}t_*\right) \\
&= kd^2\left(1 - \cos^2\left(\sqrt{\frac{k}{m_2}}t_*\right)\right) \\
&= kd^2\left(1 - \left(\frac{(m_1 + m_2)g}{kd}\right)^2\right) \\
&= kd^2 - \frac{(m_1 + m_2)^2 g^2}{k} \\
v_{2*} &= \sqrt{\frac{kd^2}{m_2} - \frac{(m_1 + m_2)^2 g^2}{km_2}}
\end{aligned}$$

From this velocity we can find a center of mass velocity

$$v_{cm,*} = \frac{m_2}{m_1 + m_2} v_{2*}.$$

What are the relative velocity and position at t_*?

$$\begin{aligned} v_{diff,*} &= v_{2*} - v_{1*} = v_{2*} \\ z_{diff,*} &= z_{2*} - z_{1*} = z_{2*} = L - \frac{m_2 g}{k} + \frac{(m_1 + m_2)g}{k} \end{aligned}$$

2.9.3 Solution afterward

We solve for a center of mass position and velocity as a function of time due to gravity. On top of this we can superimpose the oscillation of the mass spring system.

When m_1 is off the table, it and m_2 are only affected by gravity and the spring. The equations of motion are

$$m_1 \ddot{z}_1 = -m_1 g - k(z_1 - z_2 + L) \tag{39}$$

$$m_2 \ddot{z}_2 = -m_2 g + k(z_1 - z_2 + L) \tag{40}$$

We subtract these two equations to find

$$\begin{aligned} m_1 m_2 (\ddot{z}_2 - \ddot{z}_1) &= -k(m_1 + m_2)(z_2 - z_1 - L) \\ \mu \ddot{z}_{diff} &= -k(z_{diff} - L). \end{aligned} \tag{41}$$

Once m_1 is no longer touching the table, the oscillation frequency is $\sqrt{\frac{k}{\mu}}$ where $\mu = m_1 m_2/(m_1 + m_2)$ is the reduced mass.

The vibrational motion is decoupled from the motion of the center of mass. We can also add the two equations (equations 54) to find that

$$\begin{aligned} m_1 \ddot{z}_1 + m_2 \ddot{z}_2 &= -(m_1 + m_2)g \\ M \ddot{z}_{cm} &= -Mg \end{aligned} \tag{42}$$

The center of mass motion is only affected by gravity.

After m_1 leaves the table, the motion can be described as a sum of the center mass motion, which is affected by gravity alone, and an oscillation in the spring of the reduced mass and neglecting gravity.

The rest distance between m_1, m_2 is L. The relative position and velocity can be used to find the amplitude and phase of this oscillation after t_*. For example, the amplitude of the relative distance oscillation would be

$$A = \sqrt{\frac{\mu}{k} v_{diff,*}^2 + (z_{diff*} - L)^2}.$$

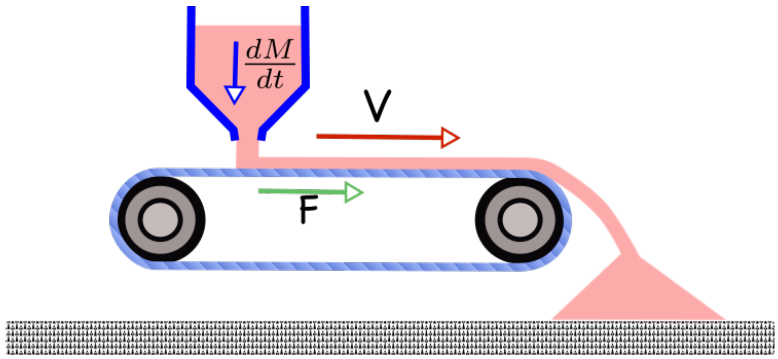

Figure 9: A conveyor belt. The belt moves at velocity V. The force on the belt is F and the mass flow rate onto the belt is $\frac{dM}{dt}$.

The initial center of mass velocity $v_{cm,*}$ can be used to find the motion of the center of mass after t_*.

The two mass system will return to hit the table surface. We could predict the time when this happens from the solution after t_*.

2.10 A conveyor belt

A hopper dumps sand on a conveyor belt at a rate of $\frac{dM}{dt}$ kilograms per second (see Figure 9). The conveyor belt is moving to the right at (non-relativistic) speed V and the sand is dumped off at the end. What force F is needed to keep the conveyor belt moving at a constant speed, assuming that the conveyor belt mechanism itself is frictionless?

In this case

$$\frac{dM_{in}}{dt} = \frac{dM_{out}}{dt}.$$

The sand on the belt and the belt holding this sand together are in a steady state. This means that together they have a constant momentum.

The sand enters the system with zero horizontal velocity, but exits the system with the horizontal velocity of the conveyor belt, V. What is the change in momentum for a small mass dM? It is $dM \times V$. This means that $\frac{dp}{dt} = \frac{dM}{dt}V$. The momentum balance equation is thus

$$0 = F - V\frac{dM}{dt}$$

and the force on the conveyor belt is

$$F = V\frac{dM}{dt}.$$

This force serves to accelerate the sand up to the velocity of the conveyor belt. This example is by David J. Raymond.

3 Thrust

Consider a rocket engine (Figure 10). The nozzle sends out a small mass of gas dm at relative velocity v_{rel} with respect to the rocket. The momentum of this small mass of gas is $dp = dm\ v_{rel}$. If it takes dt time to emit this gas then the change in momentum per unit time of the rocket is

$$\frac{dp}{dt} = \frac{dm}{dt}v_{rel}.$$

Here $\frac{dm}{dt}$ is the propellant mass outflow rate from the rocket. We have balanced the momentum changes.[1] Note that v_{rel} is the velocity of the gas in the rocket frame or relative to the rocket. This is a force exerted on the rocket known as **thrust**.

$$F_{\text{thrust}} = \frac{dm}{dt}v_{rel}.$$

The direction of the force is in the opposite direction of the velocity of the expelled gas in the rocket's frame

$$\mathbf{F}_{\text{thrust}} = -\frac{dm}{dt}\mathbf{v}_{rel}. \tag{43}$$

The thrust is applied on the mass of the rocket which is decreasing.

$$-\frac{dm}{dt}v_{rel} = M(t)\frac{dV}{dt}$$

where V is the velocity of the rocket. If the mass loss rate is constant then the mass of the rocket $M(t) = M_i - \frac{dm}{dt}t$ where M_i is the initial rocket mass. Here $\frac{dm}{dt}$ is the mass outflow rate of the expelled gas and the mass loss rate from the rocket is equal to the outflow rate but with the opposite sign, $\frac{dM}{dt} = -\frac{dm}{dt}$. We find that

$$\frac{dV}{dt} = -\frac{\dot{m}v_{rel}}{M_i - \dot{m}t}$$

where $\dot{m} = dm/dt$. This can be integrated

$$dV = -\frac{\dot{m}v_{rel}}{M_i - \dot{m}t}dt$$
$$V = v_{rel}\ln(M_i - \dot{m}t) + \text{constant}$$

[1] It is incorrect to estimate the thrust with $\frac{dp}{dt} = \frac{d(mv)}{dt} = \frac{dm}{dt}v + m\frac{dv}{dt}$.

If V_f and M_f are velocity and mass of rocket at a later time and V_i and M_i are velocity and mass at an initial time, the constant can be determined and the resulting solution written as

$$V_f = V_i + v_{rel} \ln \left| \frac{M_i}{M_f} \right| .$$

How is it that momentum is conserved? We consider a moment in time when a small mass dm is expelled from the rocket. The mass of the rocket after it is expelled is M. The velocity of the rocket and small mass before expulsion is V. The velocity of the rocket after expulsion is $V + dV$. The total momentum prior to expelling the mass is

$$p_0 = (M + dm)V. \tag{44}$$

After expelling it the small parcel of gas has velocity $V + v_{rel}$ and momentum $dm(V + v_{rel})$. The total momentum after expulsion is

$$p_1 = M(V + dV) + dm\ (V + v_{rel}). \tag{45}$$

If we set $p_0 = p_1$ we find that

$$\begin{aligned} M(V + dV) + dm\ (V + v_{rel}) &= (M + dm)V \\ MdV + dm\ v_{rel} &= 0. \end{aligned} \tag{46}$$

This gives

$$M\frac{dV}{dt} = -\frac{dm}{dt}v_{rel}$$

consistent with our previous definition of the force called thrust. Expelling the small mass dm changes its momentum. The momentum of the rocket is changed by exactly the same amount. The force exerted on the rocket exhaust is equal and opposite to that exerted on the rocket by the exhaust. Conservation of momentum is consistent with an equal and opposite force exerted by rocket and exhaust to each other.

A relativistic generalization of thrust would be

$$\mathbf{F}_{\text{thrust}} = -\frac{dm}{dt}\gamma_{rel}\mathbf{v}_{rel} \tag{47}$$

where $\gamma_{rel} = (1 - v_{rel}^2/c^2)^{-\frac{1}{2}}$ is the Lorenz factor associated with the relative velocity $\mathbf{v}_{rel}$. Here dm is the rest mass in the ejecta and dm/dt is the rate that rest mass is ejected. The momentum associated with dm is $d\mathbf{p} = dm\ \gamma_{rel}\mathbf{v}_{rel}$.

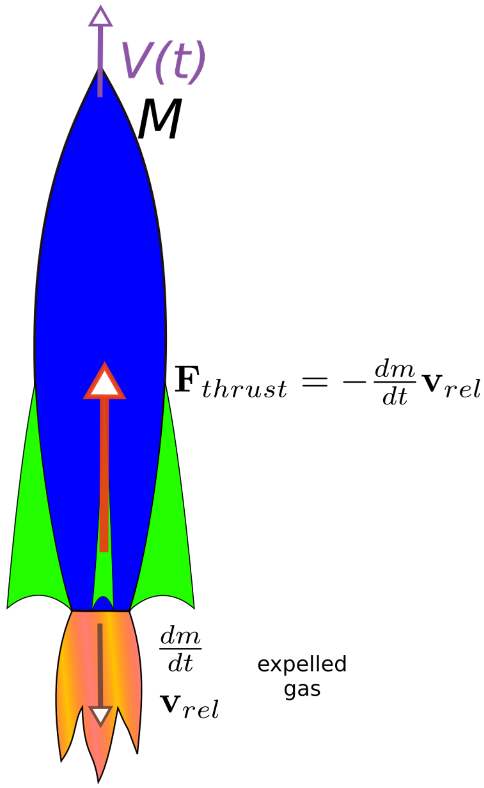

Figure 10: Thrust from a rocket engine. $\mathbf{F}_{thrust} = -\frac{dm}{dt}\mathbf{v}_{rel}$ where dm/dt is the mass loss rate and $\mathbf{v}_{rel}$ is the difference in velocity between the rocket and the emitted mass.

4 Energy of multiple particle systems

4.1 Gravitational potential energy of an extended mass

We consider an extended mass near the surface of the Earth. The total mass is

$$M = \int \rho dV$$

where $dV = dx\ dy\ dz$. The z component of the center of mass is

$$z_{cm} = \frac{1}{M} \int \rho z dV.$$

The potential energy is

$$U(z_{cm}) = \int \rho g z\ dV = M g z_{cm}.$$

This only depends on the vertical coordinate of the center of mass z_{cm} so we can define a total gravitational force

$$\begin{aligned} \mathbf{F}_T &= \int \rho dV F(\mathbf{x}) \\ &= -\frac{dU(z_{cm})}{dz_{cm}} = -Mg\hat{\mathbf{z}}. \end{aligned}$$

4.2 External force on a multiple particle system and work on the center of mass

The total momentum of a multiple particle system (non-relativistic)

$$\mathbf{P} = \sum_i m_i \mathbf{v}_i = M\mathbf{V}_{cm}$$

We apply an *external* force that might depend upon position.

$$\frac{d\mathbf{P}}{dt} = \sum_i \frac{d\mathbf{p}_i}{dt} = \sum_i \mathbf{F}_i$$

where $\mathbf{F}_i$ is the force on each particle. Let

$$\mathbf{F}_T = \sum_i \mathbf{F}_i. \tag{48}$$

Instead of computing work on each individual particle, we compute the work on the center of mass as the total force times the distance travelled by the center of mass.

$$\begin{aligned}
\int \mathbf{F}_T \cdot d\mathbf{X}_{cm} &= \int \frac{d\mathbf{P}}{dt} \cdot d\mathbf{X}_{cm} \\
&= \int M \frac{d\mathbf{V}_{cm}}{dt} \cdot \frac{d\mathbf{X}_{cm}}{dt} dt \\
&= \int M \frac{d\mathbf{V}_{cm}}{dt} \cdot \mathbf{V}_{cm} dt \\
&= M \int \frac{d}{dt} \left(\frac{V_{cm}^2}{2} \right) dt \\
&= \frac{1}{2} M V_{cm}^2 \bigg|_{V_{cm,i}}^{V_{cm,f}} .
\end{aligned}$$

The total work done on the center of mass increases the kinetic energy of the center of mass.

Note that equation 48 is the sum of forces on each particle. This is not necessarily the same as a position dependent force calculated at the position of the center of mass. If the force does not depend on position, like gravitational acceleration at the surface of the Earth, then you need not worry about this difference. For gravity, the force is proportional to mass and can be expanded as a function of position. It is convenient to describe the force on a rigid body in terms of the force integrated over the mass distribution. The first term in a multipole expansion is the force evaluated at the center of mass position. The second non-zero term in the expansion is the quadrupolar or tidal term.

4.3 Energy of multiple particle systems

Neglecting rest mass and heat, the total energy can often be decomposed into the sum of

- A center of mass kinetic term (the translational kinetic energy).
- A relative kinetic energy term (kinetic energy of the particle velocities with respect to the center of mass). This includes vibrational and rotational energy.
- The sum of potential energies for each pair of interacting particles, assuming that the interactions are due to conservative forces.
- The sum of potential energies for each particle from the external forces, assuming that they are conservative.

As long as there are no non-conservative forces or transfer of heat, application of the energy principle then implies that the total energy is conserved.

Example of particle interactions are electrostatic or gravitational forces or massless springs.

5 Summary

- The center of mass position
$$\mathbf{X}_{cm} = \frac{1}{M}\sum_i m_i \mathbf{x}_i \qquad \text{or} \qquad \frac{1}{M}\int \rho \mathbf{x} dV$$
where the total mass
$$M = \sum_i m_i \qquad \text{or} \qquad \int \rho dV$$
- The velocity of the center of mass is
$$\mathbf{V}_{cm} = \frac{1}{M}\sum_i m_i \mathbf{v}_i \qquad \text{or} \qquad \frac{1}{M}\int \rho \mathbf{v} dV$$
- It is sometimes easier to calculate the position of the center of mass, using **superposition** (or computing the center of mass positions of individual pieces).
- The kinetic energy of a non-relativistic multi-particle system
$$K = K_{translational} + K_{internal}$$
where
$$K_{translational} = \frac{1}{2} M V_{cm}^2$$
$$K_{internal} = \frac{1}{2}\sum m_i (\mathbf{v}_i - V_{cm})^2$$
- For a non-relativistic two body system
$$K_{internal} = \frac{1}{2}\mu u_{diff}^2$$
where the reduced mass
$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$
and the velocity difference $\mathbf{u}_{diff} = \mathbf{u}_1 - \mathbf{u}_2$.
- With internal forces alone, the total momentum is conserved.
- With external forces acting on a multi-particle system and total force (computed using the external forces) $\mathbf{F} = \sum_i \mathbf{F}_i$, the work done on the center of mass is equal to the change in translational kinetic energy
$$W_{cm} = \int \mathbf{F} \cdot d\mathbf{X}_{cm} = \Delta K_{translational}$$
- Thrust due to mass loss at a rate dm/dt and with relative velocity $\mathbf{v}_{rel}$ is
$$\mathbf{F}_{\text{thrust}} = -\frac{dm}{dt}\mathbf{v}_{rel}$$

6 Extra: Conservation laws

6.1 Continuum equations in 1 dimension

We consider a quantity that we expect is conserved. The amount of this quantity depends on a coordinate x and time t. An example would be density $\rho(x,t)$ or number density of cars on a road $n(x,t)$. The mass in an element of length dx would be $dm = \rho(x,t)dx$. Or similarly the number of cars on the road in a length dx would be $dN = n(x,t)dx$.

We can also consider a mean velocity $u(x,t)$. This is an average velocity of particles (or cars) in the flow.

Consider the amount of mass flowing out of a region per unit time. This would be the **mass flux** ρu evaluated at the right hand side of the region. It is a rate of mass.

$$\rho u = \frac{dm}{dx}\frac{dx}{dt} = \frac{dm}{dt}$$

The amount of mass flowing into the region is ρu but evaluated at the left hand side of the region. If the mass flowing in is equal to the mass flowing out the region does not gain mass and $\frac{d\rho}{dt} = 0$. If the mass flowing out is greater than that flowing in, then the region looses mass. The difference of the two mass fluxes determines how much mass is lost or gained in the region. Dividing both sides by dx this gives

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x}(\rho u). \tag{49}$$

An equation like this, with a time derivative on one side, equal to to a gradient of a flux on the other side, is a general form for a **conservation law**.

In three dimensions

$$\frac{\partial \rho}{\partial t} = -\boldsymbol{\nabla} \cdot (\rho \mathbf{u}).$$

6.2 Distribution functions

We describe the density of particles in phase space with a function $f(x,v,t)$. The number of particles within $dx/2$ of position x and within velocity $dv/2$ of velocity v is

$$dN = f(x,v,t)dx\ dv$$

To find number of particles per unit volume we integrate over all velocity

$$n(x,t) = \int f(x,v,t)dv. \tag{50}$$

The mean velocity is normalized

$$u(x,t) = \frac{\int f(x,v,t)v\ dv}{\int f(x,v,t)dv}. \tag{51}$$

This is equivalent to

$$nu = \int f(x, v, t) v \; dv. \tag{52}$$

Conservation of the number of particles gives

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial v}\frac{dv}{dt} + \frac{\partial f}{\partial t} = 0.$$

We integrate each term of this over all velocity space.

$$\int \frac{\partial f}{\partial x}\frac{dx}{dt} dv + \int \frac{\partial f}{\partial v}\frac{dv}{dt} dv + \int \frac{\partial f}{\partial t} dv = 0. \tag{53}$$

We evaluate each term in the sum separately.

$$\begin{aligned}\int \frac{\partial f}{\partial x}\frac{dx}{dt} \; dv &= \int \frac{\partial f}{\partial x} v \; dv \\ &= \frac{d}{dx}\int f(x, v, t) v \; dv \\ &= \frac{d}{dx}(nu)\end{aligned}$$

For the second term, we assume that the acceleration dv/dt depends only on position, not velocity

$$\begin{aligned}\int \frac{\partial f}{\partial v}\frac{dv}{dt} dv &= \frac{dv}{dt}\int \frac{\partial f}{\partial v} dv \\ &= \frac{dv}{dt} \; f|_{v=-\infty}^{v=\infty} \\ &= 0\end{aligned}$$

This is zero because we don't expect any particles to have infinite velocity. The third term

$$\begin{aligned}\int \frac{\partial f}{\partial t} dv &= \frac{d}{dt}\int f(x, v, t) dv \\ &= \frac{dn}{dt}\end{aligned}$$

Putting these together (back into equation 53) we get

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(nu) = 0$$

which is consistent with the conservation law derived in the previous section.

6.3 Conservation of momentum in 1D

Above (equation 49) we derived an equation that is equivalent to conservation of mass.

$$\frac{\partial \rho}{\partial t} = -\frac{\partial(\rho u)}{\partial x}.$$

Lets expand this

$$\frac{\partial \rho}{\partial t} = -u\frac{\partial \rho}{\partial x} - \rho\frac{\partial u}{\partial x}. \tag{54}$$

What does conservation of momentum look like? The total momentum in an element of length dx would be $\rho u\ dx$, so the momentum per unit length is ρu. In the conservation law we replace ρ with ρu.

Let's make a conservation law using ρu. The momentum flux is ρu^2 (the amount of momentum times how fast it moves). This gives

$$\frac{\partial \rho u}{\partial t} = -\frac{\partial(\rho u^2)}{\partial x}$$

We expand this and use equation 54

$$\begin{aligned}
\rho\frac{\partial u}{\partial t} + u\frac{\partial \rho}{\partial t} &= -\rho 2u\frac{\partial u}{\partial x} - u^2\frac{\partial \rho}{\partial x} \\
\rho\frac{\partial u}{\partial t} - u^2\frac{\partial \rho}{\partial x} - u\rho\frac{\partial u}{\partial x} &= -\rho 2u\frac{\partial u}{\partial x} - u^2\frac{\partial \rho}{\partial x} \\
\rho\frac{\partial u}{\partial t} &= -\rho u\frac{\partial u}{\partial x} \\
\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} &= 0
\end{aligned}$$

This last equation is known as the inviscid Burger's equation. It's non linear and shows some interesting phenomena like propagating small smooth perturbations steepening into discontinuities or shocks.

In 3D and adding in additional terms due to pressure p and viscosity ν

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot(\boldsymbol{\nabla}\mathbf{u}) = -\frac{\boldsymbol{\nabla} p}{\rho} + \nu\nabla^2\mathbf{u} \tag{55}$$

Without viscosity this is known as Euler's equation. Including viscosity this is known as the Navier-Stokes equation. This equation is equivalent to conservation of momentum in a fluid.