PHY141 Lectures 11 notes

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Contents

1 Quantization of Light and Energy

In classical physics, the energy can have any value. For example, the total orbital energy of a planet depends continuously on its semi-major axis. The classical theory of electromagnetic radiation predicts that electromagnetic waves can have any intensity and wavelength.

Previously we introduced fundamental constants c (the speed of light), and the gravitational constant G. **Planck's constant** h appears when quantum mechanical processes are important.

$$h = 6.6 \times 10^{-34} \text{ J s}$$

1.1 The photoelectric effect

The photoelectric effect is based on experiments of illuminating a surface with light. Light can liberate electrons from a material. Typically visible light for alkali metals, near ultraviolet for other metals, and extreme ultraviolet for non-metals, is required to liberate electrons from the surface.

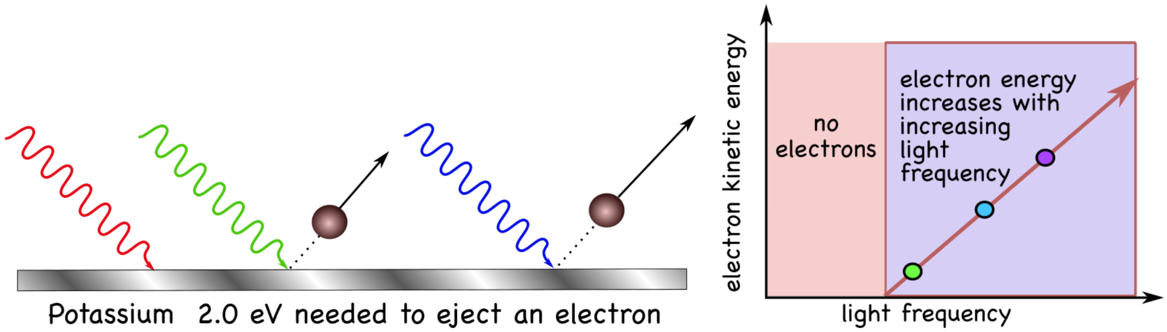

Figure 1: The photoelectric effect. A minimum frequency of light is required to liberate electrons from the surface. Above this frequency, the kinetic energy of the electrons is linearly dependent on the frequency of the light. The liberated electrons are called photoelectrons.

- No electrons are liberated if the wavelength of the incident light is larger than a critical value. (Red light emits no electrons on something like sodium). The critical wavelength value depends on the material of the surface.
- The maximum energy of each liberated electron does not depend on the *intensity* of the incident light.
- Increasing the intensity of the light increases the *number* of photoelectrons, but not their maximum kinetic energy.
- The maximum kinetic energy of the liberated electrons does depends on the wavelength of the incident light. A weak incident violet light will eject only a few electrons, but their kinetic energies are greater than those for intense incident light at longer wavelengths.

1.2 Photons

Einstein explained this effect by interpreting light as a collection of **photons** with discrete energies. Each photon is a light packet with energy

$$E = h\nu \tag{1}$$

where ν is the frequency and h is Planck's constant.

An electromagnetic wave with frequency ν has wavelength λ

$$\lambda = c/\nu \tag{2}$$

where c is the speed of light. You can think of the wavelength as the distance travelled during a single oscillation period $1/\nu$. Using the relationship between wavelength and frequency equation 1

$$E = h\nu = \frac{hc}{\lambda}. \tag{3}$$

Light behaves both like a wave and like a particle. This is known as **wave-particle duality**. Soon after the photoelectric effect experiments, other experiments showed that other particles also illustrated **wave-particle duality**. For example, electrons can show interference and diffraction.

Question: What is the momentum p carried by a photon?

Answer:

$$p = \frac{E}{c} = \frac{h\nu}{c} = \frac{h}{\lambda}. \tag{4}$$

We now show why equation 4 makes sense. Light travels at the speed of light so we need to take into account relativistic effects. For a massive particle $E = \gamma mc^2$ and $p = \gamma mv$. We compute

$$\begin{aligned} E^2 - p^2c^2 &= (\gamma mc^2)^2 - (\gamma mv)^2c^2 \\ &= m^2c^4\gamma^2(1 - v^2/c^2) \\ &= (mc^2)^2. \qquad \text{because} \qquad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}. \end{aligned}$$

What happens if a particle has zero rest mass?

$$\text{If } m = 0 \text{ then } E^2 - p^2c^2 = 0 \text{ and } p = E/c.$$

Setting $p = E/c$ for a photon (as in equation 4) is equivalent to specifying that photons have zero rest mass.

As $v \to c$, the ratio

$$\lim_{v\to c} \frac{E}{p} = \lim_{v\to c} \frac{\gamma mc^2}{\gamma mv} = c$$

and $p \to E/c$. Since photons travel at the speed of light, we might expect this to be true for them (and this is what we wrote in equation 4). Massless particles must travel at the speed of light and must obey $E = p/c$. Because photons obey $E^2 - p^2c^2 = 0$, photons can be called **null vectors**.

1.3 The de Broglie wavelength and wave/particle duality

Equation 4 for a photon gives wavelength in terms of Planck's constant and its momentum,

$$\lambda = \frac{h}{p}.$$

However, any particle can have momentum. Is the wavelength computed from the momentum and Planck's constant interesting?

The **de Broglie wavelength** for a massive particle is based on the particle's momentum

$$\lambda_{\rm deBroglie} \equiv \frac{h}{p} \tag{5}$$

where we follow the expression for the wavelength of a photon in terms of its momentum. Here, however, the particle's momentum depends on its velocity and mass, $p = \gamma m v$.

De Broglie, in his 1924 PhD thesis, proposed that just as light has both wave-like and particle-like properties, electrons, which are massive, also have wave-like properties. Later on experiments showed that massive particles can show wave-like behavior, including interference and diffraction.

With slow moving electrons, the de Broglie wavelength can be large enough to measure experimentally. The de Broglie wavelength can be inserted into the Bragg condition to predict an interference pattern. The Bragg condition relates the wavelength, inter particle spacing of a lattice to the angle for constructive interference. This was confirmed experimentally with electrons, illustrating that massive particles can exhibit wavelike behavior.

1.4 Explaining the photoelectric effect

Why do quantized light packets (photons) explain the photoelectric effect?

We assume that a single quanta of light ejects a single electron. Ignore multi-photon processes.

A critical energy is needed to liberate a single electron E_*. The energy E_* characterizes the binding energy for the electron in the material. You can also think of it as the depth of a potential well.

The photon energy must be above this critical energy to liberate a single electron.

$$\begin{aligned} E_{\rm photon} = \frac{hc}{\lambda} &> E_* \\ \lambda &< \frac{hc}{E_*} \end{aligned}$$

If the wavelength is too large, then the photons don't have enough energy to eject electrons.

If the energy of the photon is greater than the critical energy then the electron comes out with a higher kinetic energy. However the maximum kinetic energy is equal to

$$K = E_{\rm photon} - E_* = h\nu - E_*$$

This means that the maximum kinetic energy is independent of the light intensity, but is dependent on the wavelength of the light. There is a linear relation between electron kinetic energy K and ν frequency of light (see Figure 1).

1.5 The hydrogen atom - emission and absorption lines

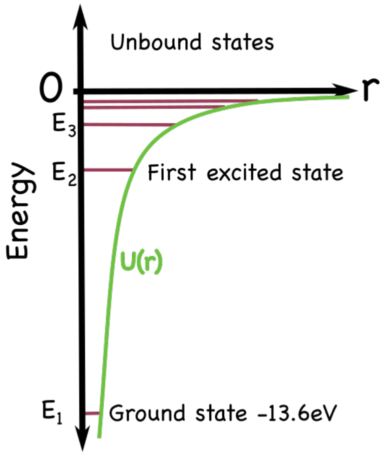

Figure 2: Atomic energy levels of the hydrogen atom. The energy levels $E_n = -\frac{E_1}{n^2}$ where integer quantum number $n \in (1, 2, ...)$. The ground state has an energy of $E_1 =$ -13.6 eV. The green line shows the potential energy of the electron/proton electrostatic interaction.

Energies for photons are quantized. Energies for other systems are also quantized. A hydrogen atom consists of a negatively charged electron in the vicinity of a positively charged nucleus. The potential energy is that due to the electromagnetic interaction.

The Rydberg formula gives the transition wavelengths, or the difference between energy levels in the hydrogen atom (or other atoms with a single electron)

$$\frac{1}{\lambda} = Z^2 R_\infty \left(\frac{1}{n_2^2} - \frac{1}{n_1^2} \right)$$

where Z is the atomic number of the nucleus. For the hydrogen atom $Z = 1$. Here n_1 is the quantum number of the upper energy level and is a positive integer. Here $n_2 > n_1$ is the quantum number of the lower energy level and is also a positive integer. R_∞ is the Rydberg constant.

If $n_2 = 1$ then transitions are from the Lyman series. These tend to be in UV.

If $n_2 = 2$ then transitions are from the Balmer series. Some of these are at visible wavelengths.

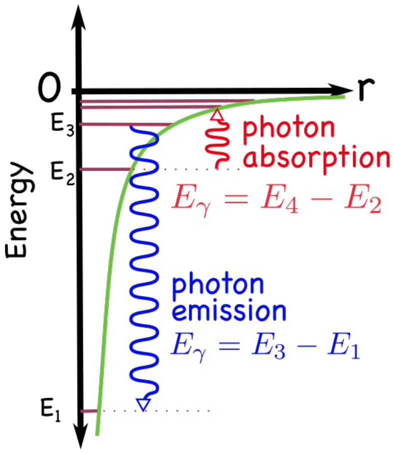

Figure 3: Bound-bound transitions of the hydrogen atom. The photon on the left, with energy $E_\gamma = E_3 - E_1$, is emitted when the atom jumps from energy level E_3 to E_1. The atom on the right jumps from level E_2 to E_4 when the photon on the right is absorbed, with energy $E_\gamma = E_4 - E_2$.

Bound-free transitions allow a bound electron to be ejected from the atom. The final electron kinetic energy can have any value, so bound-free transitions are a continuum absorption process.

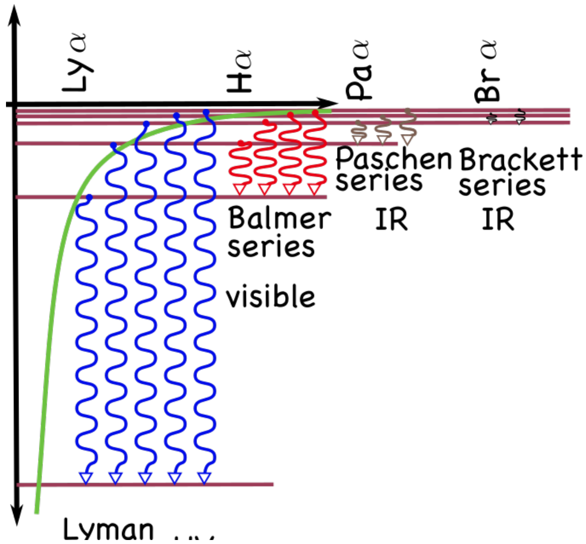

Figure 4: Different series of transitions for the hydrogen atom. The lowest energy transition for each series is denoted with the greek letter α.

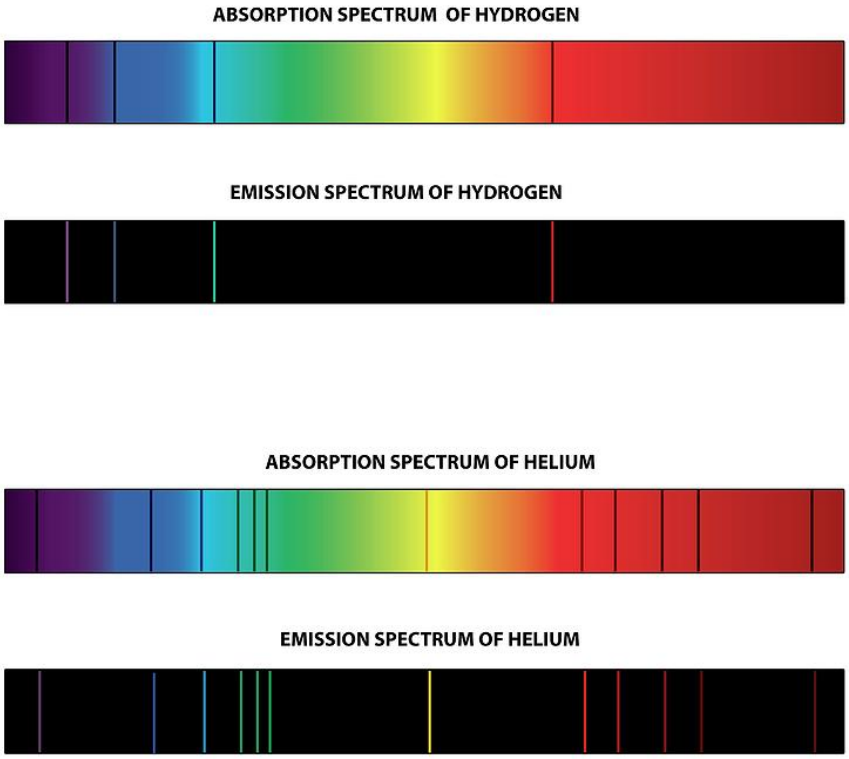

Figure 5: Absorption and emission lines. Absorption lines are seen when a cold gas is in front of a continuum radiation source.

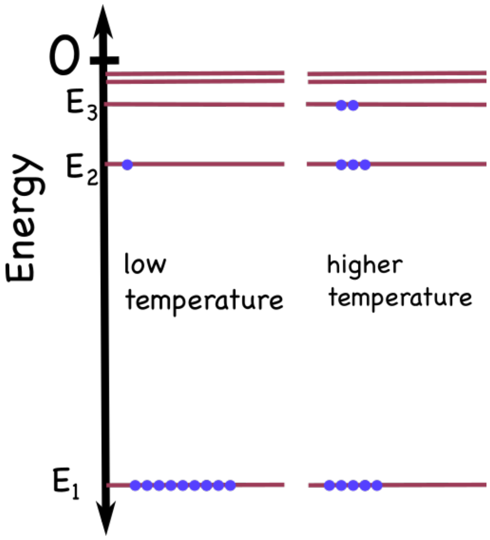

Figure 6: Temperature dependence of a spectrum. A transition between energy levels E_i and E_f cannot take place unless the initial energy level E_i is populated. The intensity ratio of two different frequencies of emitted light is often sensitive to temperature.

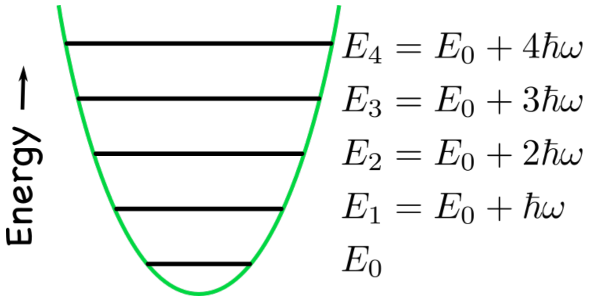

Figure 7: A quantized harmonic oscillator. The frequency $\omega = \sqrt{k/m}$ for spring constant k and mass m. The green line shows potential energy curve $U(r) = \frac{1}{2}kx^2$.

1.6 The harmonic oscillator and molecular vibrational spectra

A model for the bond between two atoms in a molecule is a spring. The model is not exact, but this approximation can be used for the low energy states. Energy levels found by solving the quantum mechanical version of a harmonic oscillator.

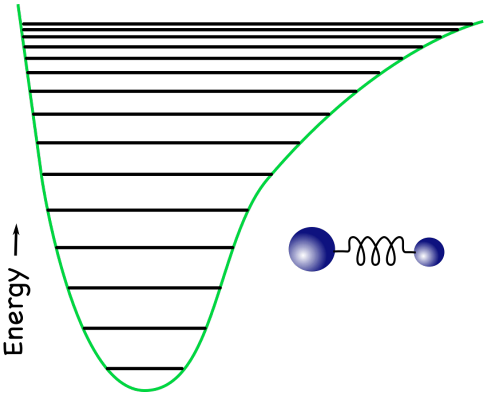

Figure 8: Because of the the attractive force between two atoms of a molecule, a molecule can vibrate. The potential energy of the force is shown in green. The energy levels are quantized. At low energies, the potential energy can be approximated with a quadratic function, giving evenly spaced energy levels, just like the quantum harmonic oscillator. The harmonic oscillator has no unbound states, however the molecule does have unbound states. For photons to be emitted or absorbed, the molecule must have a dipole moment.

The energy levels are

$$E_n = \hbar\omega_0 \left(\frac{1}{2} + n \right) \tag{6}$$

with integer $n \geq 0$ and

$$\hbar = \frac{h}{2\pi}.$$

The separation between energy levels is $\hbar\omega_0$. The possible photons for transitions have energies that are integer multiples of $\hbar\omega_0$. (Notice that $h\nu = \hbar\omega$ is consistent with $\omega = 2\pi\nu$.)

For an atom of mass m connected to a spring with spring constant k that has one fixed end, $\omega_0 \equiv \sqrt{\frac{k}{m}}$.

For a free molecule with two atoms of mass m_1, m_2 what is the frequency? The frequency $\omega_0 = \sqrt{\frac{k}{\mu}}$ where μ is the reduced mass $\mu = m_1 m_2/(m_1 + m_2)$.

1.7 Molecular rotational energy

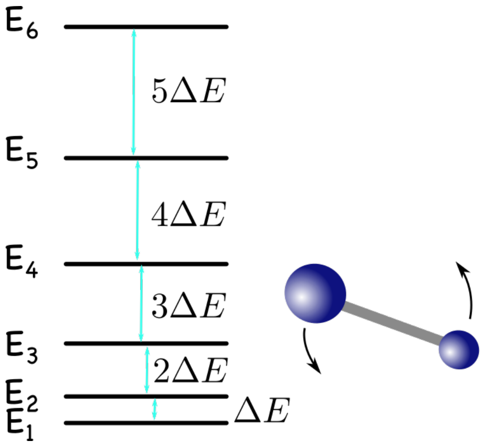

Figure 9: Rotational energy for molecules. Here $\Delta E = \hbar^2/I$.

Energy levels for a quantized rotor depend on quantum number l, an integer, $l \geq 0$,

$$E_l = \frac{\hbar^2}{2I} l(l+1) \tag{7}$$

The moment of inertia is I and $\hbar = h/(2\pi)$.

Transition energies for photons due to transitions from consecutive energy levels

$$\begin{aligned} \Delta E_{l \to l-1} &= E_l - E_{l-1} \\ &= \frac{\hbar^2}{2I} \left[l(l+1) - (l-1)l \right] \\ &= \frac{\hbar^2 l}{I} \end{aligned}$$

The transition energy depends on the quantum number l.

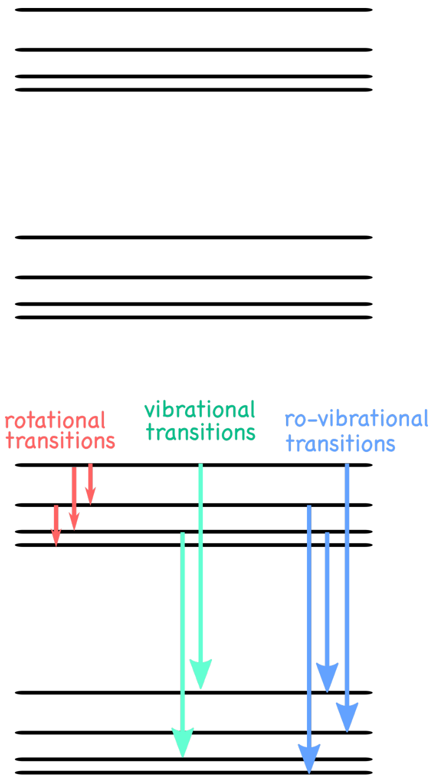

Figure 10: Ro-vibrational energy states and transitions for molecules. Depending upon the type of molecule, there can be selection rules allowing or forbidding various types of transitions.

1.8 Line and Continuum emission processes

The emission and absorption mechanisms discussed here are line emission and absorption mechanisms. There are also a number of continuum emission processes. Some of these involve quantized energy levels, such as bound-free emission or absorption. Examples of continuum radiation processes include synchrotron emission, Brehmstrallung emission and inverse Compton emission.

2 Lasers

Why are conventional lasers relevant to a discussion on quantization? They are relevant because they require an electron or atom to jump between energy levels. Power is used to create a population inversion where there are more objects in a higher energy level than a lower one. When one spontaneously drops down into the lower state it emits a photon. These photons can induce other objects to also transition to the lower energy state. The result is coherent emission. See illustrations in Figures 11, 12.

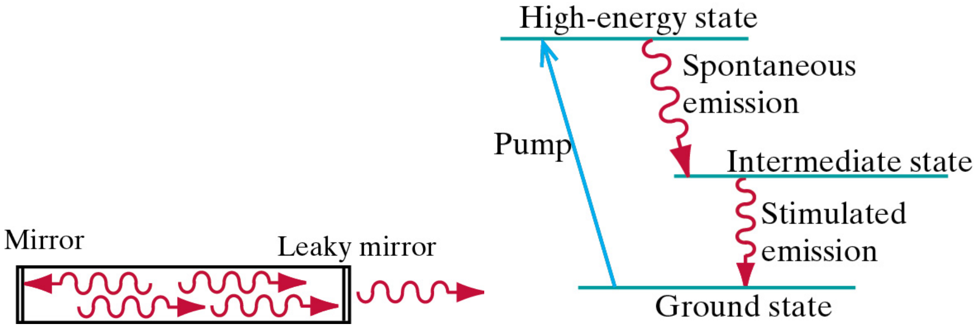

Figure 11: Lasers. The average photon will pass through the gain medium repeatedly before it is emitted from the output aperture or lost to diffraction or absorption. In most lasers, lasing begins with stimulated emission amplifying random spontaneously emitted photons present in the gain medium. Stimulated emission produces light that matches the input signal in wavelength, phase, and polarization.

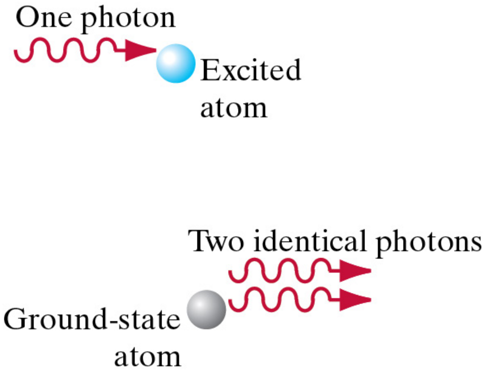

Figure 12: Stimulated emission.

Some comments: The rate of spontaneous emission is given by the Einstein A coefficient.

The rate of stimulated emission is computed from the Einstein B coefficient.

3 Summary

- Photons come in energy quanta that are packets of energy

  $$E = \hbar\omega = h\nu$$

  where ν is frequency and h is Planck's constant. Here $\hbar = h/(2\pi)$. The wavelength $\lambda = \frac{c}{\nu}$ where c is the speed of light. Photon momentum

  $$p = \frac{E}{c} = \frac{h\nu}{c} = \frac{h}{\lambda}.$$

- For a massive particle, the de Broglie wavelength, $\lambda_{\rm deBroglie} = \frac{h}{p}$.
- Energy levels are often quantized. Emitted or absorbed photons energies are equal to the difference between energy levels.
- The energy levels of a quantized harmonic oscillator are evenly spaced and separated by $\Delta E = \hbar\omega$ where $\omega = \sqrt{k/m}$. The energy levels are $E_l = \hbar\omega\left(l + \frac{1}{2}\right)$ where quantum number l is a positive integer or zero $l = 0, 1, 2, ...$

  The energy levels of a quantum rotor are $E_l = \frac{\hbar^2}{2I} l(l+1)$ where quantum number l is a positive integer ($l = 1, 2...$) and I is the moment of inertia. Transitions between these levels are rotational transitions in molecules. Consecutive energy level transitions have energy $\Delta E_l = E_l - E_{l-1} = \frac{\hbar^2 l}{2I}$.

  Together, the vibrational and rotational energy levels and transitions between them make up rovibrational spectra of molecules.