Lab #5.

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Exploring properties of deformable and stretchy objects. Elastic, anelastic or anharmonic?

Equipment list

For each station: Red box of sensors for PHY141. Blue box of equipment for PHY141. Stand. Silver track. Rulers. Calipers.

At the front of the room: spare hardware, extra boxes with spare equipment. Rubber bands. String. Twist-ties. Scissors. Other elastic objects such as deformable or squishable toys, balloons.

1 Objectives

The goal in this lab is to explore how force and displacement deviates from that predicted by Hooke's law in some stretchy or deformable objects. You can bring in your own elastic objects to explore!

Since the class now has some experience with the labs, the experiments and goals of this lab should be decided by each lab group. This presents additional challenges. A challenge for this lab is that you will need to decide what you want to study and what you want to measure. You will also design your own experiments. It may take some trial and error to get nice measurements! You will need to decide which data to take and how to analyze your data. The lab report should describe what you learned from your experiments.

Here are some possible ideas for experiments:

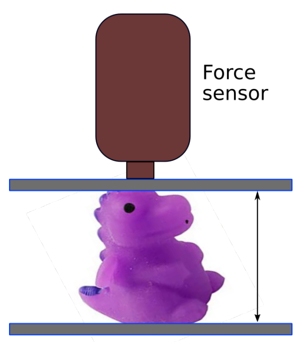

Figure 1: An example of an experiment. Here the force displacement curve is measured for a deformable toy. The displacement would be measured using a ruler or calipers for different levels of applied force.

- Measure the force displacement law for a compressed or stretched rubber ball, a deformable toy or a balloon. See Figure 1.

- Explore whether force for rubber band depends on whether the displacement is increasing or decreasing. Is the force dependent on velocity? Present a work loop diagram. An example of such an experiment is in Figure 2.
- Explore how and whether an elastic object permanently deforms or returns to its initial state. (We have small rubber bands that just keep getting longer).
- Compare the behavior of different rubber bands.
- Characterize the elasticity of an elastic fabric such as lycra.
- Measure the sensitivity the period of an oscillating system to oscillation amplitude. This would test how the system deviates from a harmonic oscillator. An example of such an experiment is in Figure 3.
- Your idea here!

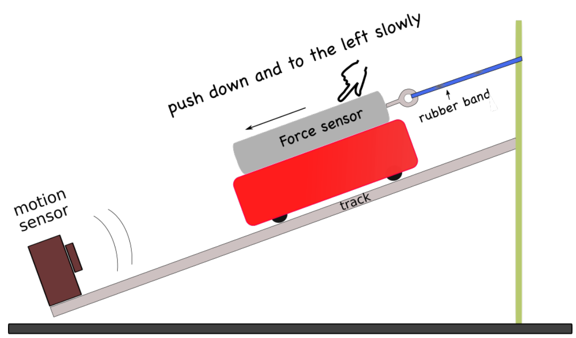

Figure 2: An example of an experiment. Here the force displacement curve is measured for a rubber band. The force displacement curve can depend upon the speed and direction of motion and the past history of deformation.

Notes on designing experiments:

You can measure force with the Pasco force sensor (and lab 4 has instructions on how to calibrate it). The motion sensor can measure the position of a cart, as shown in Figure 2. If you place the force sensor on the cart and put the cart on the track you can restrict motion to 1 dimension (along the track) and simultaneously measure displacement using the motion sensor. I have found that I could measure the force and displacement simultaneously for a rubber band using the Pasco

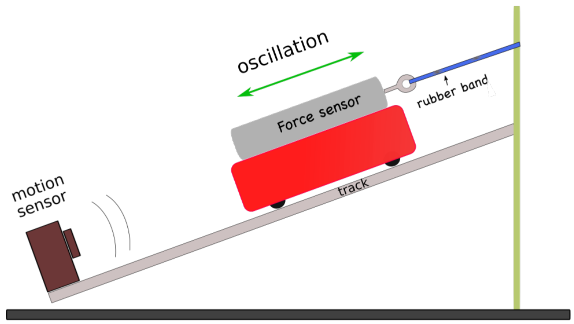

Figure 3: An example of an experiment. As the system oscillates back and forth, you can measure the sensitivity of oscillation period to the amplitude of motion.

software if I put the cart on the track and inclined the track. The incline helped me adjust the strength of the force on the rubber band. Recall that the force sensor has a **limit of 20 N**.

For some experiments you might get better measurements using the force sensor alone and measuring displacement with a ruler or calipers. In this case you would not bother using the motion sensor to measurement displacement.

If you have a vibrating or oscillating system, then you may not need to measure displacement at all. You could instead focus on exploring how the amplitude of oscillation is sensitive to amplitude.

2 Elastic phenomena: Deviations from Hooke's law

In this section we introduce different types of elastic behavior and phenomena. This section is designed to inspire your measurements and analysis.

2.1 Hooke's law and the harmonic oscillator

The harmonic oscillator consists of a mass m connected by a spring to a fixed point. The spring obeys Hooke's law, which is a linear relation between force F and displacement x from the rest position;

$$F(x) = -kx.$$

The coefficient that relates the force to the displacement is the spring constant k. The spring constant can be measured by comparing force to displacement.

When the force is applied to mass m, the equation of motion is

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x$$

which has sinusoidal solutions. An example of a solution is $x(t) = A\cos(\omega t)$ with angular frequency $\omega = \sqrt{k/m}$ and amplitude A. Notice that the period of oscillation

$$T = \frac{2\pi}{\omega}$$

is independent of the amplitude A.

A conservative force can be written as the derivative of a potential energy function U. The relation between the potential energy and force

$$F(x) = -\frac{dU}{dx}.$$

The potential energy of a spring is

$$U(x) = \int^x kx'\, dx' = \frac{kx^2}{2}.$$

This equation for potential energy is consistent with Hooke's law. With amplitude A, the total energy of mass m is the sum of potential U and kinetic energy K, giving

$$\begin{aligned} E &= K + U(x) = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 \\ &= m\omega^2 A^2 = kA^2. \end{aligned} \quad (1)$$

The energy is dependent upon the square of the amplitude.

2.2 Non-linear forces laws and anharmonic motion

Many springs are well approximated by Hooke's force law. However, other bending objects such as a bend paper clip, and short, conical or extremely stretched or compressed springs can deviate from this linear force law. One way a 1 dimensional system can deviate from Hooke's law is if the force is not linearly proportional to the displacement. For example

$$F(x) = -kx(1 + ax + bx^2)$$

where the dimensionless coefficients a, b describe deviations from a linear relation between force and displacement. These coefficients could be positive or negative.

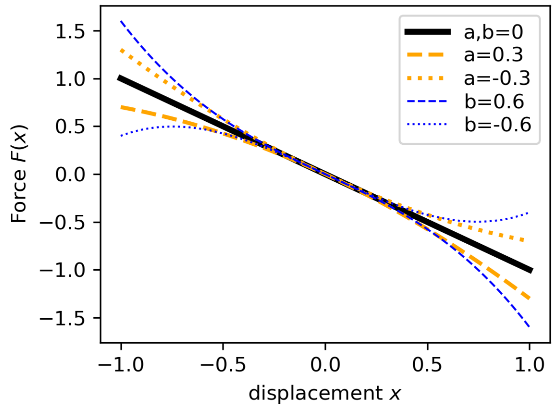

Figure 4: Illustration of different polynomial force laws. We show force law $F(x) = -kx(1 + ax + bx^2)$ with $k = 1$ and for different values of dimensionless coefficients a, b. The orange lines have $b = 0$ and the blue lines have $a = 0$.

Another example of a system that deviates from Hooke's law is known as the Hertzian contact model. This describes the force between an elastic sphere and another elastic object such as a flat surface as shown in Figure 5. The relation between force F and displacement d is $F \propto d^{\frac{3}{2}}$. This model is used in **contact mechanics**, which is the study of the deformation of solids that touch each other. The Hertzian contact law is also used to construct models and simulations for granular systems.

If a spring obeys a non-linear force law, then the motion of mass connected to the spring is **anharmonic**. In this case the period of oscillation is dependent upon the amplitude. Anharmonic motion is characterized by a dependence of the oscillation period on the amplitude of motion. You can determine whether a system is anharmonic by measuring the force displacement curve $F(x)$ or by measuring how the oscillation period depends on amplitude. What do we mean by amplitude if the spring is anharmonic. We can measure extreme positions of the motion, x_- and x_+ and take

$$A = \frac{x_+ - x_-}{2}$$

to estimate an amplitude. You could also characterize the amplitude with the difference between maximum and minimum velocity or the difference between maximum and minimum force. For an anharmonic spring the function $T(A)$ would be curved instead of a constant, as it would be for a spring that obeys Hooke's law. In this case the period of an anharmonic spring might obey something like this

$$T(A) = c + dA$$

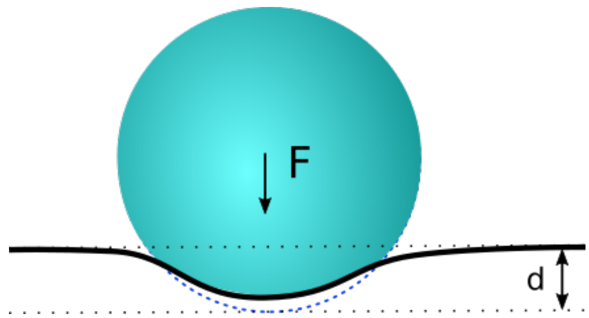

Figure 5: When an elastic sphere is pressed into a flat elastic material, the force F depends on the displacement d of the center of sphere. The Hertzian contact model gives $F \propto d^{\frac{3}{2}}$. Deformation of both surface and sphere can be used to predict the force law.

which is described by a different set of coefficients c, d. The coefficient d is related to the anharmonic terms in the force law (those described by a, b) but in a non-trivial way. It is possible to relate the two descriptions via numerical integrations or expansions.

The system shown in Figure 6 decreases in oscillation period at the same time as the amplitude decreases. This would be an example of an anharmonic system.

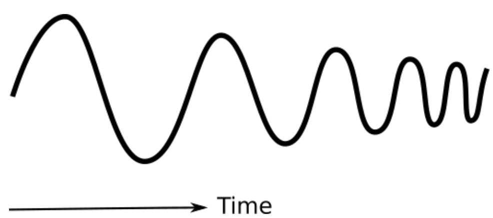

Figure 6: A system that decreases in amplitude and simultaneously decreases in period.

2.3 Anelastic behavior

So far we have assumed that the force does not depend upon the velocity or dx/dt or on the direction of motion (the sign of dx/dt) or on the past history of deformation. A symptom of anelastic behavior is that energy is lost by the system. This can be seen via a work-loop.

2.4 Work loops and hysteresis

Consider a path in one dimension starting at position x_1 and ending at x_2. Work is the integral of force times distance and the work done along the path is

$$W = \int_{x_1}^{x_2} F(x)\, dx.$$

On a plot of force vs displacement, the work is the area under the curve; see Figure 7.

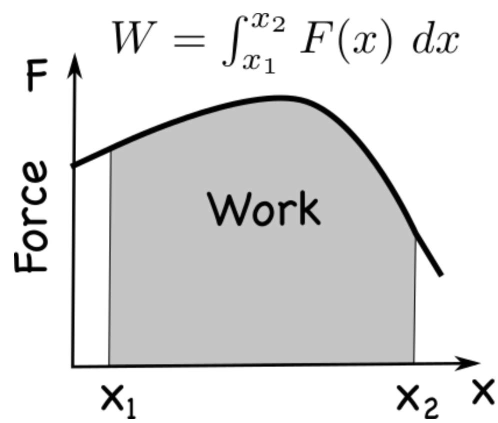

Figure 7: Work done on a trajectory from position x_1 to x_2 is equal to the area under the curve.

If the force is conservative

$$W = \int_{x_1}^{x_2} F(x)\, dx = \int_{x_1}^{x_2} -\frac{dU}{dx}\, dx = -\int_{U(x_1)}^{U(x_2)} dU = U(x_1) - U(x_2)$$

and only depends on the difference in potential energy at the beginning and end of a trajectory. Since $U(x)$ depends on x, a trajectory in x coordinate space that starts and ends at the same position will have a total work of $W = 0$.

If the force obeys Hooke's law then a slow stretch of the spring followed by a slow release will look like that shown in Figure 8 on a force vs displacement plot.

Again consider a trajectory that begins at x_1, goes to x_2, and then back to x_1. If the force is **not** conservative, then the trajectory may form a loop in a force vs displacement plot, as shown in Figure 9. Effectively the elastic coefficient of the rubber band is not the same when it is loading and unloading tension.

Ideally, a rubber band should immediately return to its original state when released. However, rubber deviates from pure elastic behavior.

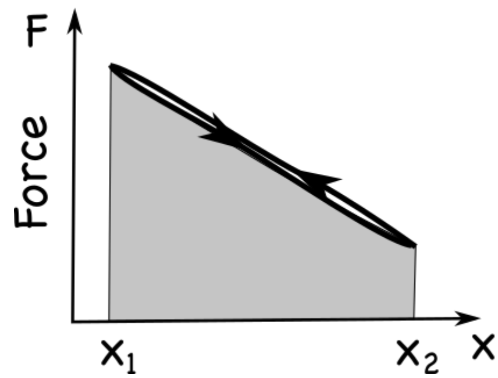

Figure 8: Work done as the spring is stretched is computed from the area under the lower trajectory. If the force obeys Hooke's law then the force is linearly dependent on displacement. The spring returns to its rest position along the upper trajectory. The area under the returning curve is the same as that under the outgoing curve. There is no work done. The energy put into stretching the spring is regained when the spring goes back to its rest position. The force is a conservative force.

Rubber bands only slowly return to their original state after being released, leading to hysteresis or a delay in tension measurements. The tension measured when a rubber band is stretched to a length L is not the same as that measured when it is stretched past L and then allowed to relax to L. The lag between tension and length (or stress and strain) is sometimes called *hysteresis*. The force displacement law can depend upon the direction and speed of motion and on the past history of deformation.

The word *resilience* can be used to describe the percent of energy required for a deformed piece of rubber to rebound back to its original shape after a deformation.

Where does the energy go? When a rubber band is extended, the polymers inside it become untangled. This lowers the entropy. Heat is lost during extension/release cycle. The length of a rubber band under a fixed tension depends on temperature. Most materials expand when heated, however heating causes a rubber band to contract and cooling causes expansion. Work done on rubber tends to be released as heat.

2.5 Ductile behavior

We have some short rubber bands in the lab that I have found show large deviations from Hooke's law. They exhibit really nice work loops!

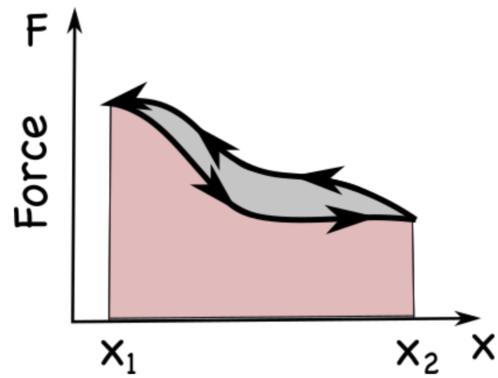

Figure 9: The system starts at x_1 and moves along the lower trajectory to x_2. The system then moves along the upper trajectory to x_1. The net work done is the difference between the work done on the lower trajectory and that done on the upper trajectory. This is the difference between the areas under the two curves. The net work done is equal to the area in the grey region. The system has returned to its original position but work was done! The force is **not** a conservative force. Rubber bands can behave in this manner. The force displacement law can depend upon the direction and speed of motion and on the past history of the rubber band's deformation.

However, they do not necessarily return to their original length after stretching. If an object is slowly and permanently deformed then it is **ductile**. This complicates measurement as the behavior of the rubber band can depend upon how long it has become. It is also interesting for its own sake! One possibility is to compare the characteristics of these rubber bands before and after they have stretched.

3 Analysis

Because your experiments will differ between lab groups, the details of how you set up your experiments are particularly important in your lab reports. Likewise you will need to clearly describe how you made measurements and what models you use to interpret your data.

We will be looking forward to hearing about your experiments!

4 Feedback

This is a new lab, so comments are particularly appreciated!