PHY141 Some order of magnitude physics relevant to our last lab!

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Contents

1 The coefficient of restitution of a bouncing sphere

1.1 The times and velocities of a series of bounces

Consider a ball moving vertically under downward gravitational acceleration, g. The vertical coordinate is z and $z = 0$ when the ball touches the floor. At $t = 0$ the ball touches the floor and its upward velocity is v_0. The trajectory

$$z(t) = -\frac{gt^2}{2} + v_0 t.$$

The ball returns to impact the ground at time

$$t_g = \frac{2v_0}{g}. \tag{1}$$

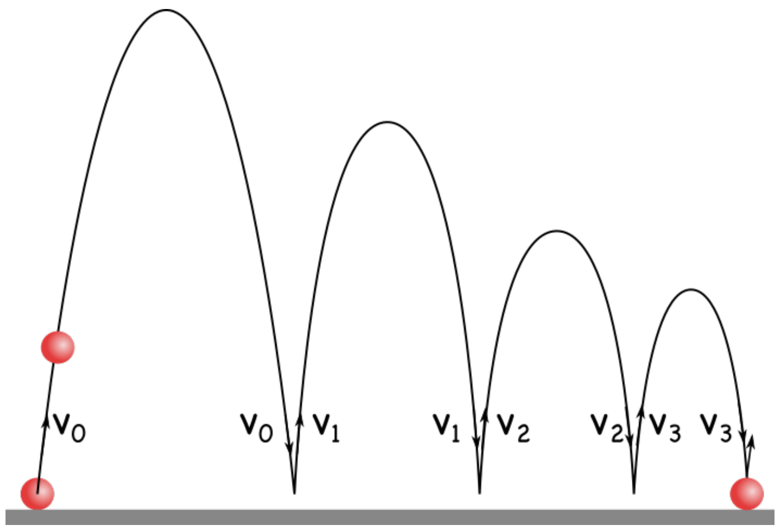

Figure 1: An elastic ball bounces off the floor. The ratio of velocity after and before each bounce is the coefficient of restitution. This figure shows some horizontal motion, but we will only consider a ball that bounces up and down vertically.

We neglect air resistance. The upward velocity at $t = 0$ is the same as the downward velocity at time $t = t_g$. We invert equation 1 to find the upward velocity at $t = 0$

$$v_0 = \frac{gt_g}{2}. \tag{2}$$

The elastic ball undergoes a series of bounces, as shown in Figure 1. We call t_i the time of the i-th bounce and v_i the upward velocity after the bounce at time t_i. Using equation 2 we can find v_i from the time between two bounces

$$v_i = \frac{g(t_{i+1} - t_i)}{2}. \tag{3}$$

We can use this equation to propagate errors. Suppose the error to measure the time is the same for all bounces and has a standard deviation σ_t. The standard deviation in each measurement of v_i is

$$\sigma_v = \sqrt{2}\frac{g\sigma_t}{2}. \tag{4}$$

Using our audio recording of a ball bouncing, we will measure the times of a series of bounces. We will use equation 3 to find the velocity of the ball just after each bounce, and equation 4 will be used to estimate the standard deviation of these velocities.

1.2 The coefficient of restitution

The coefficient of restitution, COR, is defined as the ratio of velocity after and before a collision or bounce,

$$COR \equiv \frac{v_{\rm after}}{v_{\rm before}}. \tag{5}$$

If $COR = 1$ the bounce is perfectly elastic and no energy is lost during the collision.

The definition of the coefficient of restitution implies that

$$v_{i+1} = (COR)\ v_i \tag{6}$$

where v_i is the velocity just after the i-th bounce.

With three bounces, we can compute two time intervals and this would give us two impact velocities. That's enough information to compute an estimate for the coefficient of restitution.

Inverting equation 6

$$COR = \frac{v_{i+1}}{v_i}. \tag{7}$$

We can propagate errors to estimate the standard deviation of COR from the standard deviations in the velocity measurements, σ_v,

$$\frac{\sigma_{COR}}{COR} = \sqrt{\frac{\sigma_v^2}{v_{i+1}^2} + \frac{\sigma_v^2}{v_i^2}}.$$

If the coefficient of restitution COR is independent of velocity then

$$v_i = (COR)^i v_0. \tag{8}$$

If we take the natural log of this equation

$$\ln\left(\frac{v_i}{v_0}\right) = i \ln(COR). \tag{9}$$

On a plot of $\ln\left(\frac{v_i}{v_0}\right)$ vs index i, we expect all the points to lie on a line and the slope of the line would be the log of the coefficient of restitution. If the points don't lie on a line then the coefficient of restitution probably depends on velocity.

Models for the coefficient of restitution can take into account the spherical shape of the ball, the contact area, the extent of deformation in elastic and plastic regimes for both ball and surface that the ball bounces on. Many models predict a weak dependence on velocity. If you are curious about physical models of the coefficient of restitution for a bouncing elastic sphere, see On predicting the coefficient of restitution .

1.3 Order of magnitude approaches

What physics is likely to be important and how do we decide?

We can make a list of parameters that are important or might be important in the problem.

- Radius of sphere R
- Mass of sphere M
- Initial height of sphere when you drop it h.
- Gravitational acceleration $g = 9.8\text{m/s}^2$.
- Density of air $\rho_{air} \sim 1.2 \text{ kg/m}^3$
- Elastic modulus of sphere material E
- Yield strength of sphere material Y
- Viscoelastic time-scale for sphere material τ_{visco} ($\tau_{visco} = E/\mu$) where μ is a viscosity.
- Elastic modulus of table top or ground E_{table}
- Speed of sound in air $c_{s,air}$
- Angle of impact
- Moment of inertia of sphere
- Friction coefficients

Material Units	Young's modulus (GPa)	Yield strength (MPa)
Glass	50	50
Rubber	0.01	
Wood	10	40
Hard plastic	2	50

From Young's modulus and tensile and compressive yield strengths of some common materials

Some computed quantities

- Velocity of first impact

$$v_0 = \sqrt{2hg} = 4.5 \text{ m/s} \left(\frac{h}{1 \text{ m}} \right)^{\frac{1}{2}} \tag{10}$$

- Free fall time

$$t_{freefall} = \sqrt{2h/g} = 0.44\text{s} \left(\frac{h}{1 \text{ m}}\right)^{\frac{1}{2}} \tag{11}$$

- Time between bounces

$$t_{bounce} \sim 2t_{freefall} = 2\sqrt{2h/g} \sim 1\text{s} \left(\frac{h}{1 \text{ m}}\right)^{\frac{1}{2}} \tag{12}$$

- Radius crossing time

$$t_{cross} = \frac{R}{v_0} = 2 \text{ ms} \left(\frac{R}{1 \text{ cm}}\right) \left(\frac{v_0}{5 \text{ m/s}}\right)^{-1} \tag{13}$$

- For many solids, the density of sphere $\rho \sim M/R^3 \sim 2\text{g/cc}$. If the sphere is hollow then its mean density could be much lower!

- The speed of elastic waves in the sphere

$$v_{p,sphere} \sim \sqrt{\frac{E}{\rho}} \sim 2000 \text{ m/s} \left(\frac{E}{10\text{GPa}}\right)^{\frac{1}{2}} \left(\frac{\rho}{2000 \text{ kg m}^{-3}}\right)^{-\frac{1}{2}} \tag{14}$$

- The time it takes sound waves to cross the sphere

$$t_s = \frac{R}{v_{p,sphere}} = 5\mu s \left(\frac{R}{1 \text{ cm}}\right) \left(\frac{v_{p,sphere}}{2000 \text{ m/s}}\right)^{-1} \tag{15}$$

1.4 The Buckingham Pi theorem

Suppose you have N physical quantities q_i where $i \in 1 \to N$ and they depend on k independent physical units. Then the number of independent dimensionless quantities that can be formed is $p = N - k$. The dimensionless numbers can be constructed as products

$$\pi_j = q_1^{a_{1j}} q_2^{a_{2j}} \ldots . q_N^{a_{nj}}$$

where the exponents are rational numbers.

Why are dimensionless numbers important?

Dimensionless numbers help you classify the physical regime.

Dimensionless numbers let you carry out experiments in the lab that are relevant and predictive for another setting where you cannot do experiments.

For example, wind tunnels are used to understand aerodynamics of airplanes. The hydrodynamics of a boat can studied with a small model in a tank, prior to building an expensive full sized one. The Reynolds number of the real object is matched by the model.

Let's show this with the Navier-Stokes equation in 1 dimensino which is derived using momentum conservation.

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial x^2} \tag{16}$$

where u is velocity, p is pressure and ν is viscosity. We divide all terms by velocity V distance L.

$$\frac{\partial u/V}{\partial t/(L/V)}\frac{V^2}{L} + \frac{u}{V}\frac{\partial u/V}{\partial x/L}\frac{V^2}{L} = -\frac{1}{\rho}\frac{\partial p}{\partial x/L}\frac{1}{L} + \nu\frac{\partial^2 u/V}{\partial (x/L)^2}\frac{V}{L^2} \tag{17}$$

We put distance in units of L with $X = x/L$ and us a dimensionless time $T = Vt/L$ and put velocity in units of V, with $\tilde{u} = u/V$. We use the sound speed c_s

$$\frac{dP}{dx} = \frac{dP}{d\rho}\frac{d\rho}{dx} = \frac{d\rho}{dx}c_s^2$$

$$\frac{V^2}{L}\left(\frac{\partial \tilde{u}}{\partial T} + \tilde{u}\frac{\partial \tilde{u}}{\partial X}\right) = -\frac{\partial \ln\rho}{\partial X}\frac{c_s^2}{L} + \frac{\nu V}{L^2}\frac{\partial^2 \tilde{u}}{\partial X^2} \tag{18}$$

We divide both sizes by V^2/L

$$\frac{\partial \tilde{u}}{\partial T} + \tilde{u}\frac{\partial \tilde{u}}{\partial X} = -\frac{\partial \ln\rho}{\partial X}\frac{c_s^2}{V^2} + \frac{\nu}{VL}\frac{\partial^2 \tilde{u}}{\partial X^2} \tag{19}$$

We define the Mach number as

$$M \equiv \frac{V}{c_s}. \tag{20}$$

We define the Reynolds number as

$$Re \equiv \frac{VL}{\nu}. \tag{21}$$

Equation 19 becomes

$$\frac{\partial \tilde{u}}{\partial T} + \tilde{u}\frac{\partial \tilde{u}}{\partial X} = -M^{-2}\frac{\partial \ln\rho}{\partial X} + Re^{-1}\frac{\partial^2 \tilde{u}}{\partial X^2} \tag{22}$$

The first two terms are of order unity if we have chosen V, L appropriately. Only when the Reynold's number is small does the last term become important. Only when the Mach number is large can there be density variations (as the first term on right must be of order 1!)

The point here is that we can rescale the equations describing our real system in terms of dimensionless numbers, in this case the Reynolds number. The physical equations would be the same in systems that are vastly different in size, as long as we match dimensionless numbers describing the regime.

1.5 Dimensionless numbers

Let's make some dimensionless parameters for the problem of the bouncing sphere.

Density ratio

$$\pi_\rho \equiv \frac{\rho}{\rho_{air}} \sim 2000 \left(\frac{\rho}{2\text{g/cc}}\right) \tag{23}$$

Acceleration parameter or Froude number

$$\pi_{Fr} \equiv \sqrt{\frac{v_0^2}{Rg}} \sim 16 \left(\frac{v_0}{5\text{m/s}}\right) \left(\frac{R}{1\text{ cm}}\right)^{-\frac{1}{2}} \tag{24}$$

We have a ratio of distances (radius to drop height)

$$\pi_R \equiv \frac{R}{h} \sim 10^{-2} \left(\frac{R}{1\text{ cm}}\right) \left(\frac{h}{1\text{ m}}\right)^{-1} \tag{25}$$

We have a ratio of velocities (impact velocity to elastic wave speed in the sphere)

$$\pi_v \equiv \frac{v_0}{v_p} \sim 2 \times 10^{-3} \left(\frac{v_0}{5\text{m/s}}\right) \left(\frac{v_p}{2000\text{m/s}}\right)^{-1} \tag{26}$$

We have a ratio of yield strength to Young's modulus

$$\pi_Y \equiv \frac{Y}{E} \sim 10^{-3} \tag{27}$$

We have a ratio of timescales - time for a sound wave to cross sphere divided by viscoelastic time

$$\pi_\nu \equiv \frac{t_s}{\tau_{visco}} \tag{28}$$

1.6 Is air resistance important?

To determine this we estimate a stopping time due to drag from air. The drag force

$$F_D \sim \rho_{air} \pi R^2 v_0^2 \sim M \frac{dv}{dt} \sim M \frac{v_0}{t_{stop}} \sim \pi \rho R^3 \frac{v_0}{t_{stop}}$$

This gives

$$t_{stop} \sim \frac{\rho}{\rho_{air}} \frac{R}{v_0} \sim 2000 \times t_{\text{cross}} \sim 4\text{ s} \left(\frac{R}{1\text{ cm}}\right) \left(\frac{v_0}{5\text{ m/s}}\right) \left(\frac{\rho}{2\text{ g/cc}}\right) \tag{29}$$

How does this compare to the time between bounces? It exceeds the time between bounces but not by a large factor. We have somewhat overestimated the stopping time because the trajectory is not at v_0 the entire time. Because acceleration is constant the velocity linearly depends on time. If $v(t) = \frac{v_0}{T}t$ then the average value of v^2 in an interval of time T is $\frac{1}{T}\int_0^T \frac{v_0^2}{T^2}t^2 dt = \frac{v_0^2}{T^3}\frac{T^3}{3} = v_0^2/3$. We have only overestimated by a factor of about 3. Air resistance may not be entirely negligible and it might really be important if you are doing this experiment with a lightweight ball. Using an equation for drag one can probably roughly estimate the energy loss between bounces due to drag from air.

What dimensionless parameter might tell us if air resistance is important? I would take

$$\begin{aligned} \pi_{air} \equiv \frac{t_{stop}}{t_{bounce}} &= \frac{\rho}{\rho_{air}}\frac{t_{cross}}{t_{bounce}} = \frac{\rho}{\rho_{air}}\frac{R}{4h} \\ &\sim 4\left(\frac{\rho}{2\text{g/cc}}\right)\left(\frac{R}{1\text{ cm}}\right)\left(\frac{h}{1\text{ m}}\right)^{-1}. \end{aligned} \tag{30}$$

If this dimensionless ratio is large then the stopping time is long and air resistance is negligible. Note that this ratio is primarily dependent on the density of the ball! So if the ball is hollow then air resistance might be important.

Note that dimensionless ratio π_{air} can be written as a produce of π_ρ and π_R. However we constructed π_{air} to delineate physical regimes and so it might be more useful than π_ρ or π_R. Physical arguments might allow you to determine which dimensionless ratios are likely to be important.

1.7 Does the ball significantly deform during the bounce?

Young's modulus E has units of pressure. Stress σ has the same units and is a force per unit area. In a linear regime, stress is Young's modulus times strain where strain $\epsilon = dL/L$ which is the amount of deformation. Potential energy is $\frac{1}{2}E\epsilon^2$ integrated over the volume. We can balance potential energy with kinetic energy to estimate the maximum strain during a bounce,

$$\frac{1}{2}E\epsilon_{max}^2 R^3 = \frac{1}{2}mv_0^2.$$

This gives

$$\epsilon_{max} \sim \sqrt{\frac{v_0^2}{E}\frac{m}{R^3}} \sim \frac{v_0}{v_p} \sim 2\times 10^{-3}\left(\frac{v_0}{5\text{ m/s}}\right)\left(\frac{v_p}{2000\text{ m/s}}\right)^{-1} \tag{31}$$

Note that $\epsilon_{max} \sim \pi_v$ is related to our dimensionless velocity ratio. For a solid sphere made of glass or hard plastic, it will not deform much but a rubber sphere will.

1.8 Is the yield strength important during the bounce?

The ball is likely to dissipate more energy if the yield strength is exceeded. We have an estimate for the maximum strain in the bounce from equation 31. If the maximum stress exceeds the yield strength, or

$$E\epsilon_{max} > Y$$

then it is likely that the yield strength is important.

Can we find a dimensionless parameter that tells us whether the yield strength is exceeded during the bounce? We can use

$$\pi_\epsilon \equiv \epsilon_{max}\frac{E}{Y} \sim \frac{v_0}{v_p}\frac{E}{Y} \sim \frac{v_0\sqrt{E\rho}}{Y} \tag{32}$$
$$\sim 1 \times \left(\frac{v_0}{5 \text{ m/s}}\right)\left(\frac{v_p}{2000 \text{ m/s}}\right)^{-1}\left(\frac{Y/E}{10^{-3}}\right)^{-1}$$

If this dimensionless parameter is greater than 1 then the yield strength is likely exceeded during the bounce. We could also write

$$\pi_\epsilon = \frac{\pi_v}{\pi_Y}$$

in terms of dimensionless ratios π_v, π_Y.

What does it mean to go above the yield strength? We could assume that the material deforms elastically until it reaches a maximum strain where it is at the yield strength. Past this point we could assume that the remaining kinetic energy goes into irreversible plastic deformation and is not regained. This description is backwards to what we might see if we could look at the deformation in real time. At the contact side, the ball would deform plastically but the rest of it might behave elastically. The fraction of energy lost depends on how much plastic deformation takes place.

A real material does not necessarily have a quick transition between elastic and plastic behavior. The material might be slightly plastic and mostly elastic in some intermediate regime. A single stress strain curve may not be good enough to describe the material. Compression and release might lie on different curves and the stress can be strain rate dependent.

Suppose the bounce is well below the yield strength where deformation is plastic. A material can deform elastically and reversibly like a spring and in this case no energy is lost. If the material loses energy while compressed then we could modify the model to give it a velocity dependent or strain rate dependent force. This is a 'viscoelastic' model and could be applied to estimate energy loss in a predominantly elastic regime. Unfortunately typical viscosity estimates for solids are not all that commonly estimated or measured.

So what might we expect for the coefficient of restitution? Because we might expect more energy lost at higher strain rates and at higher stresses, we would expect a larger fraction of energy is lost at higher bounce velocities than at lower velocities. The coefficient of restitution would be larger for smaller bounces than larger bounces.

1.9 Contact time. When do the bounces stop?

As the ball deforms it maintains contact with the ground or table top. Approximately what is the duration of the contact?

Suppose I have a spring with spring constant k and I compress it with a mass m by a distance d. Once released, how long does it take for the spring to reach its rest length? The time is of order $\sqrt{m/k}$ which is the oscillation period and is independent of how far the spring is compressed. With this analogy the time of contact is

$$t_{contact} \sim \frac{R}{v_p} \sim t_{cross} \frac{v_0}{v_p}$$

The crossing time is a few ms. If the sphere is hard then v_p exceeds the impact velocity by a hundred or so we expect the contact time to be of order 10^{-5} s. If the sphere is soft then the contact time could be longer.

We previously made the assumption that bounces are instantaneous. Happily we find here that this assumption is pretty reasonable, at least for the first few bounces!

The contact time is seems independent of the impact velocity. However, as the ball slows down, eventually the contact time will be the same as the time between bounces. We can make another dimensionless ratio

$$\pi_{contact} = \frac{t_{contact}}{t_{bounce}} = \frac{Rg}{2v_p v_0}$$

As the velocity gets slower, this dimensionless parameter gets larger. Eventually when it nears 1, the sphere can no longer get off the table or ground.

We haven't really discussed the role of the Froude number but it is important anywhere where gravitational acceleration g is important. The dimensionless parameter $\pi_{contact}$ can be written in terms of π_{Fr} and π_v.

1.10 The role of internal dissipation

At some point discuss the viscoelastic relaxation time. xxxxx

1.11 A comment on dimensionless numbers

Not all dimensionless numbers are necessarily meaningful. Physical arguments can help you decide which ones might be useful or important. Dimensionless numbers might help you determine which physical processes are important in a particular setting or problem.

In this discussion I neglected to discuss the elasticity of the table or ground and the sound speed. The bounce velocity is well below the sound speed in air, (though elastic waves in solids tend to be above the sound speed in air). The fraction of energy lost into sound is probably small. However, some energy must be lost through generating sound as we are using sound to make our measurements. Over much of this discussion we have ignored the elasticity of the table, but if the table is softer than the ball, then the coefficient of restitution would be more strongly dependent on the material properties of the table than of the ball.

We have also neglected horizontal motion, friction and spin.