PHY141 Lectures 10 notes

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Contents

1 Potential energy in 1 dimension

In the previous lecture we considered the work done by constant forces and we showed that the work was equal to the change in kinetic energy. Now we consider the work done on a mass by **position dependent** forces.

Consider a force in 1 dimension $F(x)$. The work done along a particle trajectory that goes from x_1 to x_2 is

$$W = \int_{x_1}^{x_2} F dx.$$

Let's define a function $U(x)$ that satisfies

$$\frac{dU(x)}{dx} = -F(x).$$

We integrate this along a path

$$\begin{aligned} \int_{x_1}^{x_2} dx \frac{dU(x)}{dx} &= -\int_{x_1}^{x_2} F(x) dx = -W \\ U(x)\Big]_{x_1}^{x_2} &= -W \\ U(x_2) - U(x_1) &= -W \end{aligned}$$

The function $U(x)$ is known as the **potential energy**. In the last step we see that that the work, which is the integral of the force along a path from x_1 to x_2, only depends on the end points and is independent of the path taken. However this is only true if we can write the force as a gradient or derivative of a function.

2 Conservative forces in three-dimensions

For a force $\mathbf{F}$ we can integrate

$$W = \int_{\mathbf{x}_1}^{\mathbf{x}_2} \mathbf{F}(\mathbf{x}) \cdot d\mathbf{x}$$

Consider two paths both going from $\mathbf{x}_1$ to $\mathbf{x}_2$. We can reverse the second path and make a loop.

$$\text{If} \qquad \oint_{loop} \mathbf{F}(\mathbf{x}) \cdot d\mathbf{x} = 0$$

then we say that the force is **conservative**.

If the force is a function of position in 1-dimension then it is a derivative of some other function and the force is conservative.

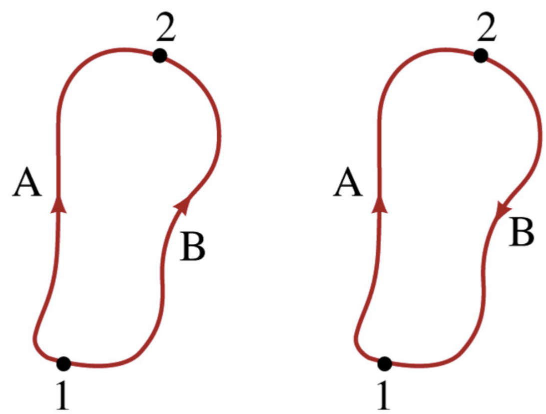

Figure 1: On the left we show two paths A and B going from point 1 to point 2. On the right we reverse the direction of path B so to form a loop.

In 3-dimensions, not all functions of position are gradients of some other function. If there is a function $U(\mathbf{x})$ such that

$$\boldsymbol{\nabla} U(\mathbf{x}) = \left(\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial U}{\partial z} \right) = -\mathbf{F}(\mathbf{x})$$

then the force is **conservative**.

3 Conservation of energy

We consider a conservative force with potential U. The force acts on a point mass m.

With $v = dx/dt$ (non relativistic setting), let's take the time derivative of the sum of kinetic and potential energy

$$\begin{aligned} E &= K + U \\ E &= \frac{1}{2} m v^2 + U(x) \\ \frac{dE}{dt} &= m v \dot{v} + \frac{dU}{dx}\frac{dx}{dt} \qquad = \frac{dK}{dt} - P \\ &= vF - Fv = 0, \end{aligned}$$

where I used Newton's law $F = m\frac{dv}{dt}$. In the non-relativistic setting and with a conservative force, energy which is a sum of kinetic plus potential energy, is conserved.

We can prove the same thing in three dimensions as long as the force is conservative so that we can find a potential such that $-\boldsymbol{\nabla} U(\mathbf{x}) = \mathbf{F}(\mathbf{x})$. We repeat the previous steps

$$\begin{aligned} E &= \frac{1}{2}mv^2 + U(\mathbf{x}) \\ \frac{dE}{dt} &= m\mathbf{v}\cdot\frac{d\mathbf{v}}{dt} + \boldsymbol{\nabla} U \cdot \frac{d\mathbf{x}}{dt} \\ &= \mathbf{v}\cdot(m\mathbf{a}) + \boldsymbol{\nabla} U \cdot \mathbf{v} \\ &= \mathbf{v}\cdot\mathbf{F} - \mathbf{F}\cdot\mathbf{v} \\ &= 0. \end{aligned}$$

Consider a velocity dependent force, such as

$$F(v) = -\alpha v.$$

If I integrate over a loop

$$\begin{aligned} W &= \oint F(v)\cdot dx \\ &= \oint -\alpha v \frac{dx}{dt} dt \\ &= \oint -\alpha v^2 dt \\ W &< 0 \end{aligned}$$

This must be less than zero for any trajectory. Hence work done $W < 0$ for the loop and the force is *not conservative*.

Velocity dependent forces, (even in 1d) are not *conservative* forces. Velocity dependent forces, such as friction and drag, dissipate energy. The energy does not disappear, it goes into heat or vibrations or fluid motions.

For non-relativistic particles and with potential energy U

$$E = K + U = \frac{1}{2}mv^2 + U(x)$$

and E is conserved. In this case we have neglected the rest mass (and heat). For relativistic particles the energy is more generally

$$E = \gamma mc^2 + U(x)$$

When integrating $W = \int \mathbf{F}\cdot d\mathbf{x}$ in classical systems we can arbitrary chose a constant offset for potential energy U without affecting the force or work done on a path.

4 Examples

In this set of examples I ignore heat flux as a source of energy.

4.1 Velocity from a potential energy difference

Consider a particle with initial position x_0 and velocity $v_0 \ll c$ that is in a force field with potential energy $U(x)$. Its initial energy is the sum of potential and kinetic energy

$$E = \frac{1}{2}mv_0^2 + U(x_0)$$

Suppose the particle moves to x_1. Energy is conserved so after moving to x_1

$$E = \frac{1}{2}mv_1^2 + U(x_1)$$

We can solve for v_1 from only the position x_1.

$$v_1 = \sqrt{v_0^2 + \frac{2}{m}\left(U(x_0) - U(x_1)\right)}$$

The resulting velocity is independent of the particle trajectory and only dependent upon the endpoint.

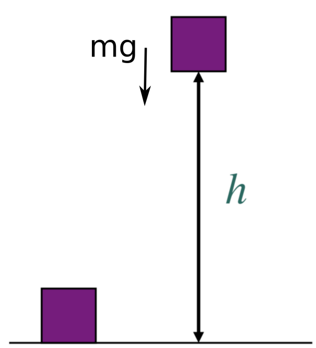

Figure 2: The difference in gravitational potential energy depends on the height of the mass.

4.2 Gravitational potential energy on the surface of the Earth

We consider a mass m initially on the surface of the Earth. The gravitational force is

$$\mathbf{F} = -mg\hat{\mathbf{z}}.$$

Here $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$ are unit vectors pointed in the +x, +y and +z directions, respectively. The potential energy associated with this is

$$U(z) = mgz$$

which we find by integrating a constant force. We check

$$\frac{\partial U}{\partial z} = mg$$

$$\mathbf{F} = -\left(\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial U}{\partial z}\right) = -\frac{\partial U}{\partial x}\hat{\mathbf{x}} - \frac{\partial U}{\partial y}\hat{\mathbf{y}} - \frac{\partial U}{\partial z}\hat{\mathbf{z}} = -mg\hat{\mathbf{z}}.$$

a) We move the mass from $z = 0$ to $z = h$. What is the work done by gravity? $\int_0^h dz = h$ and dz is positive but $\mathbf{F}$ is downward so

$$W = \int_0^h F dz = -mgh.$$

The work done has the same size (but opposite sign) as the change in potential energy.

b) A mass initially at rest and at $z = h$ falls to $z = 0$. What is its velocity at $z = 0$?

Because the velocity is initial zero, the kinetic energy is initially 0. The potential energy initially is mgh. The potential energy at the end is 0 but the kinetic energy is $mv^2/2$. Its energy is

$$E = K + U = 0 + mgh = mv^2/2 + 0$$
$$mgh = mv^2/2$$

We solve for

$$v = \sqrt{2gh}.$$

c) A mass initially at rest and at $z = h$ falls to $z = 0$. What is the work on the mass done by gravity? The direction travelled is down and so is the gravitational force. This means the work done by gravity $W = mgh$ is positive.

4.3 Work done by friction

Friction is not a conservative force. However I want to talk about it here so as to try to make clear what it means to have positive or negative work. A mass is on a horizontal surface (see Figure 3). It is initially at velocity v_0. It moves a distance d before coming to rest. The surface has kinetic coefficient of friction μ_k. What is the work done by friction? The friction force opposes the direction of motion so force times distance is negative. The work done by friction onto the mass is

$$W = -\mu_k M g d$$

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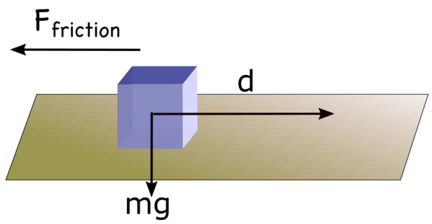

Figure 3: A block on a table initially with velocity v_0 is brought to rest by friction. It travels a distance d before coming to rest. The work done by friction is $W = F_{\text{friction}} d = -\mu_k M g d$. If it is initially moving, the kinetic energy of the block would decay because of friction. The work done on the block by the friction force is negative.

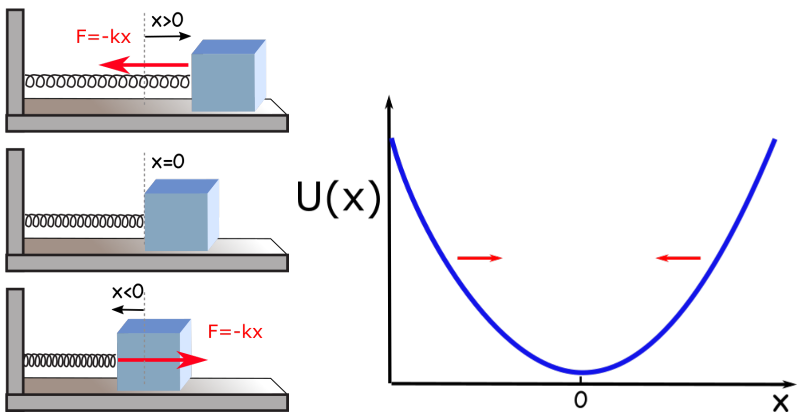

Figure 4: A spring force. On the right we show the potential energy function $U(x) = \frac{1}{2}kx^2$. For $x > 0$, the slope $\frac{dU}{dx} > 0$ and $F = -\frac{dU}{dx} < 0$ so the force pushes the mass to the left. For $x < 0$, the slope $\frac{dU}{dx} < 0$ and $F = -\frac{dU}{dx} > 0$ so the force pushes the mass to the right. The bottom of the potential well is a stable fixed point or equilibrium position.

4.4 Potential energy of a spring

The force of a spring onto a block of mass m is

$$F(x) = -kx$$

we integrate this

$$\begin{aligned} U(x) &= -\int^x F(x)\ dx = \int^x kx\ dx \\ &= \frac{kx^2}{2} + \text{constant}. \end{aligned}$$

The block is moved from its equilibrium position to position $x = A$. What is the work done?

The work

$$W = -\Delta U = -\frac{kA^2}{2}.$$

The spring force is in the -x direction, the displacement in the +x direction. So force times distance is negative.

4.5 Gravitational potential energy

We consider moving an object with mass m from position $\mathbf{r}_1$ to position $\mathbf{r}_2$ from the center of a planet with mass M. We take origin at the center of the planet. What is the work required to move the mass m from $\mathbf{r}_1$ to $\mathbf{r}_2$? Here the distance travelled Δr is not small compared to r_1 or r_2. For small distances we can use a constant gravitational acceleration (g) but for larger distances this would be a bad approximation.

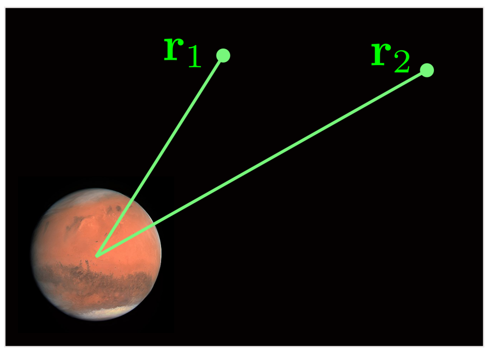

Figure 5: The gravitational potential energy function of a planet depends on radius from the planet's center.

The force is

$$\mathbf{F}(\mathbf{r}) = -\frac{GmM}{r^2}\hat{\mathbf{r}}.$$

Here the unit vector $\hat{\mathbf{r}} = \frac{\mathbf{r}}{r}$. This is a vector so we are looking for $U(\mathbf{r})$ such that

$$\boldsymbol{\nabla} U(\mathbf{r}) = -\mathbf{F}(\mathbf{r})$$

It makes sense to work in spherical coordinates. In spherical coordinates a function that depends on radius $U(r)$ has gradient

$$\boldsymbol{\nabla} U(r) = \frac{dU}{dr}\hat{\mathbf{r}}$$

So we need only integrate

$$\begin{aligned} U(r) &= -\int^r F(r)\ dr \\ &= \int^r \frac{GMm}{r^2} \\ &= -\frac{GMm}{r} \end{aligned}$$

To find the work required to move m from $\mathbf{r}_1$ to $\mathbf{r}_2$, we could integrate the force directly with

$$W = \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F} \cdot d\mathbf{x} = \int_{r_1}^{r_2} F(r)\ dr.$$

The integral does not depend on the path angle because the force is radial. Equivalently we can take the difference in the potential energies.

$$W = -\left(U(\mathbf{r}_2) - U(\mathbf{r}_1)\right) = GMm\left(\frac{1}{r_2} - \frac{1}{r_1}\right)$$

4.6 Electrostatic potential energy

The electric force from charge q_2 at position $\mathbf{r}_2$ onto charge q_1 which is at position $\mathbf{r}_1$ is

$$\mathbf{F}(\mathbf{r}_{21}) = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r_{12}^2}\hat{\mathbf{r}}_{12}$$

where $\mathbf{r}_{12} = \mathbf{r}_1 - \mathbf{r}_2$. The vector points from q_2 to q_1. The force is repulsive if the charges are the same and attractive otherwise. The associated potential energy is

$$U(r_{12}) = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r_{12}}.$$

It is positive if the charges are the same, otherwise negative if they are opposite.

Here ϵ_0 is the vacuum permittivity, or the permittivity of free space. The constant

$$k_0 = \frac{1}{4\pi\epsilon_0} = 8.897551787 \times 10^9 \text{N m}^2\ \text{C}^{-2}.$$

A single electron has charge

$$e = 1.602176565 \times 10^{-19}\text{C}$$

where C is a coulomb.

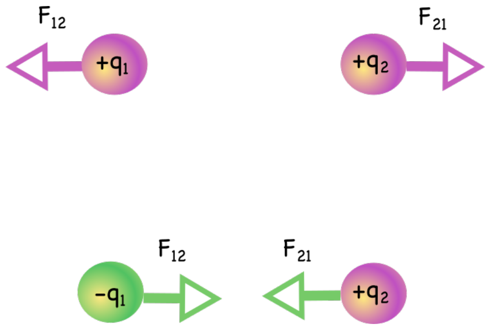

Figure 6: Electrostatic force depends on charge. The force on two charges is equal and opposite, $|F_{12}| = |F_{21}| = k\frac{q_1 q_2}{r^2}$ where r is the distance between the two charges, and q_1, q_2 are their charges. The force is attractive if the charges are opposite and is repulsive if the charges have the same sign. Here k is a constant.

What is the electric field $\mathbf{E}$? Taking a small test charge q, the electric field at a particular position is the force divided by q or $\mathbf{E} = \mathbf{F}/q$ from all other charges. This means

$$\mathbf{E} = -\frac{\boldsymbol{\nabla} U}{q}$$

where U/q is sum of the potential energies from all other charges.

4.7 Nuclear potential energy

We consider the force between a free proton and an atomic nucleus. At large distances, the force is dominated by the repulsive electric force as the charges are the same. At small distances, the force is dominated by the attractive nuclear force. A free particle usually cannot overcome the barrier. Only at high velocity can it overcome the barrier. Quantum tunneling makes it possible for a particle that is at an energy below the barrier peak to tunnel into the nucleus. This picture is relevant to estimating nuclear burning rates in the Sun.

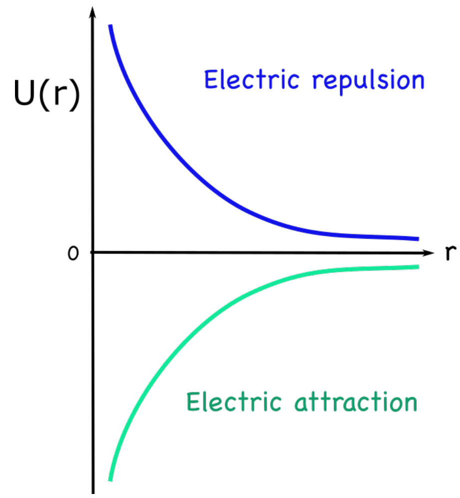

Figure 7: Top: Electric potential for two repulsing and same sign charges. $U(r) > 0$ and $\lim_{r\to\infty} U(r) = 0$. Bottom: Electric potential for two attracting and opposite sign charges. $U(r) < 0$ and $\lim_{r\to\infty} U(r) = 0$.

4.8 The Brachistochrone curve

We have two points at different horizontal positions and different heights. A mass starts at rest at $x = x_1$ and $z = h$. The mass slides down a curved frictionless surface, and reaches $x = x_2$ and $z = 0$. While sliding the mass is accelerated downward by the gravitational acceleration g.

Does the shape of the surface affect the final velocity?

Answer: No. The final velocity only depends on h because it only depends on the difference in potential energy at start and end points.

What shaped curve minimizes the travel time?

The answer is here: The brachistochrone curve.

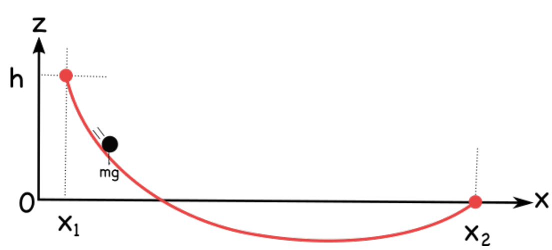

Figure 8: Consider a mass sliding down a curve starting at the red point on the left and ending at the right point on the right. Gravitational acceleration is in the vertical direction. The curve of fastest descent is the red curve.

5 Energy diagrams

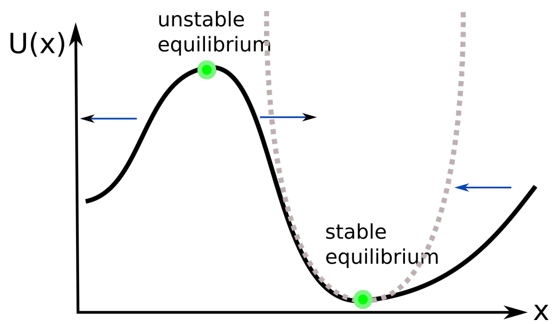

Figure 9: The maxima and minima of $U(x)$ are locations where the force is zero. This means they are equilibrium positions. Where the slope is negative the force is to the right. Where the slope is positive the force is to the left. The bottom of the valley (a green point on the right) is a stable equilibrium point whereas the top of the mountain (the green point on the left) is an unstable equilibrium point. The gray dotted parabola shows the second derivative at the equilibrium point and this sets the frequency of oscillations near equilibrium.

We consider the maxima and minima of a potential $U(x)$. At an extremum

$$\frac{dU(x)}{dx} = 0.$$

Because

$$F = ma = m\frac{dv}{dt} = -\frac{dU(x)}{dx} = 0$$

at an extremum, $dv/dt = 0$ at the extremum. A particle with $v = 0$ and $dv/dt = 0$ will stay a the same position and will not move.

Let x_* be an extremum and consider a particle with energy $E = U(x_*)$. Because $E = U + K$, the kinetic energy K must be zero at x_* for this particle when it has $x = x_*$. The velocity is zero and the velocity stays at zero. This point is a fixed or an *equilibrium* point.

It is either a stable or unstable equilibrium point dependent upon whether $U''(x)$ is positive or negative or zero. If $U''(x_*) > 0$, the equilibrium point is stable. Consider a point $x_* + y$ with small y. The force on the particle pushes it back toward the equilibrium point. If $U''(x_*) < 0$ the equilibrium point is unstable. The force on a particle that is near x_* would move it away from the equilibrium point.

We show some pictures with $E = K + U$ as horizontal lines on a plot showing $U(x)$. On these plots $K = E - U(x)$ is the distance between horizontal E lines and the $U(x)$ curves.

Kinetic energy $K > 0$, so if the horizontal line for E is always above $U(x)$ the particle is *free* and can reach large or small values of x. Otherwise the particle cannot cross the $U(x)$ curve. If the horizontal line intersects U on left and right sides, the particle cannot cross the x values of these intersections and it would be *bound*.

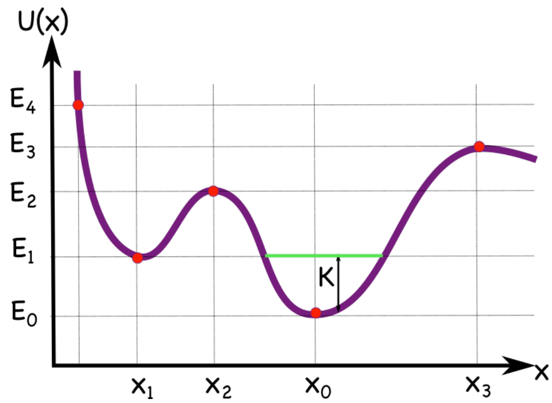

Figure 10: There are two regions (potential wells) where particles can be bound. There is a larger region where particles might not be bound (energies $E \geq E_3$). There are two stable equilibrium points (x_0, x_1) and two unstable equilibrium points(x_2, x_3). A particle with energy $E_2 \leq E < E_3$ could be found in either potential well. A particle with $E_1 \leq E < E_2$ is trapped either in the left well or the right one. If $E_0 \geq E < E_1$ then it can only be in the right potential well.

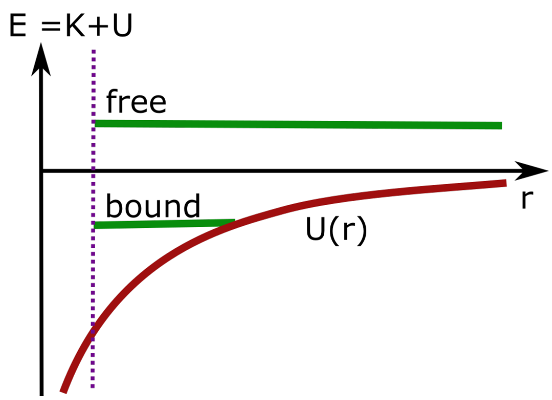

Figure 11: We show potential energy for a planet with radius shown by the dotted vertical line. The top horizontal line shows and energy that allows a particle to escape the gravitational field. The bottom horizontal line shows a particle that is bound. The difference between total energy and potential energy is the kinetic energy. Here the force is attractive and $U < 0$. The potential energy approaches zero as $r \to \infty$.

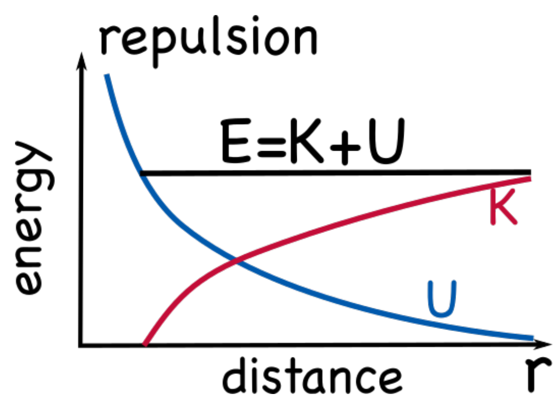

Figure 12: Two free protons approach each other. The sum of kinetic and potential energy remains fixed. They cannot get closer than where the kinetic energy reaches zero. Here the force is repulsive and $U > 0$. The potential energy approaches zero as $r \to \infty$.

5.1 Attracting particles at small radius

Consider an atomic nucleus, which is positively charged, and an electron which is negatively charged. They attract because they have opposite charges. What prevents them from getting very close together? If the problem was purely classical, angular momentum would prevent the two from getting close together and the electron would orbit the nucleus. But then the electron would emit radiation causing it to spiral in. Quantum mechanics gives a theory in which this does not happen.

Consider two black holes approaching each other. The gravitational force is attractive. Due to angular momentum they wind up orbiting each other. In this case gravitational wave emission can lead to loss of angular momentum and they can merge.

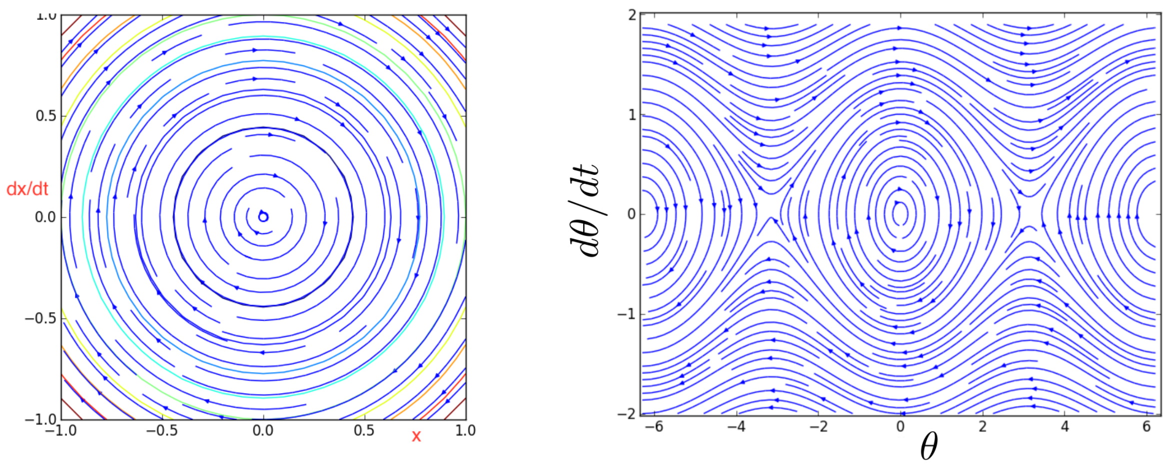

Figure 13: Left: Level curves and phase flow of the Harmonic oscillator. Right: Level curves and phase flow of the pendulum. The orbits are level curves. It can be easier to find the orbits by plotting energy contours than solving the equations of motion.

6 Level curves in phase space

The harmonic oscillator has potential energy $U(x) = \frac{1}{2}kx^2$. The total energy per unit mass

$$\frac{E(v, x)}{m} = \frac{1}{2}v^2 + \frac{1}{2}\frac{k}{m}x^2$$

where $v = \frac{dx}{dt} = \dot{x}$. On a plot of $\dot{x}$ versus x (or equivalently v versus x) and known as **phase space**, the curves of constant energy are ellipses (see Figure 13a). These are also the orbits.

6.1 Potential energy and energy level curves of a pendulum

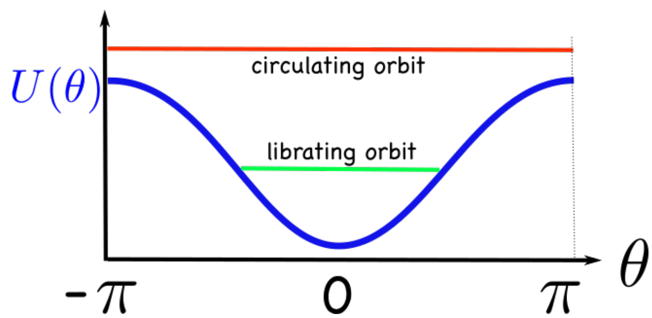

Figure 14: Potential energy of a pendulum. Oscillating orbits are those where the pendulum swings until it is inverted and then falls down to swing back up again. Librating orbits go back and forth about the hanging vertical position with $\theta = 0$.

For the pendulum the force is

$$F(\theta) = -mgL \sin \theta.$$

The potential energy is

$$U(\theta) = -mgL \cos \theta.$$

The kinetic energy is $K = \frac{m}{2}(L\dot{\theta})^2$ where $\dot{\theta} = \frac{d\theta}{dt}$. The total energy satisfies

$$\frac{E(\dot{\theta}, \theta)}{mL^2} = \frac{1}{2}\left(\frac{d\theta}{dt}\right)^2 - \frac{g}{L}\cos\theta$$

The curves of constant energy are also orbits and they are plotted in Figure 13b.

It is easy to plot contours of $E(\theta, \dot{\theta})$ and these show you the orbits in phase space. This can be easier than solving the equations of motion. For a pendulum, the equations of motion depend on elliptic functions unless θ is small. Even though we may not have a simple analytic form for the equations of motion, we do know how the orbits move in phase space.

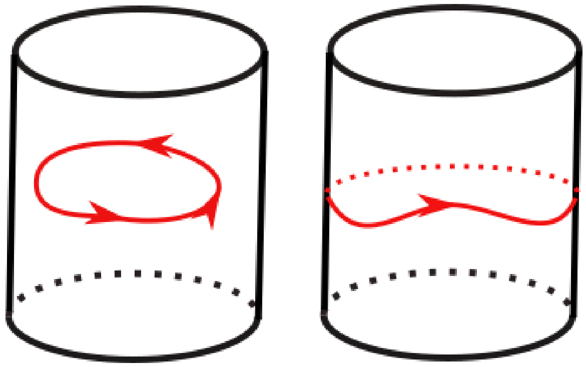

Figure 15: The difference between librating and oscillating orbits for the pendulum. Here the vertical axis is $\dot{\theta}$ and the position azimuthally on each cylinder is determined by θ.

7 Summary

- The relationship between potential energy and force.
  $U = -\int \mathbf{F} \cdot d\mathbf{x}$ and $\boldsymbol{\nabla} U = \frac{dU}{d\mathbf{x}} = -\mathbf{F}(\mathbf{x})$.
  Work $W = -\Delta U$.
- How to use potential energy differences to compute kinetic energy or velocity.
- How to find stable and unstable fixed points from a potential energy function (in 1d).
- How to identify regions where there are bound states from a potential energy function (in 1d).
- The meaning of conservative and dissipative forces.