PHY141 Lectures 1,2 notes

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Contents

1 Physics is an experimental science

Physics is an experimental science. No matter how beautiful the theory, if it is not relevant for experiments, then it should not become an important part of physics. That is not to say that a beautiful but irrelevant theory has no value necessarily. It may become important someday in a new physical setting or it may become important in another field, such as math.

Experiments are part of physics. This class traditionally has labs and in these labs you will take quantitative measurements and understand how well you can measure physical quantities. Experiments confirm and challenge theories but also can be sources for discovery and innovation. Because of this, we are starting the lectures with an introduction to units and error estimates.

2 Units

Physics research is primarily done in SI units. SI = International System of Units or *Système Internationale d'Unités*. Note that these units are *international*. Groups of physicists from all over the world have organized and collaborated (and are continuing to do this) to carefully define our system of units so that we have a common language for numerical quantities.

See: NIST on Units or SI units

In the beginning of this class we will primarily be concerned with lengths in meters, time in seconds, and mass in kg. This is known as **MKS**. Note: astronomers and cosmologists sometimes work in *cgs* or cm/g/s. Either one is easier than working in units such as inches, feet, pounds or cubits.

From the MKS system of units we will construct units of force (Newtons N), energy (Joules J), momentum (kg m/s), pressure (Pascals Pa), acceleration (m/s^2), velocity (m/s), or power (Watts $W = J/s$).

Then we will add in units of temperature K (Kelvin).

Physical constants like the Boltzmann constant, the gravitational constant, and the speed of light will be given in these units.

An example of converting from one set of units to another. We convert a velocity that is 50 miles/hour into m/s.

$$50\frac{\text{miles}}{\text{hour}} \times \frac{5280 \text{ feet}}{\text{mile}} \times \frac{12 \text{ inches}}{\text{foot}} \times \frac{2.54 \text{ cm}}{\text{inch}} \times \frac{\text{hour}}{60 \text{ minutes}} \times \frac{\text{minute}}{60 \text{ second}} \times \frac{\text{m}}{10^2 \text{ cm}} = 22.2\frac{\text{m}}{\text{s}}$$

Notice that the same unit on the top cancels one on the bottom, just like it would if it were natural numbers in a fraction.

I redo estimate quickly to make sure we got it roughly correct

$$10^2 \times 10^4 \times 10 \times 1 \times 10^{-2} \times 10^{-2} \times 10^{-2} = 10^{2+4+1+0-2-2-2} = 10^1 = 10$$

It is a good idea to check units and order of magnitude with every calculation. Sometimes arithmetic errors can be caught with a unit check.

2.1 Scientific Notation

$5.322275345345345 \times 10^8$ yr = 530 Myr = 5.3E8 yr.

Try not to use more decimal places than you need or are accurate.

2.2 SI unit definitions

The seven defining constants of the SI (système internationale) are:

- the Cesium hyperfine frequency $\Delta\nu_{Cs}$;
- the speed of light in vacuum c;
- the Planck constant h;
- the elementary charge e;
- the Boltzmann constant k_B;
- the Avogadro constant N_A; and
- the luminous efficacy of a defined visible radiation K_{cd}.

The units themselves

- **Unit of length: meter**. The meter, symbol m, is the SI unit of length. It is defined by taking the fixed numerical value of the speed of light in vacuum c to be 299,792,458 when expressed in the unit m s^{-1}, where the second is defined in terms of the cesium frequency $\Delta\nu_{Cs}$, the unperturbed ground-state hyperfine transition frequency of the cesium 133 atom.
- **Unit of time: second**. The second, symbol s, is the SI unit of time. It is defined by taking the fixed numerical value of the cesium frequency $\Delta\nu_{Cs}$, to be 9,192,631,770 when expressed in the unit Hz, which is equal to s^{-1}. One oscillation period is the inverse of the frequency in Hz.
- **Unit of mass: kilogram**. The kilogram, symbol kg, is the SI unit of mass. It is defined by taking the fixed numerical value of the Planck constant h to be $6.62607015 \times 10^{-34}$ when expressed in the unit J s, which is equal to kg m^2 s^{-1}, where the meter and the second are defined in terms of the speed of light c and cesium frequency $\Delta\nu_{Cs}$.

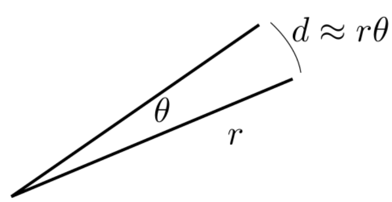
- **Unit of electric current: ampere**. The ampere, symbol A, is the SI unit of electric current. It is defined by taking the fixed numerical value of the elementary charge e to be $1.602176634 \times 10^{-19}$ when expressed in the unit C (coulombs), which is equal to A s, where the second is defined in terms of the cesium frequency $\Delta\nu_{Cs}$.

- **Unit of thermodynamic temperature: kelvin**. The kelvin, symbol K, is the SI unit of thermodynamic temperature. It is defined by taking the fixed numerical value of the Boltzmann constant k_B to be 1.380649×10^{-23} when expressed in the unit J K^{-1}, which is equal to kg m^2 s^{-2} K^{-1}, where the kilogram, meter and second are defined in terms of Planck constant h, speed of light c and cesium frequency $\Delta\nu_{Cs}$. Here J is Joules.

- **Unit of amount of substance: mole**. The mole, symbol mol, is the SI unit of amount of substance. One mole contains exactly $6.02214076 \times 10^{23}$ elementary entities. This number is the fixed numerical value of the Avogadro constant, N_A, when expressed in the unit mol^{-1} and is called the Avogadro number. The amount of substance, symbol n or N, of a system is a measure of the number of specified elementary entities. An elementary entity may be an atom, a molecule, an ion, an electron, any other particle or specified group of particles.

- **Unit of luminous intensity: candela**. The candela, symbol cd, is the SI unit of luminous intensity in a given direction. It is defined by taking the fixed numerical value of the luminous efficacy of monochromatic radiation of frequency 540×10^{12} Hz, to be $\mathrm{K}_{cd} = 683$ when expressed in the unit lm W^{-1}, which is equal to cd sr W^{-1}, or cd sr kg^{-1} m^{-2} s^3, where the kilogram, meter and second are defined in terms of Planck constant h, speed of light c and cesium frequency $\Delta\nu_{Cs}$. Here sr is steradian, lm is lumen and W (watts) is J/s.

2.3 Angles



The small angle approximation $\sin\theta \sim \theta$ is only correct if θ is in radians.

$$1^\circ \times \frac{\pi \text{ radians}}{180^\circ} = 0.01745 \text{ radians}.$$

It is often better to do calculations in radians rather than degrees.

In astronomy we often work in units of arcseconds or ” and arcminutes or ’. There are 60’ per degree and 60” per arcminute.

$$1'' \times \frac{1'}{60''} \times \frac{1^\circ}{60'} \times \frac{\pi \text{radians}}{180^\circ} = 4.8 \times 10^{-6} \text{radians}$$

3 Errors and why might we care about them?

How accurate are your measurements? Are you sure you are measuring anything sensible? Do you need better measurements to measure something interesting? Should you spend time by improving your experimental setup? Do your experiments merit spending your time on complicated error analysis? Similar questions are relevant when you make your own decisions about the reliability or quality of other people's work or when you are trying to convince other people that you have measured something significant.

3.1 AGN variability story

In the late 90's images taken from the Hubble Space Telescope revealed structure in galaxies at higher angular resolution than possible at that time from the ground. See Figure 1 for a Hubble Space Telescope image of a nearby diffuse galaxy. Using images from the Hubble Space Telescope, around 1998, we measured brightness variations in the centers of some galaxies. Are these evidence of variability and so from accretion onto a massive black hole? Or are the brightness variations due to photometric errors from read out noise, point spread function variations, cosmic rays and other instrument related issues? How do we tell the difference?

To answer this question we found (using images in the archive) a set of similarly observed galaxies that we were sure did not contain active black holes. We repeated the same types of measurements on this sample. The differences in brightness of the active set of galaxies were larger than the differences in brightness of the non-active set of galaxies. We argued that the brightness variations in the active set were due to variability in the luminous sources associated with accretion onto massive black holes (aka Active Galactic Nuclei or AGN). What do we mean by ‘larger’? We estimated a standard deviation, σ, from the non-active galaxy measurements. The differences from the active set were many times this standard deviation σ. We are going to discuss why this is important below.

The point of this story is that sometimes one can be inspired to devise an experiment solely for the purpose of measuring an error or uncertainty. We took the time to measure the uncertainty in this case because its size affected our interpretation.

Interestingly, this study relied upon galaxies that were observed **twice** by the Hubble Space Telescope, during a time when the Space Telescope had a draconian and unscientific policy of removing all possible duplicate observations (even if they were not actually duplicates in position or wavelength) from the observation schedule.

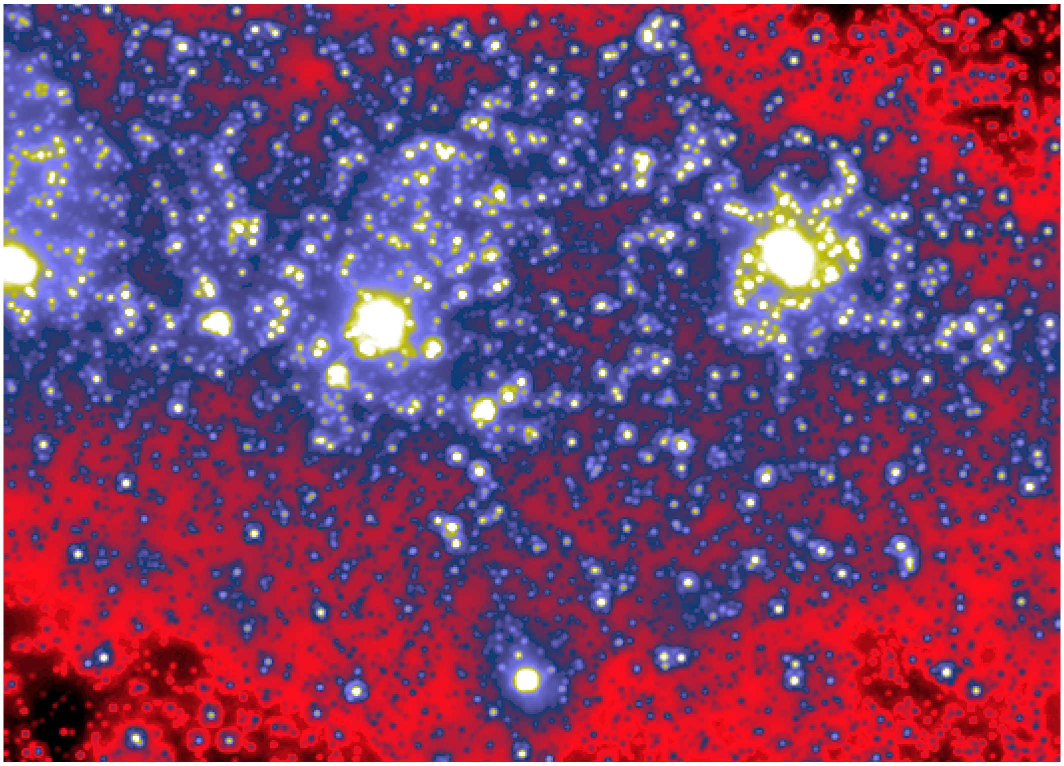


Figure 1: A J-band image (from the Hubble space telescope and using the NICMOS camera) of dwarf irregular galaxy NGC 1569. The galaxy is quite diffuse. Star clusters and stars show diffraction rings. This image illustrates that determining whether a source in a galaxy varies is complicated by the crowded field. This image is a piece of an image from StarFinder PSF modeling examples

3.2 Mean and standard deviation

Most measurements if taken over and over again would **scatter** about a **mean** value (see Figure 2). We implicitly assume that errors associated with measurements are taken from a random process that is described via a probability distribution.

The simplest way to describe a probability distribution is with a **mean** and a **standard deviation** or **variance**. By simplest way I mean with the least amount of required information. The standard deviation is often written as σ and the dispersion or variance is σ^2.

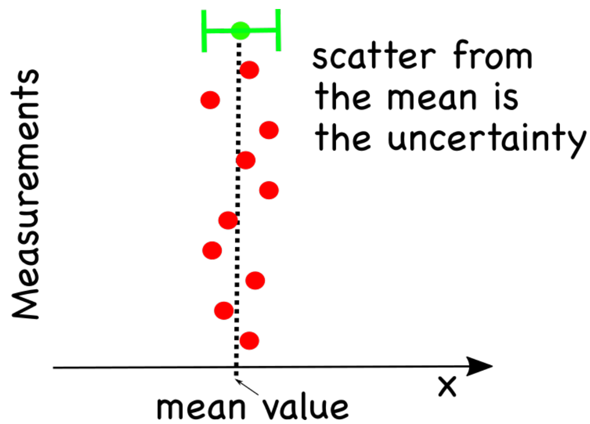


Figure 2: We show a series of measurements. The measurements all are different numbers. The average value is shown with the dotted line. The standard deviation describes the size of the scatter about the mean value. An estimate for the measurement is shown with the green data point and an errorbar. The error bar gives the width of the scatter or uncertainty in the measurement.

3.3 The Gaussian probability distribution

The Gaussian probability distribution is specified by 2 parameters, the mean and the standard deviation. Lacking any other information, we usually adopt or assume Gaussian errors. The other reason the Gaussian is popular is the **central limit theorem**.

This is what the Gaussian looks like, see Figure 3.

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp^{-\frac{(x-\mu)^2}{2\sigma^2}} \tag{1}$$

This is probability distribution so it is normalized to sum (or integrate) to 1;

$$\int_{-\infty}^{\infty} f(x)\ dx = 1$$

Here $f(x)\ dx$ gives the probability that you get x but within a width dx of possible values. As shown in Figure 3, you can think of the integral as a sum of individual regions of width dx.

A problem with the Gaussian is that $x \in \{-\infty, \infty\}$ but in reality we don't have infinite ranges for our variables. The Gaussian function slightly underestimates the probability of a variable that has a truncated range for the possible values.

The mean μ is at the peak of the distribution. The standard deviation σ describes the **width** of the distribution.

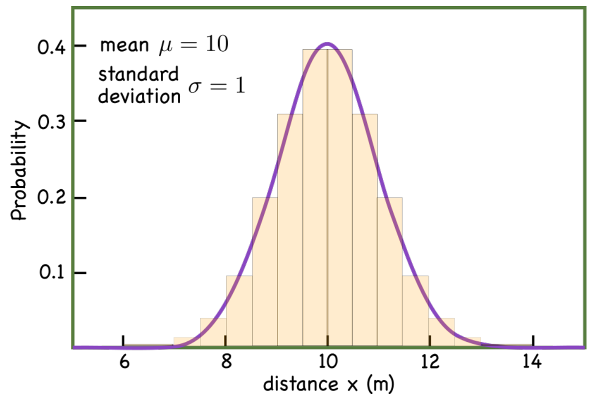


Figure 3: A Gaussian probability distribution of position with mean in distance x of $\mu = 10$ m and a standard deviation $\sigma = 1$ m, and following equation 1. We also show a histogram that approximates the distribution. The each orange bar represents the probability of x being within a specific range of values. The sum of these probabilities is 1.

The peak value is also the mean value, μ. (This is not true for all probability distributions). The mean value is the most likely value

$$\mu \equiv \int_{-\infty}^{\infty} x f(x)\ dx.$$

Here the thing we are interested in is x and we weight by $f(x)$ which is the probability distribution. Then we integrate (sum) over all possible x values to give the average value μ. The variance

$$\sigma^2 \equiv \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\ dx.$$

Here we are interested in the distance of x from the mean, and again we weight by the probability distribution to give a mean value for $(x-\mu)^2$.

Consider an experiment where the expected value of a measurement is 3 and the error (standard deviation) is 1 and you measure 4, which is 1σ away from the mean. Does your measurement agree with expected value or not?

One way to answer this is by giving a probability. What is the probability that a single measurement, taken from a Gaussian distribution is more than 1 σ away from the mean?

The following table shows the probability of a single measurement lying within $\pm 1, 2,$ or 3σ of the mean. These are computed using equation 2.

A single measurement	
	probability
within $\pm 1\sigma$	0.6827
within $\pm 2\sigma$	0.9545
within $\pm 3\sigma$	0.9973
within $\pm 4\sigma$	0.99993666
within $\pm 5\sigma$	0.999999426697

The function that gives these numbers is known as the error function (erf) and it depends on the integral of the Gaussian probability distribution.

The error function is defined as

$$\mathrm{erf}(x) \equiv \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\ dt.$$

The probability that a measurement is within $\pm n\sigma$ of the mean is

$$\begin{aligned}
P_{n\sigma} &= \int_{\mu-n\sigma}^{\mu+n\sigma} dx\ \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \\
&\text{by symmetry} \\
&= 2\int_0^{\mu+\sigma} dx\ \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \\
&\text{let} \qquad y = (x-\mu)/\sigma \\
&= 2\int_0^n dy\ \frac{\sigma}{\sqrt{2\pi\sigma^2}} e^{-\frac{y^2}{2}} = \int_0^n dy\ \frac{2}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} \\
&\text{let} \qquad t = y/\sqrt{2} \\
&= \int_0^{n/\sqrt{2}} dt\ \frac{2\sqrt{2}}{\sqrt{2\pi}} e^{-t^2} = \frac{2}{\pi}\int_0^{n/\sqrt{2}} dt\ e^{-t^2} = \mathrm{erf}(n/\sqrt{2})
\end{aligned}$$

The probability that a measurement is within $\pm n\sigma$ of the mean is

$$P_{n\sigma} = \mathrm{erf}(n/\sqrt{2}). \tag{2}$$

When the results of a series of measurements are described by a normal distribution with standard deviation σ and mean $\mu = 0$, then $\mathrm{erf}(a/(\sigma\sqrt{2}))$ is the probability that the error of a single measurement lies between $-a$ and $+a$, for positive a.

The above probabilities (in the Table) for measurements $\pm 1, \pm 2, \pm 3, \pm 4, \pm 5\sigma$ are calculated using the error function. The error function is a special function that can be computed numerically or with a series expansion.

It is not unlikely (i.e. it is common) to have a measurement outside $\pm 1\sigma$ from the mean but it is unlikely to have a measurement outside $\pm 3\sigma$. If you take enough measurements

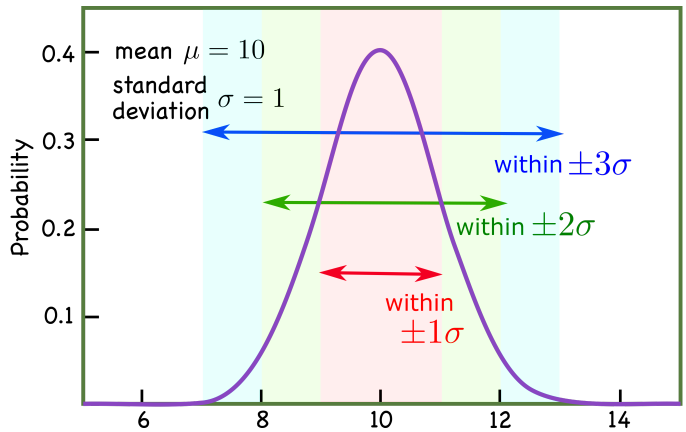


Figure 4: The regions of a Gaussian probability distribution that are with ± 1, 2 and 3 σ.

then it is likely to have one outside $\pm 3\sigma$. For example with a million pixels in an image and if the noise in the pixels is Gaussian, a bunch of them might be above 3σ from the mean. See figure 5 for an image showing 10^4 pixels each with value taken from a normal probability distribution.

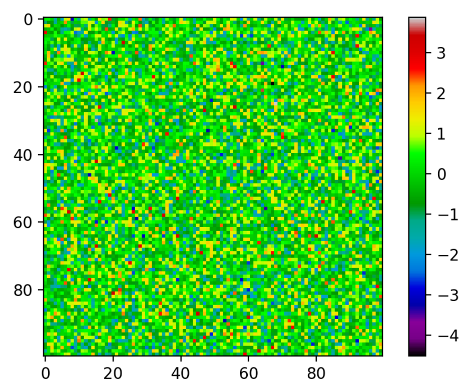


Figure 5: A camera has $100 \times 100 = 10{,}000$ pixels. A picture of a blank field is taken. Each pixel has value given with a normal distribution. With 10,000 pixels there are a few that are 4 sigma from the mean.

3.4 Errors add in quadrature

Consider two measurements, x_1, and x_2 each taken with a measurement error of σ (standard deviation). Add the two measurements to obtain

$$z = x_1 + x_2.$$

What is the error of z?

$$\sigma_z = \sqrt{\sigma^2 + \sigma^2} = \sqrt{2}\sigma.$$

By *quadrature* we mean quadratically as in a quadratic equation.

We could prove this by starting with a particular z. The probability of x_1 is given by $f(x_1)$ with the Gaussian function from equation 1. Given z we know that $x_2 = z - x_1$ and x_2 has probability $f(x_2)$. The probability of z is then $f(x_1)f(z - x_1)$ for all possible values of x_1. If we integrate this we can find the probability function for z. The integral is a convolution function and happily the convolution of a Gaussian with another Gaussian is also a Gaussian function. The new Gaussian will have the standard deviation given above.

3.5 Standard deviation of a sum

We now compute the standard deviation of a sum of N measurements, of x. We take N measurements, x_i with $i \in 1$ to N and a quantity z that is the sum

$$z = \sum_{i=1}^{N} x_i.$$

Generalizing from the sum of two measurements we find that the standard deviation

$$\sigma_z = \sqrt{\sum_{i=1}^{N} \sigma^2} = \sqrt{N}\sigma.$$

This can be made more general with N variables x_i each with their own standard deviation σ_i giving

$$\sigma_z = \sqrt{\sum_{i=1}^{N} \sigma_i^2}.$$

3.6 Measuring the mean value from measurements

We consider estimating a quantity from *multiple* measurements of the quantity x.

We estimate the mean of x with a sum of N measurements

$$z = \frac{1}{N} \sum_{i=1}^{N} x_i.$$

The sum is an average of the N measurements. This sum is an estimate for the mean of the distribution of the quantities x_i.

If the standard deviation of each individual measurement is σ what is the error in your computed estimate of the mean value?

Each measurement x_i has standard deviation σ but the quantity x_i/N has standard deviation σ/N. The variance is the sum of N things. Each of them has variance σ^2/N^2. Your computed mean has standard deviation

$$\sigma_z = \sqrt{N} \times \frac{\sigma}{N} = \frac{\sigma}{\sqrt{N}}. \tag{3}$$

Happily we see that the more measurements we have the smaller the error in the measurement of the mean value.

The error computed in equation 3 is sometimes called **standard error**.

If you combine individual measurements you can improve the quality of your measurement. What is meant by quality? A measurement with a small error is better than one with a big error. We can reduce the size of the uncertainty (or error) by combining measurements.

Example: Suppose you make ten measurements of a length and the mean of these 10 measurements is $\mu = 1.528$ m. The uncertainty of a single measurement is $\sigma = 0.510$ m. The standard error is $0.510/\sqrt{10} = 0.161$ m. Your measurement, along with its uncertainty would be

$$1.528 \pm 0.161 \text{ m}.$$

3.7 Measuring mean and standard deviation from a series of measurements

Consider 5 individual measurements of the gravitational acceleration g.

9.0 m/s^2
8.8 m/s^2
9.1 m/s^2
8.9 m/s^2
9.1 m/s^2

The mean is estimated with the average

$$\mu_m = \bar{x} = \langle x \rangle = \frac{1}{N} \sum_{i=1}^{N} x_i$$

where x_i are the individual measurements.

For our 5 measurements we estimate a mean value of

$$\mu_m = \frac{1}{5}\,(9.0 + 8.8 + 9.1 + 8.9 + 9.1)\,\mathrm{m/s^2} = 8.98\ \mathrm{m/s^2}$$

What is the size of the uncertainty in a single measurement?

We can measure the size of the uncertainty using the data measurements themselves. Using our mean value we can estimate the standard deviation of our measurements (aka our measurement error) with

$$\sigma = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(x_i - \mu)^2},$$

Usually we replace N with $N - 1$ so that the estimate for σ is not 'biased'.

We use this to estimate the standard deviation of a single measurement

$$\sigma_m = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(x_i - \mu)^2}, \tag{4}$$

For our 5 measurements

$$\begin{aligned}\sigma_m &= \sqrt{\frac{1}{5-1}\left[(9.0-9.0)^2 + (8.8-9.0)^2 + (9.1-9.0)^2 + (8.9-9.0)^2 + (9.1-9.0)^2\right]}\\ &= 0.12\ \mathrm{m/s^2}.\end{aligned} \tag{5}$$

Here I am using 9.0 for the mean as it is very close to 8.98. This gives us an estimate for the standard deviation of each **individual** measurement in $\mathrm{m/s^2}$.

Our measurement is then

$$8.98 \pm 0.12\ \mathrm{m/s^2}?$$

No. This is not correct. The error is smaller because we have added 5 measurements together. Each of the individual measurements has an error of $0.12\ \mathrm{m/s^2}$, and by combining the measurements we should get a smaller error.

Our measurement is

$$8.89 \pm \frac{0.12}{\sqrt{5}}\mathrm{m/s^2} = 8.98 \pm 0.05\ \mathrm{m/s^2}$$

because we have taken the average of 5 measurements each with standard deviation 0.12 $\mathrm{m/s^2}$.

Does our measurement value agree with the expected value of $9.80\ \mathrm{m/s^2}$?

The measurement is many σ away from the expected value. We would say that our measurement is *not consistent* with the known value of g.

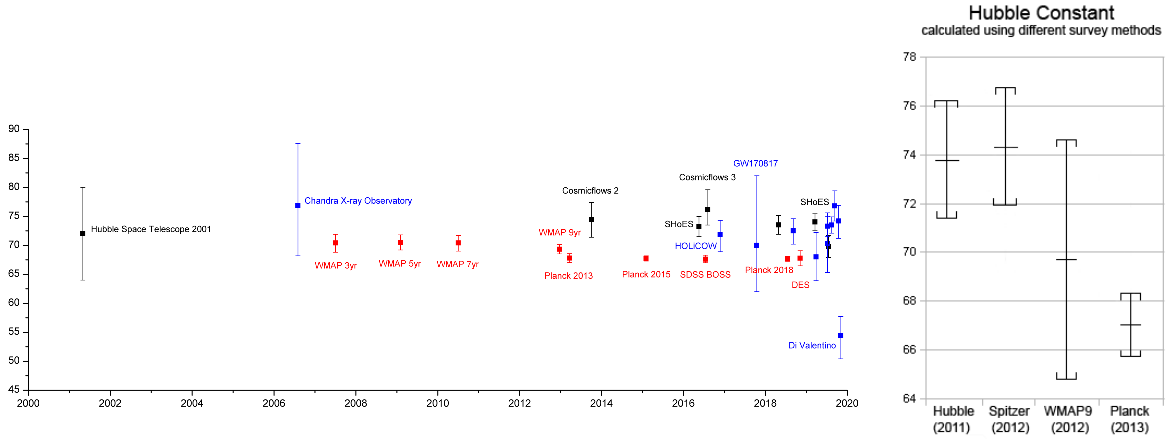


Figure 6: Recent measurements of the Hubble constant along with their 1σ uncertainties. Measurements made from the cosmic microwave background are significantly lower than those made from the nearby universe. This discrepancy is called the *Hubble tension*. This Figure is from Hubbleconstants_color.png and with license Creative Commons Attribution-Share Alike 4.0 International and Constant_Values.png by Alexander Stohr with license Creative Commons Attribution-Share Alike 3.0 Unported.

3.8 Tension in the Hubble constant

What do we mean when we say our measurement is **consistent with** or **discrepant** with somebody else's measurement?

A nice example might be the Hubble constant. Some recent measurements of the Hubble constant along with their 1σ uncertainties are shown in Figure 6. Measurements made from the cosmic microwave background (in red on the left and the WMAP and Planck measurements on the right) are lower than those made from tracers in the nearby universe, such as Cepheids variable stars. The error-bars are 1 σ uncertainties. Some measurements have small errorbars that do not overlap those of other measurements.

Consider two measurements, one for x with standard deviation σ_x and one for y with standard deviation σ_y. We take the difference $z = x - y$ which has standard deviation $\sigma_z = \sqrt{\sigma_x^2 + \sigma_y^2}$. The triangle inequality $\sigma_z < \sigma_x + \sigma_y$. Errorbars show σ_x, σ_y. If the difference between measurements is larger than the sum of the errorbars then the measurements are discrepant. The probability that such a large difference is seen would be low. With a Gaussian model, the likelihood can be estimated using the error function.

The Hubble constant measurements are many sigma apart which is unlikely if they were all measuring the same thing. The discrepancies between the measurements of the Hubble constant imply that we might be missing some physics in the interpretation or in making the measurements. The **Hubble Tension** is currently unresolved. Astronomers are motivated to resolve this tension, by exploring new scenarios or theories and improving the observational measurements.

3.9 Bias

A bias is when you tend to overestimate or underestimate a quantity. Bias is a quantitative term describing the difference between the average of measurements made for an object and its true value.

An example of bias might be an astronomical survey where it might be easier to measure brighter or nearer objects. If the bright or near objects are not typical of the population then your measurements can be biased.

3.10 Error as a percentage

It is sometimes useful to describe errors as a percentage. Suppose you measure a quantity $Y = 2.0 \text{ m} \pm 0.1 \text{ m}$. You can also write this as

$$\begin{aligned} Y &= 2.0 \text{ m} \pm 0.1 \text{ m} \\ &= 2.0 \text{ m} \times (1 \pm 0.05) \\ &= 2.0 \text{ m} \pm 5\%. \end{aligned}$$

The **fractional error** (or fractional standard deviation) is 5%.

The fractional error can also be computed as σ_Y/Y.

3.11 Frequency Measurement story

You are working on an experiment and you detect an unknown signal. You want to find out what is causing it. Maybe its frequency would give you a clue to where it is coming from. Unfortunately you are working in a lab that is full of clunky pieces of junk, including some ancient counters from the 1950s. You are lucky to have a functional oscilloscope.

Which is more accurate? Measuring a frequency with an oscilloscope (by eye) or measuring the frequency with one of those archaic counters?

Suppose you use the counter to count the number of cycles per second. If the signal is not horribly swamped by noise and the trigger levels are set appropriately, this gives an error of about $\Delta f = \pm 1$ Hz. Hz (Hertz) is the same as cycles per second. If the frequency is high then this gives a fractional error $\frac{\Delta f}{f}$ that is small. However if the frequency is low then ± 1 Hz gives a large fractional error.

Suppose we estimate the measurement uncertainty from using the oscilloscope display at about 1/4 box on the screen and that is about 1/4 period if you set the horizontal

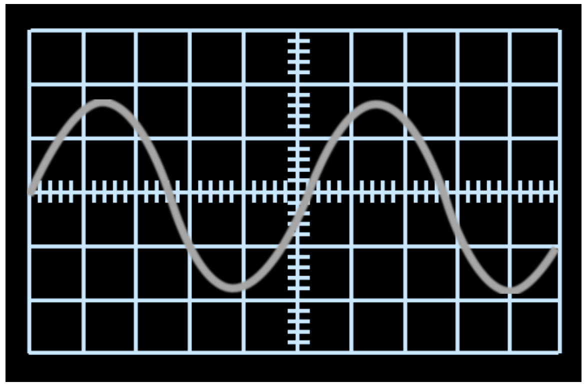


Figure 7: To measure frequency from an oscilloscope, you count the number of boxes in a single period. The horizontal axis is in units of time. The number of boxes gives you the period. The inverse of the period gives you cycles per second or the frequency of the signal.

time axis so that a full period is displayed. This is an uncertainty of about 25 % in a measurement of the frequency. This is not very accurate.

A frequencies larger than a few Hz and in terms of the fractional error, the counter would be more accurate. Even though the counter appears low tech compared the to oscilloscope, it can give quite accurate frequency measurements compared to the oscilloscope screen.

What would you do? You measure the frequency with more than one counter and with the oscilloscope. You check that the measurements are consistent. If the measurements are consistent, the counters (despite their appearance) are probably working. The counter measurement is then likely the most accurate, unless f is very low.

Question You measure the frequency and find that it is about 60 Hz. What is probably

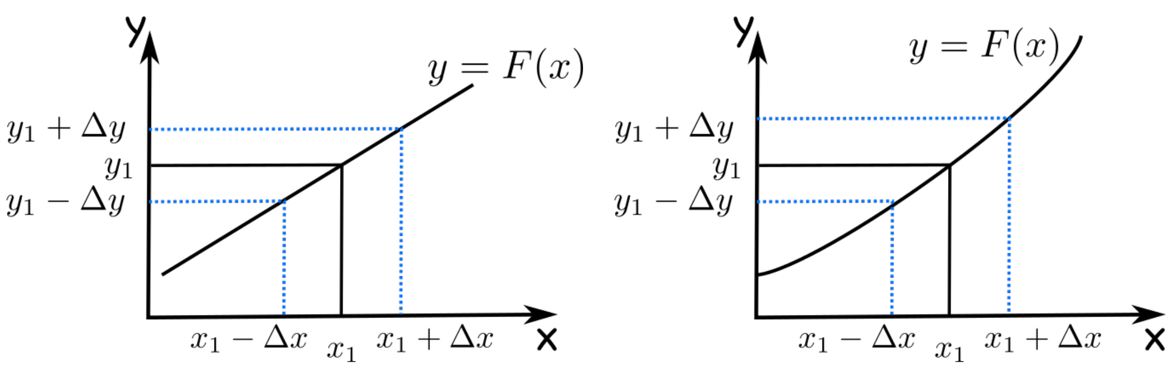


Figure 8: The errors in y depends on the error in x and on the slope.

causing this signal?

4 Propagation of Errors

4.1 An error in one quantity affects the error in another that depends on it

Consider a situation where you measure x but you actually want to measure a different quantity $y = F(x)$. You have a standard deviation σ_x and you need a standard deviation for your measurement of y or σ_y.

The degree to which an error in one measurable quantity affects the error in the another is driven by the functional dependence of the variables or the slope (see Figure 8).

How do you estimate the error for y with $y = F(x)$ from the error in x or σ_x? Following Figure 8 we use the slope

$$\sigma_y = \left| \frac{dF(x)}{dx} \right| \sigma_x. \tag{6}$$

As the derivative $\frac{dF(x)}{dx}$ depends on x, your estimate for σ_y is a function of the x value of the measurement. Figure 8 could have a curvy line for $F(x)$ instead of a straight one.

For example suppose you want to measure the gravitational acceleration g from measurements of the period of small oscillations of a pendulum. The period of small amplitude oscillations is

$$P = 2\pi\sqrt{\frac{L}{g}}.$$

We solve for g

$$g = L\left(\frac{2\pi}{P}\right)^2. \tag{7}$$

We check units and find that the equation is good! Our function is $g(P)$ as we measure P and we want an estimate for the gravitational acceleration g.

We use equation 7 to compute the derivative

$$\frac{dg(P)}{dP} = -2L\frac{(2\pi)^2}{P^3}$$

We plug this into equation 6 to compute the error

$$\sigma_g = 2L\frac{(2\pi)^2}{P^3}\sigma_P. \tag{8}$$

Suppose we measure a period of 6.3 ± 0.1 s on a pendulum of length 10.0 m. Using equation 7 we estimate that

$$g = L\left(\frac{2\pi}{P}\right)^2 = 10\text{ m} \times \left(\frac{2\pi}{6.3\text{ s}}\right)^2 = 9.95\text{ m/s}^2.$$

We compute the error using equation 8

$$\sigma_g = 2L\frac{(2\pi)^2}{P^3}\sigma_P = 2\times 10\ \mathrm{m}\frac{(2\pi)^2}{(6.3\ \mathrm{s})^3}0.1\ \mathrm{s} = 0.32\ \mathrm{m/s}^2$$

Our measurement is then

$$g = 9.95 \pm 0.32\ \mathrm{m/s}^2.$$

4.2 Propagation of Errors with partial derivatives

We look again at the formula for g

$$g(L,P) = L\left(\frac{2\pi}{P}\right)^2. \tag{9}$$

This depends on P but also depends on L and we may also have measurement errors in L.

We consider a more general setting $z = F(x,y,....)$. The partial derivative of a function $F(x,y,...)$ with respect to x is the derivative of the function w.r.t. x while keeping other variables fixed.

$$\frac{\partial F(x,y,...)}{\partial x} = \left.\frac{dF(x,y,...)}{dx}\right|_{y,...\ \text{fixed}}$$

With errors in both x, y and $z = F(x,y)$

$$\sigma_z = \sqrt{\left(\frac{\partial F}{\partial x}\right)^2\sigma_x^2 + \left(\frac{\partial F}{\partial y}\right)^2\sigma_y^2}.$$

This can be extended to more variables!

Going back to our equation for g (equation 9) we compute partial derivatives

$$\begin{aligned}
g(L,P) &= L\left(\frac{2\pi}{P}\right)^2 \\
\frac{\partial g(L,P)}{\partial L} &= \left(\frac{2\pi}{P}\right)^2 \\
\frac{\partial g(L,P)}{\partial P} &= -2L\frac{(2\pi)^2}{P^3} \\
\sigma_g &= \sqrt{\left(\frac{2\pi}{P}\right)^4\sigma_L^2 + \left(\frac{2L}{P}\frac{(2\pi)^2}{P^2}\right)^2\sigma_P^2}
\end{aligned}$$

Check units!

Applying this to our example of the period of a pendulum where we measured a period of 6.3 ± 0.1 s on a pendulum of length 10.0 m. Let's assume we have a measurement error on the length of the pendulum of a cm or ± 0.01 m.

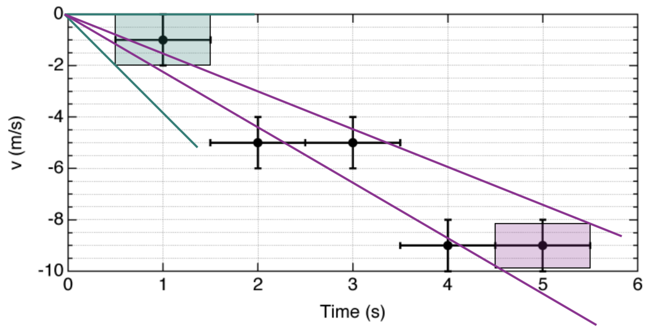


Figure 9: This shows velocity measurements as a function of time for a decelerating object. We assume a linear relation between velocity and time $v = at$ and want to estimate acceleration a. The slope is more sensitive to points with large velocities and times.

We compute

$$\frac{\partial g(L,P)}{\partial L}\sigma_L = \left(\frac{2\pi}{P}\right)^2 \sigma_L = \left(\frac{2\pi}{6.3 \text{ s}}\right)^2 \times 0.01 \text{ m} = 0.01 \text{ m/s}^2$$

We should add this in quadrature to our previous error. However our previous error was 0.32 m/s^2 and is so much larger than 0.01 m/s^2. We can justifiably ignore the contribution to the error due to the uncertainty in length. You can justify ignoring some error contributions if you can show they do not 'significantly' affect your measurements. The word significantly is quantitatively related to the affect on the standard deviation of your measurement.

Taking into account errors in measurement of length our measurement remains

$$g = 9.95 \pm 0.32 \text{ m/s}^2.$$

4.3 The weighted mean and its error

In Figure 9 we show times and velocities measured for a decelerating object. Our goal is to estimate the acceleration a using a linear model with $v = at$ or

$$a = \frac{v}{t}.$$

Using error propagation we find that the standard deviation in acceleration for a single data point is

$$\sigma_a = \sqrt{\left(\frac{\sigma_v}{t}\right)^2 + \left(\frac{\sigma_t v}{t^2}\right)^2} = \frac{v}{t}\sqrt{\frac{\sigma_v^2}{v^2} + \frac{\sigma_t^2}{t^2}}. \tag{10}$$

With standard deviation in σ_v independent of v and σ_t independent of t we see that σ_a is smallest for large t. Each point has a different error for its value of σ_a.

To compute the acceleration using all the data points we **weight** the individual measurements. Points with small errors should contribute more to the calculated value for the acceleration than points with large errors.

Our points are v_i, t_i. Each point is used to compute $a_i = v_i/t_i$. We use equation 10 to estimate σ_i for each acceleration value a_i.

The **weights** are the inverse of the square of the standard deviation.

With a series of measurements x_i and errors σ_i, the weights w_i are

$$w_i = \frac{1}{\sigma_i^2}, \tag{11}$$

and the mean value is estimated as

$$\mu_m = \frac{\sum_{i=1}^N w_i x_i}{\sum_{j=1}^N w_j} \tag{12}$$

Why chose this type of weight? This weight is derived from what is called a *maximum likelihood estimator*. The method of maximum likelihood is only applicable if the form of the theoretical distribution from which the sample is taken is known and here it is assumed to be Gaussian. In the absence of knowledge about the distribution, this weighting choice is a reasonable one, and it is 'most likely' to give you a computed value that is close to the actual one.

To compute the error in the measurement μ_m we need to again use error propagation. Consider a single term (the ith term) in the sum

$$\frac{w_i x_i}{\sum_{j=1}^N w_j} = \frac{x_i}{\sigma_i^2(\sum_{j=1}^N w_j)}$$

This single term has error

$$\frac{\sigma_i}{\sigma_i^2(\sum_{j=1}^N w_j)} = \frac{1}{\sigma_i(\sum_{j=1}^N w_j)}$$

because x_i has standard deviation σ_i. The error in μ_m is then the sum of these in quadrature or

$$
\begin{aligned}
\sigma_m^2 &= \sum_{i=1}^{N} \frac{1}{\sigma_i^2} \left(\frac{1}{\sum_{j=1}^{N} w_j} \right)^2 \\
&= \left(\sum_{i=1}^{N} w_i \right) \frac{1}{\left(\sum_{j=1}^{N} w_j \right)^2} = \frac{1}{\sum_{j=1}^{N} w_j} \\
\sigma_m &= \sqrt{\frac{1}{\sum_{j=1}^{N} w_j}} = \sqrt{\frac{1}{\sum_{j=1}^{N} \sigma_j^{-2}}}.
\end{aligned} \tag{13}
$$

Here we have assumed that we should weight each data point x_i with weight $w_i = 1/\sigma_i^2$. In this example we are estimating a slope from a few data points. By considering a least squares minimizer, our weighting system is a desirable or possibly an optimal way to weight data points. In statistics, a linear approach to modeling the relationship between a scalar response (or dependent variable) and one or more explanatory or model variables is referred to as a **linear regression**.

Let's go back to original problem. Using points in Figure 9 we made a list of acceleration measurements and each measurement has its own error. We use equation 12 to compute a combined measurement for the acceleration and equation 13 to compute the uncertainty in this measurement.

5 Summary

What do you need to remember:

- Work in MKS (SI) units.
- Express numerical results at a relevant precision level and with units.
- Check units in every equation you write down.
- What is meant by $\pm 1\sigma$, $\pm 2\sigma$ etc. What is the probability that a single measurement has a particular distance from the mean value using a Gaussian probability distribution.
- How to estimate the mean and standard deviation of a single measurement from a series of measurements.
- How to compute the error of an average of N measurements.
- Error propagation: Estimating the error in a variable from how it depends on another variable.

- Error propagation using partial derivatives and summing standard deviations in quadrature.
- What is meant by fractional error.
- Estimating a mean with weighted errors and knowing how to compute the uncertainty of this mean.

5.1 Handy formulas:

The mean of a series of measurements each with the same standard deviation:

$$\mu = \frac{1}{N} \sum_{i=1}^{N} x_i$$

The standard deviation computed from a series of measurements (this is the standard deviation of a single measurement).

$$\sigma = \sqrt{\frac{1}{N-1} \sum_{i=1}^{N} (x_i - \mu)^2}$$

The standard deviation of an average that is computed from N measurements with standard deviation σ (also known as the standard error)

$$\sigma_\mu = \frac{\sigma}{\sqrt{N}}$$

Computing an average of measurements, where each measurement has a different standard deviation (and we weight the measurements)

$$\mu = \frac{\sum_{i=1}^{N} w_i x_i}{\sum_{j=1}^{N} w_j} \qquad w_i = \sigma_i^{-2}$$

The standard deviation of this average

$$\sigma_m = \sqrt{\frac{1}{\sum_{j=1}^{N} w_j}}$$

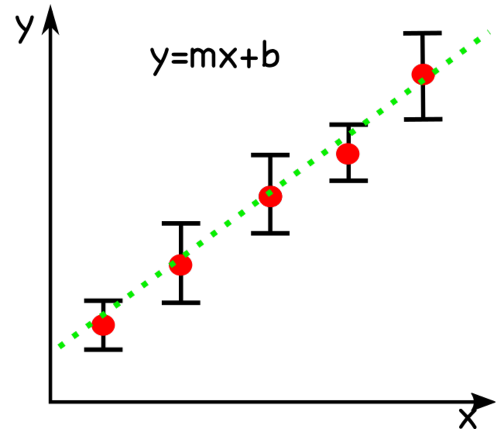


Figure 10: A line fit to data.

5.2 Least squares fit to a line

We consider a set of N measured points x_i, y_i with i going from 1 to N (see Figure 10). We assume x_i are given (no errorbars). We have standard deviations for the y_i measurements which are σ_i. We fit a line

$$y = mx + b$$

to these data. Here the slope is m and the intercept is b. We want a measurement for m, b and the standard deviations for these measurements.

The coefficients and their standard deviations can be computed with

$$\Delta = \sum_j \left[\frac{1}{\sigma_j^2} \left[\sum_i \frac{x_i^2}{\sigma_i^2} \right] \right] - \left[\sum_i \frac{x_i}{\sigma_i^2} \right]^2$$

$$m = \frac{1}{\Delta} \left[\left[\sum_i \frac{1}{\sigma_i^2} \right] \left[\sum_j \frac{x_j y_j}{\sigma_j^2} \right] - \left[\sum_i \frac{x_i}{\sigma_i^2} \right] \left[\sum_j \frac{y_j}{\sigma_j^2} \right] \right]$$

$$b = \frac{1}{\Delta} \left[\left[\sum_i \frac{x_i^2}{\sigma_i^2} \right] \left[\sum_j \frac{y_i}{\sigma_j^2} \right] - \left[\sum_i \frac{x_i}{\sigma_i^2} \right] \left[\sum_j \frac{x_j y_j}{\sigma_j^2} \right] \right]$$

$$\sigma_m = \sqrt{\frac{1}{\Delta} \sum_i \frac{1}{\sigma_i^2}}$$

$$\sigma_b = \sqrt{\frac{1}{\Delta} \sum_i \frac{x_i^2}{\sigma_i^2}}$$

where all sums go from 1 to N.

These formulae are from a linear regression model that minimizes the sum of the square of the weighted distances in y from the line $y = mx + b$. These formula can be derived using a 'least squares' maximum likelihood model.

You can use these formula to compute coefficients if you would like to fit a line to data and each data point has its own vertical error bar (in y).

If all the vertical error bars (in y) are the same size, these formula simplify to give slope and intercept

$$\begin{aligned} m &= \frac{N\sum_i x_i y_i - \sum_i x_i \sum_j y_j}{N\sum_i x_i^2 - (\sum_i x_i)^2} \\ &= \frac{\sum_i (x_i - \bar{x})(y_i - \bar{y})}{\sum_i (x_i - \bar{x})^2} \qquad (14) \\ b &= \frac{\sum_i x_i^2 \sum_j y_j - \sum_i x_i \sum_j x_j y_j}{N\sum_i x_i^2 - (\sum_i x_i)^2} \\ &= \bar{y} - m\bar{x} \qquad (15) \end{aligned}$$

with m from computed as in the first equation. Here the averages

$$\begin{aligned} \bar{x} &= \frac{1}{N}\sum_i x_i \\ \bar{y} &= \frac{1}{N}\sum_i y_i. \end{aligned}$$

These expressions are called *estimators*. These estimators can be derived by minimizing the function

$$Q(m, b) = \sum_i (y_i - b - mx_i^2).$$

Here $Q(m, b)$ is a sum of squares of distances in y of the data from the line $y = mx + b$ (this sum is also called χ^2). Standard deviations in the slope and intercept can be estimated

$$\sigma_m = \sqrt{\frac{N\sigma^2}{N\sum_i x_i^2 - (\sum_i x_i)^2}} = \frac{\sigma}{\sqrt{\sum_i (x_i - \bar{x})^2}} \qquad (16)$$

$$\sigma_b = \sqrt{\frac{\sigma^2 \sum_i x_i^2}{N\sum_i x_i^2 - (\sum_i x_i)^2}} = \sqrt{\frac{\sigma^2 \sum_i x_i^2}{N\sum (x_i - \bar{x})^2}} \qquad (17)$$