Connections between Classical and Quantum Complexity

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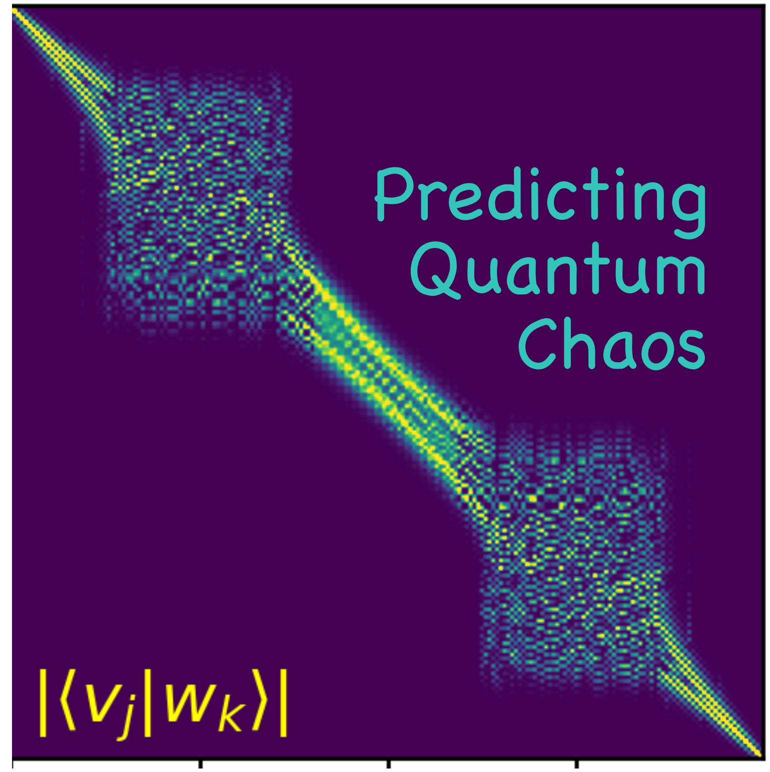
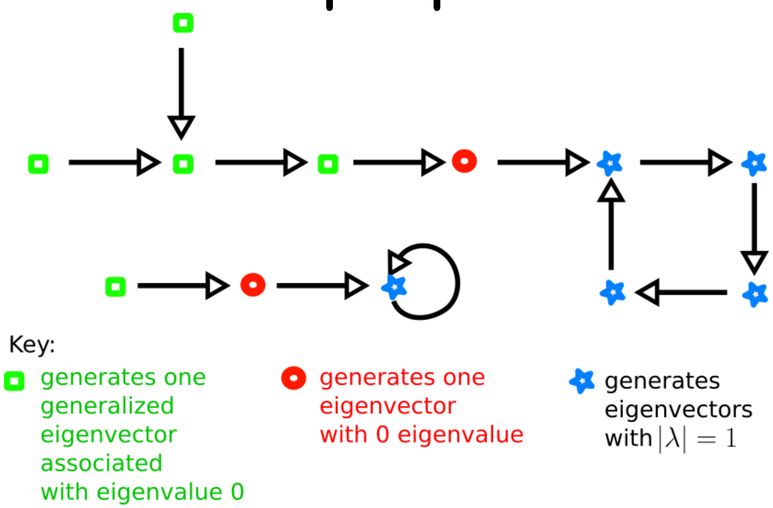
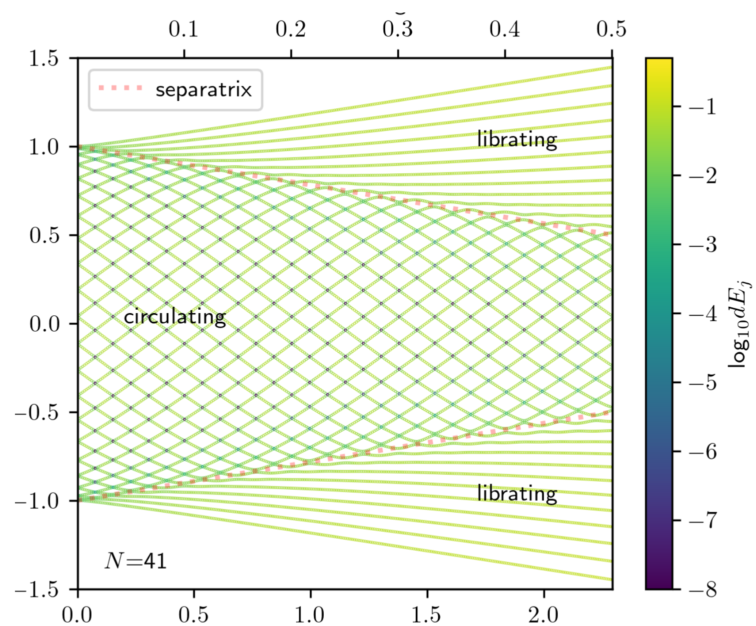


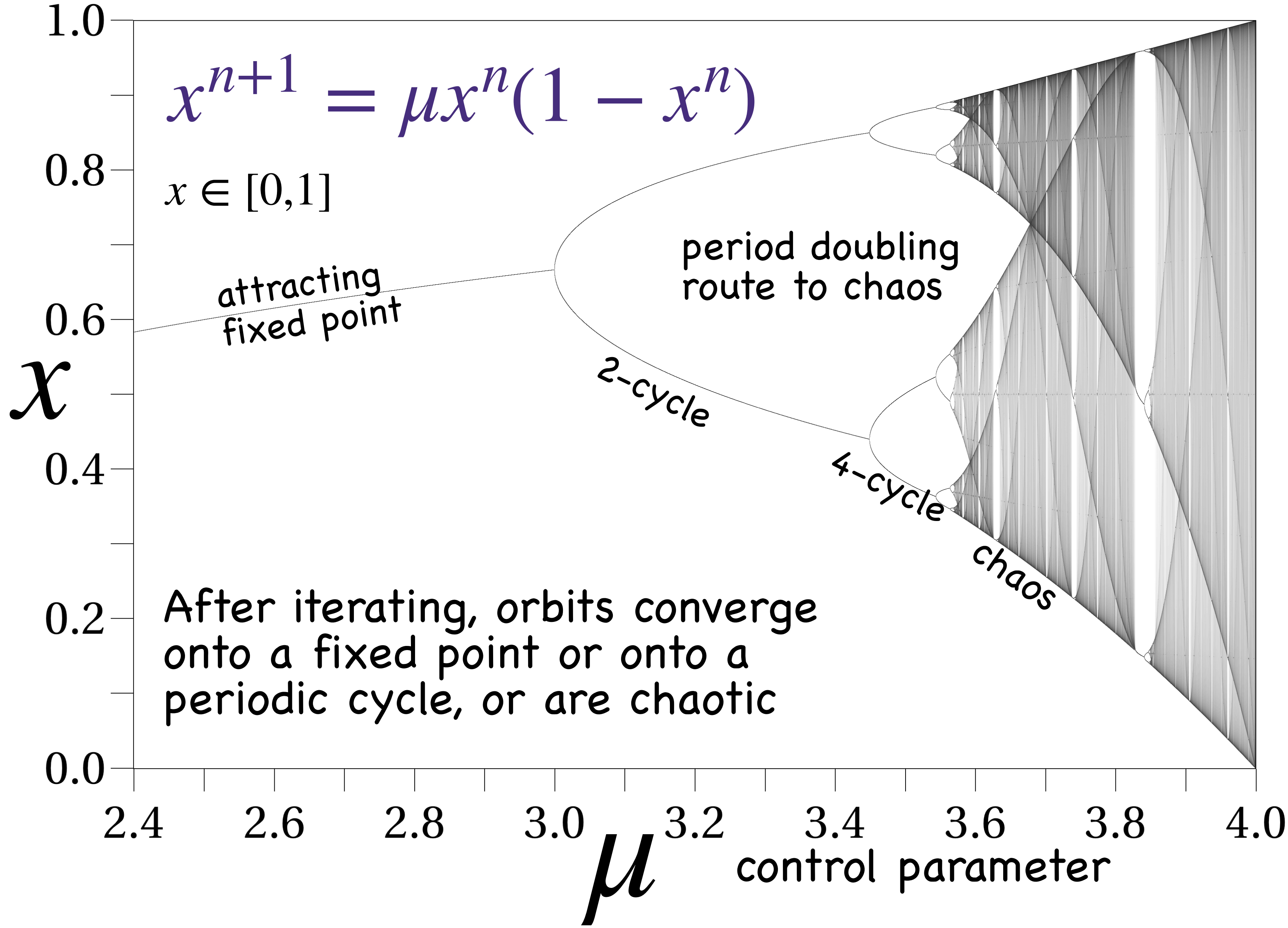
Quantum iterators that have both quantum and classical properties



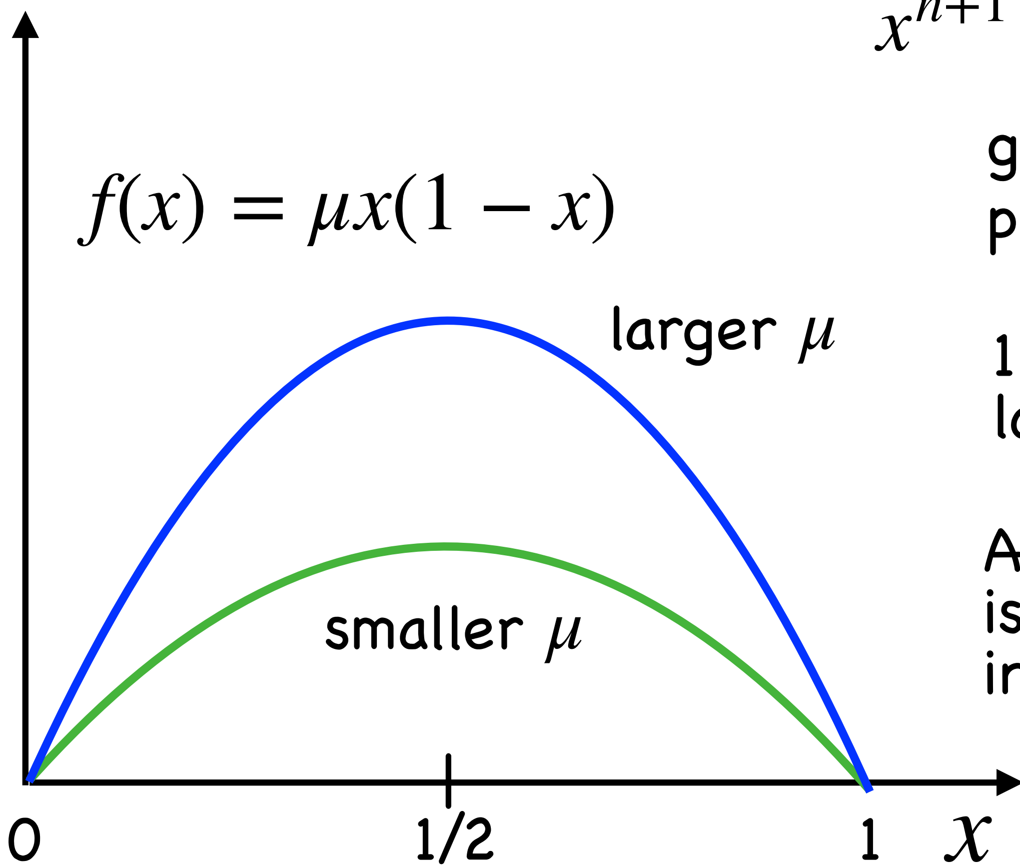
Notions of adiabatic drift



Iterates of the logistic map



The logistic function is not invertible



$$x^{n+1} = \mu x^n(1-x^n)$$

given x^n, there are two possible values of x^{n-1}

1 bit of information loss per iteration

After iterating, little is known about the initial conditions

Iterators $x_{n+1} = f(x_n)$ discrete time dynamical system

Classical world

- Dynamics can be dissipative. Attracting fixed points or cycles
- Information about initial conditions is lost
- Possibility of Ergodicity/chaos
- Functions need not be invertible
- Optimization can be done via iteration

Quantum world

- Unitary transformations (preserve probability of a quantum state)
- Invertible
- Act like rotations when iterated
- Number of operations needed to construct any desired unitary operator from a set of quantum gates is low (universal gate sets/Kitaev Thm)

Hybrids:

- Algorithms involving both classical and quantum portions
- Open quantum systems/quantum channels
- Quantum error correction

Reset/Initialization on a quantum computer

Put a specific quantum subsystem into a particular pure state
Needed for initialization in quantum computing

$$|\psi\rangle \xrightarrow[\mathscr{E}]{} |\psi_0\rangle$$ initialize a state

$$\mathscr{E}_{\text{reset}}(\rho) = |\psi_0\rangle\langle\psi_0|$$ Reset quantum channel

ρ density matrix

Quantum state initialization can be accurate and efficient

Information is lost

We look at operations that can be constructed from unitary operators and reset operators.
In the realm of hybrid quantum/classical algorithms

Quantum oracle

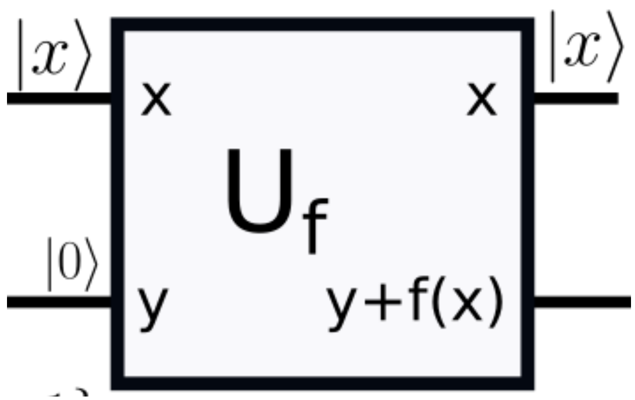
A function $f(x)$ where $x, f(x) \in \{0,1,\ldots,N\}$ $(\mathbb{Z}_N)$

Complexity of a black box algorithm is the number of queries of the oracle function required to learn something about the oracle function

To implement a non-invertible function via a **unitary** operation

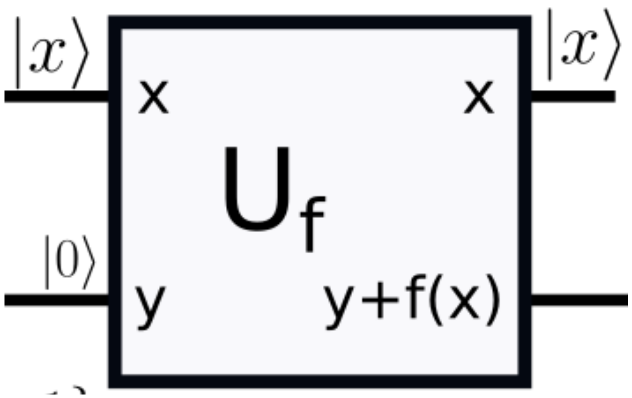
register of qubits

another register with same number of qubits



because information for the input quantum register is available in the output, the operator can be inverted

Quantum oracle



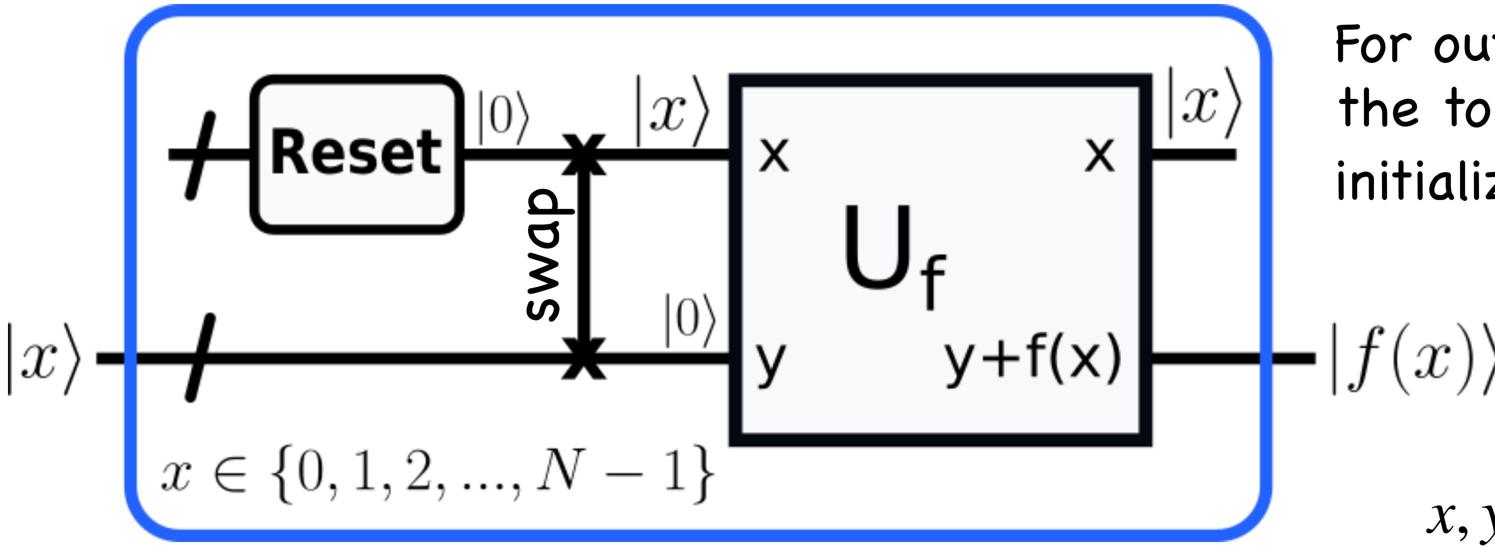
$|f(x)\rangle$ is in the output register, only if the input of the lower register is the $|0\rangle$ state

To iterate $f(x)$, we need to reset the lower register and swap registers

On notation for quantum circuits:

left to right is increasing in time
horizontal lines represent registers of qubits
x, y label pure states (here natural numbers) in a particular basis
it is assumed that you could input a superposition state

Creating an iterator from a quantum oracle



For output $|f(x)\rangle$ the top register is initialized to $|0\rangle$

$x, y, f(x) \in \mathbb{Z}_N$

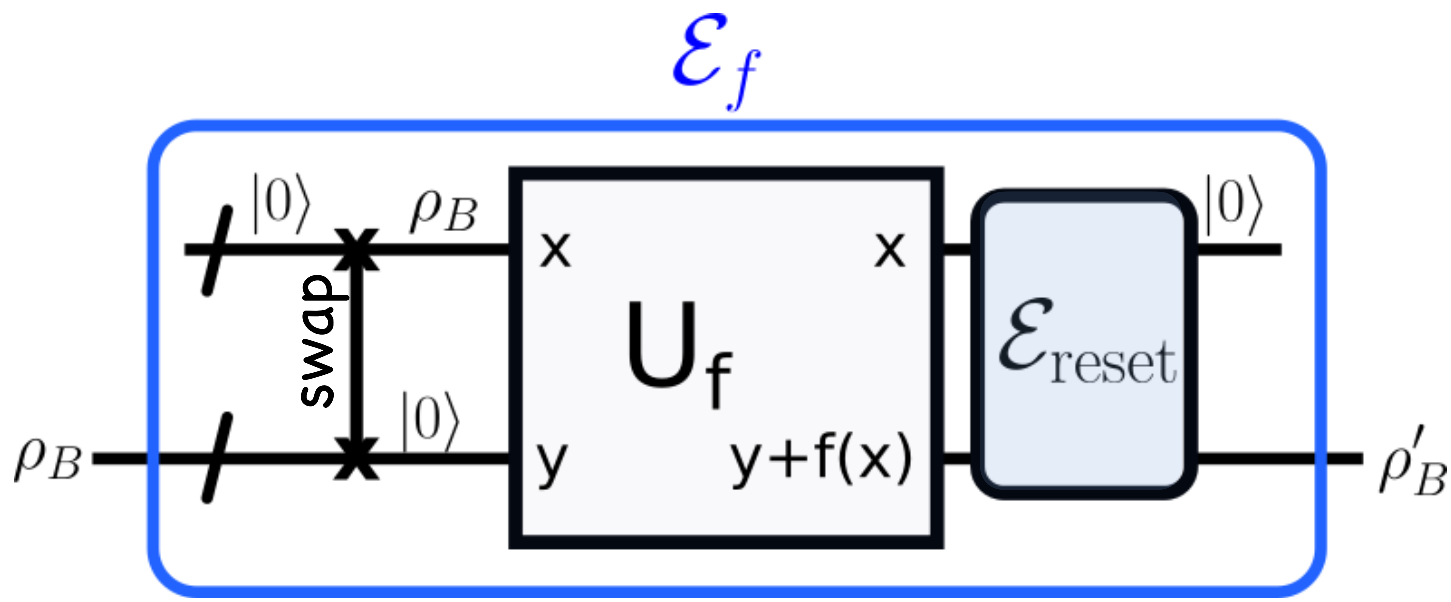
$U_f : |x\rangle_A |y\rangle_B \rightarrow |x\rangle_A |f(x) + y\rangle_B$ oracle operator (unitary)

$$\text{SWAP} = \sum_{x,y=0}^{N-1} |x\rangle_A |y\rangle_B \langle y|_A \langle x|_B$$ swap operator (unitary)

$$\mathcal{E}_{\text{reset}}(\rho_{AB}) = |0\rangle_A\langle 0|_A \otimes \text{tr}_A \rho_{AB}$$ reset channel (not unitary)

entanglement breaking

The iterator



Composed of unitary and non-unitary components

Input a pure state and iterate j times

$$|x\rangle \xrightarrow{\mathscr{E}_f^j} |f^j(x)\rangle \quad x \in \mathbb{Z}_N$$

An interesting function that can be iterated

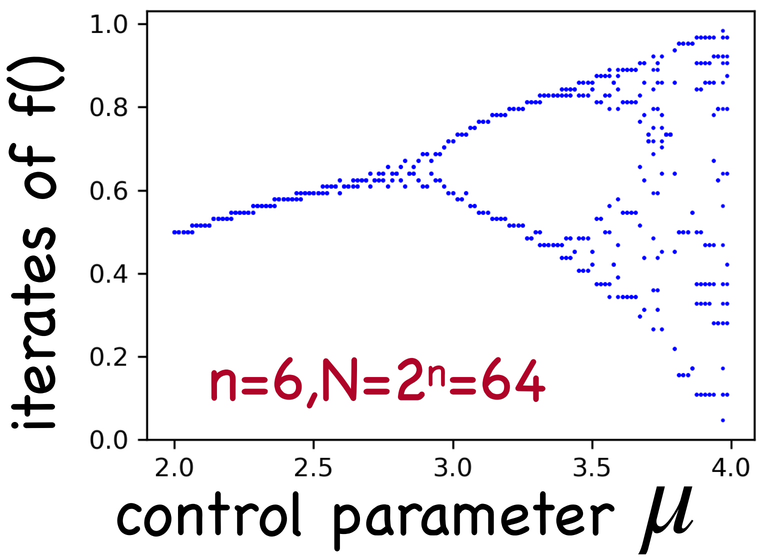
Truncated logistic map $g(y) = \mu y(1-y)$

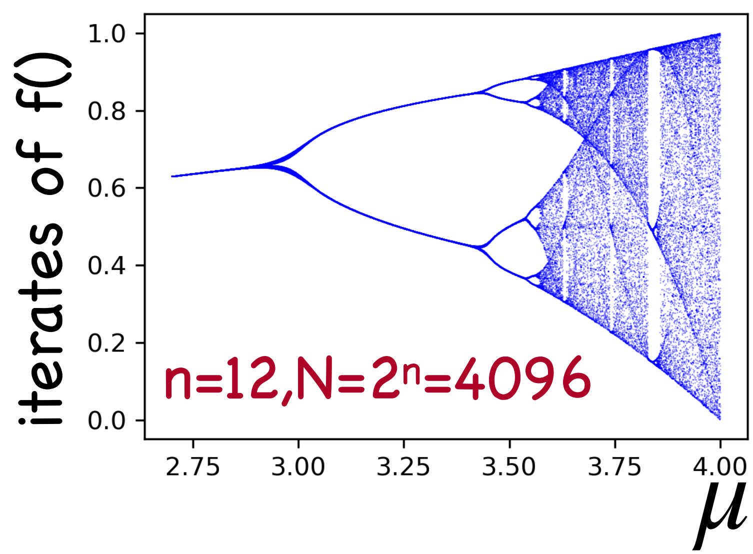
$g()$ a function of real numbers on the unit interval to itself

$$f(x) = \text{floor}\left[g\left(\frac{x}{2^n}\right)2^n\right] \quad \text{for } x \in \{0,1,...,2^n-1\}$$

$N = 2^n$

n = number of qubits
N = number of states



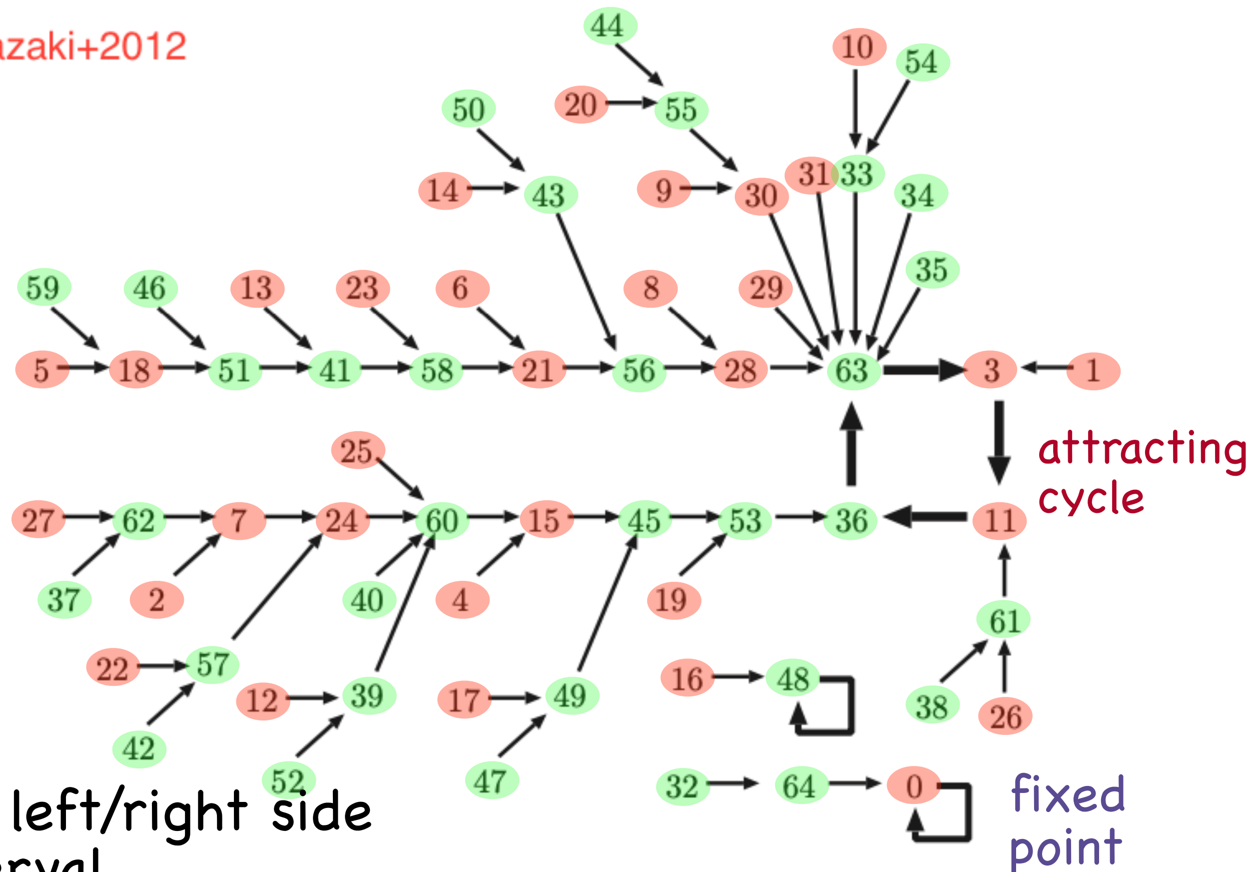


A quantum channel that converges to fixed states or cycles

Iterates of the truncated logistic map

From Miyazaki+2012

interest in creating a digital random number generator (Ulam/von Neumann)



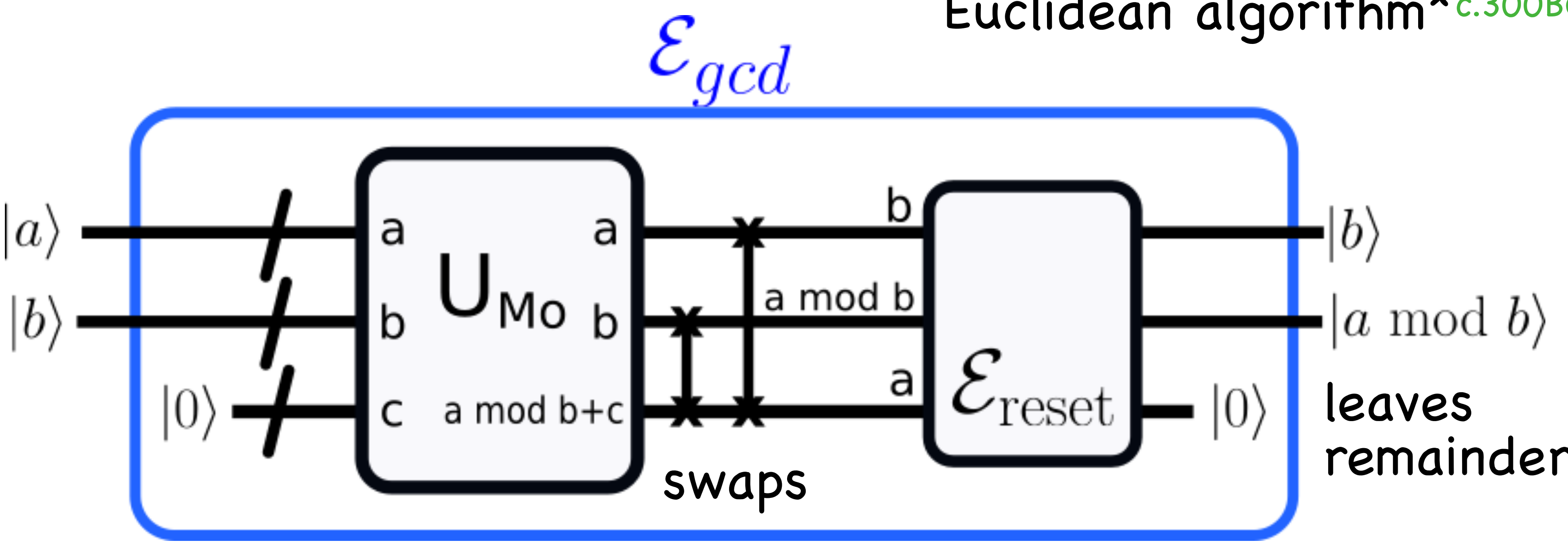
Color based on left/right side of the unit interval

Fig. 2. Trajectory for the sequences generated by $LM^{(6)}_{Int}(X)$ represented as a directed graph

fixed points are present in the truncated map even though they are a set of measure zero in the actual map

Iterate to compute the greatest common divisor of integers a,b

Euclidean algorithm* c.300BC

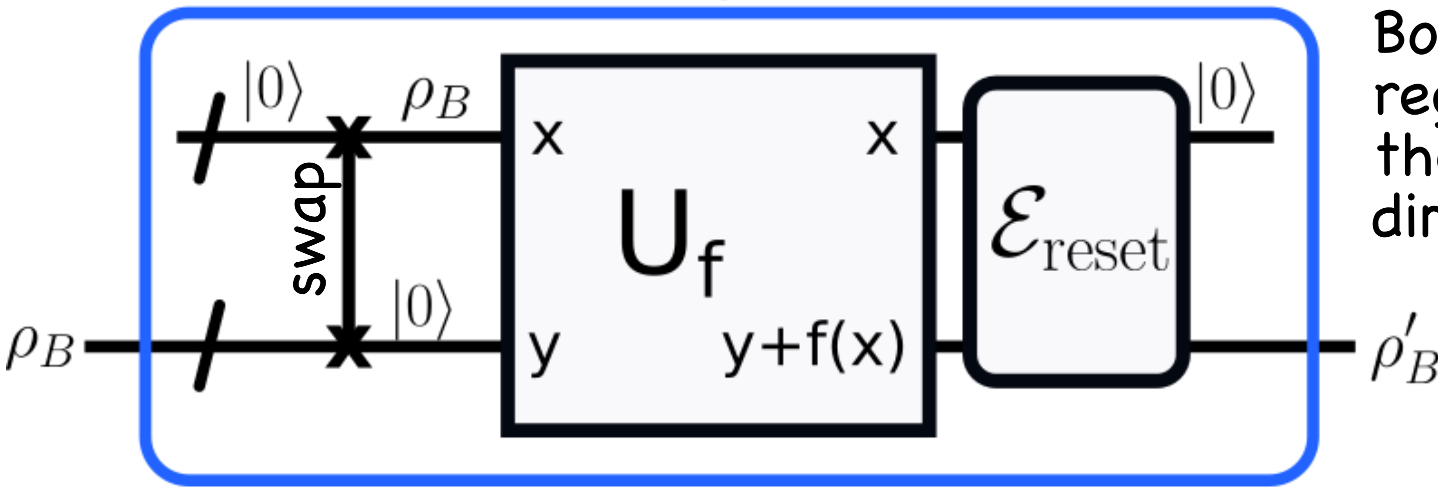


An ingredient of the Shor factoring algorithm that could be done in place on a quantum computer

Iterate for 5 times the number of digits base 10 ($\mathcal{O} \ln N$) and you are guaranteed to have the desired result (Lamé's Theorem)

$$g_{gcd}(a,b) = \begin{cases} (b, a \bmod b) & \text{if } b \geq 1 \\ (a,0) & \text{if } b = 0 \end{cases}$$

What does the oracle channel do to a mixed state? $\mathcal{E}_f$



Both quantum registers have the same dimension

The channel is **dephasing**

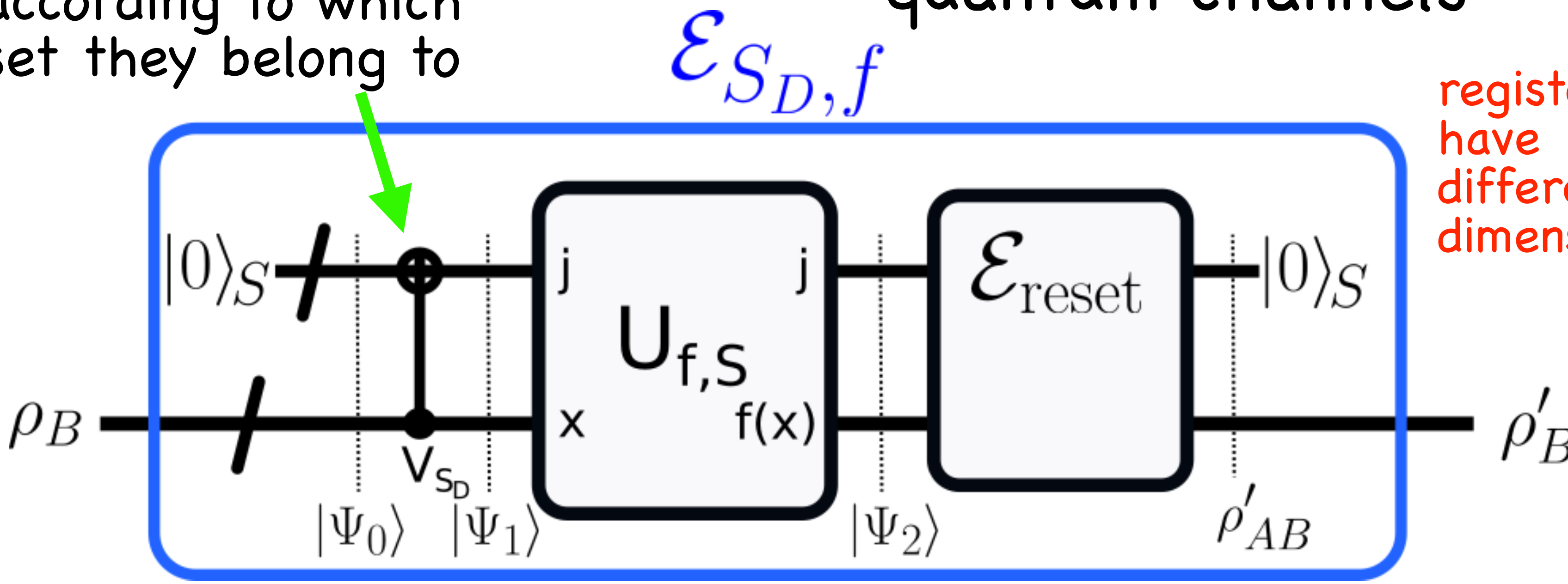
If you input $a|x_a\rangle + b|x_b\rangle$ the result is

$|f(x_a)\rangle$ with probability aa^* and $|f(x_b)\rangle$ with probability bb^*

Superposition is lost (decoherence)
Result is described by classical probabilities
Information loss is associated with classical behavior

control operator marks states according to which set they belong to

A more general class of quantum channels



registers have different dimensions

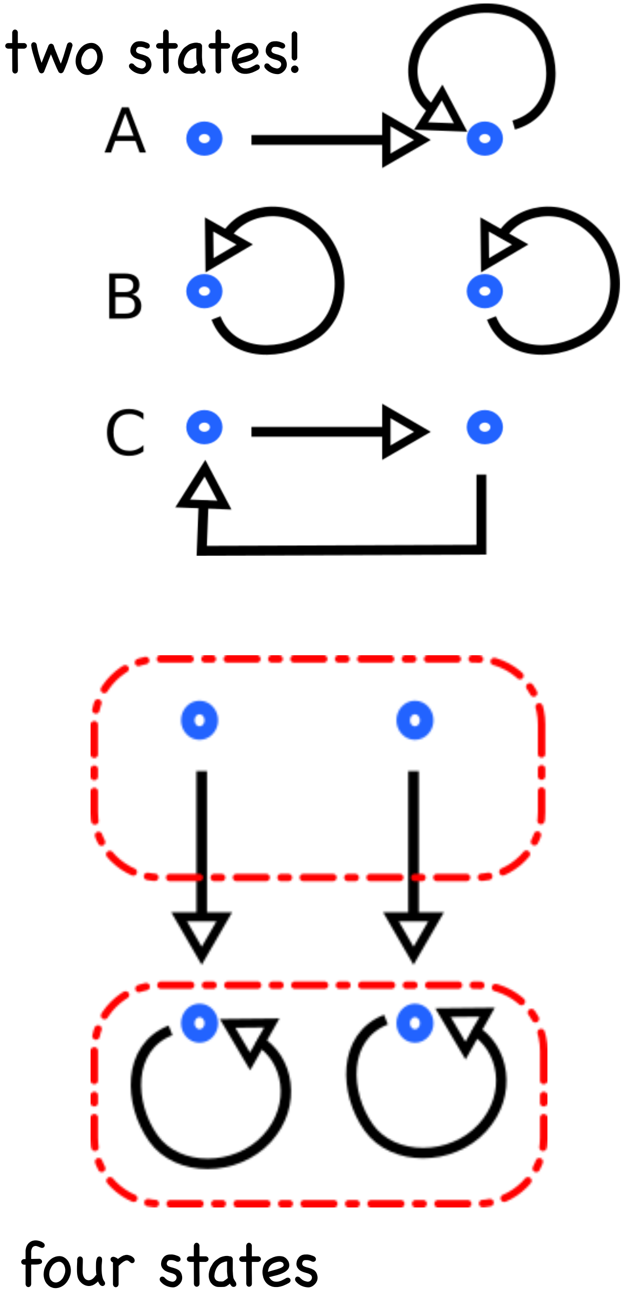
Partition the set of basis states into disjoint sets

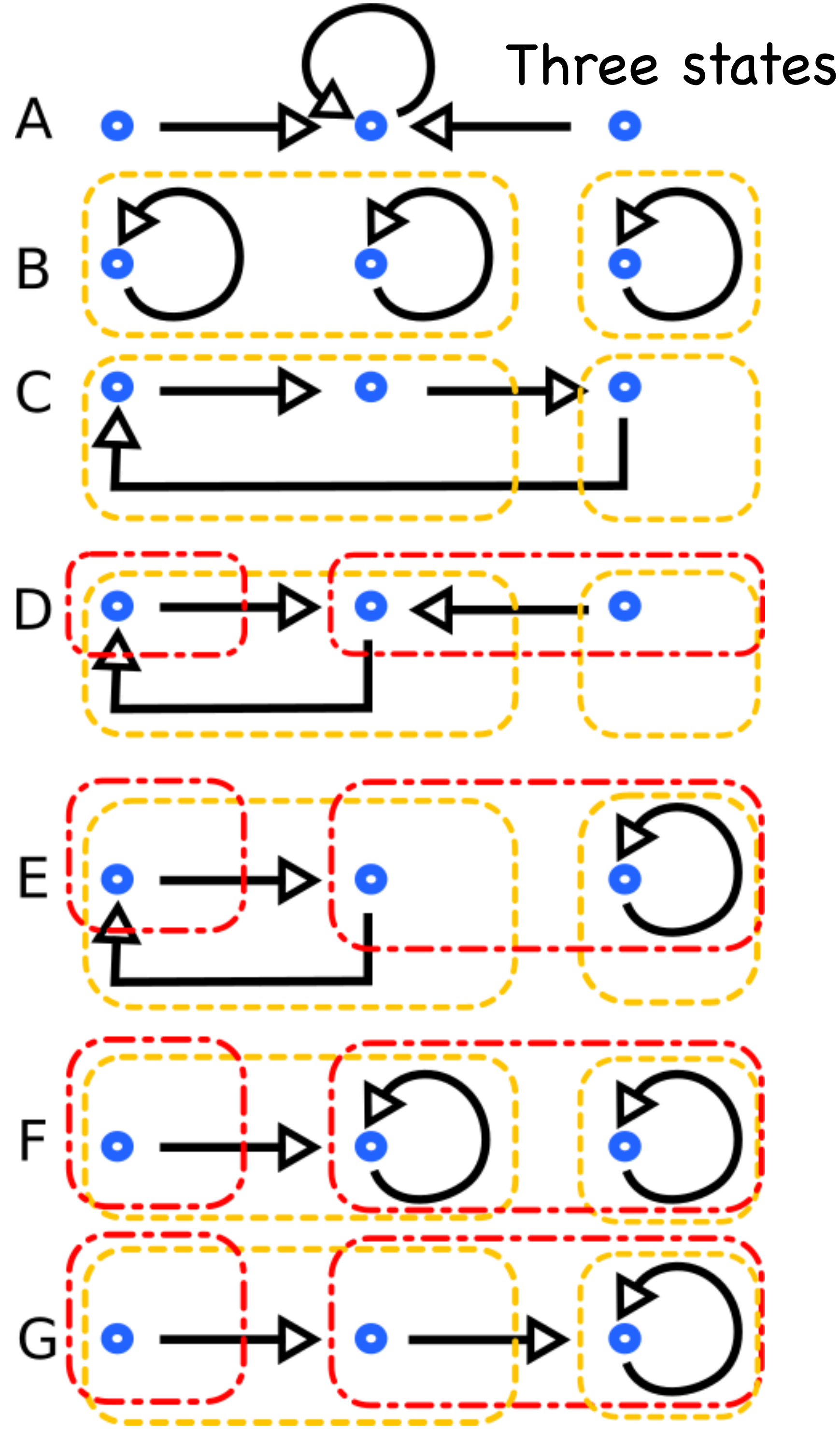
Each element in set has a unique function that can be inverted
Do resets on the register labeling the disjoint sets

Amount of information loss per iteration depends on the number of disjoint sets

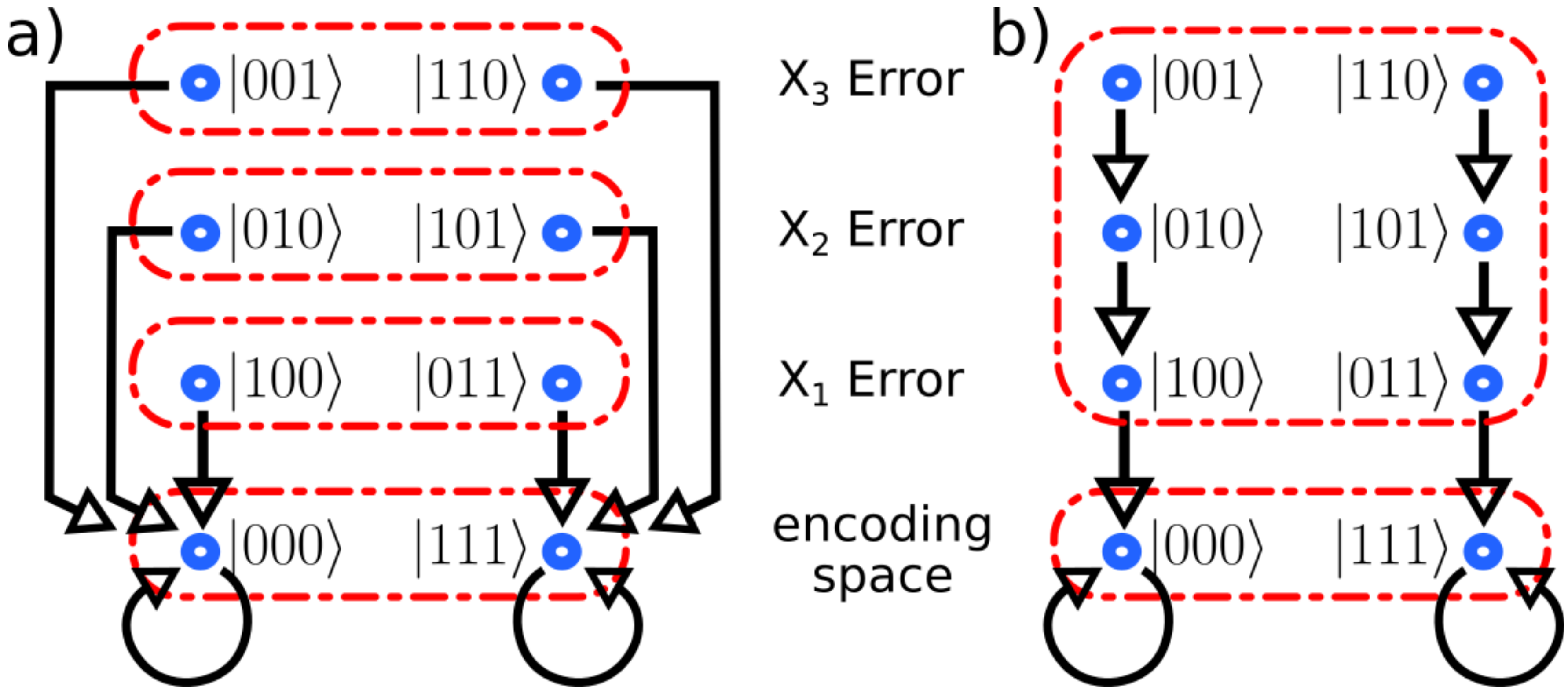
Superposition for states constructed from members of a single disjoint set is not lost!

Classification spree

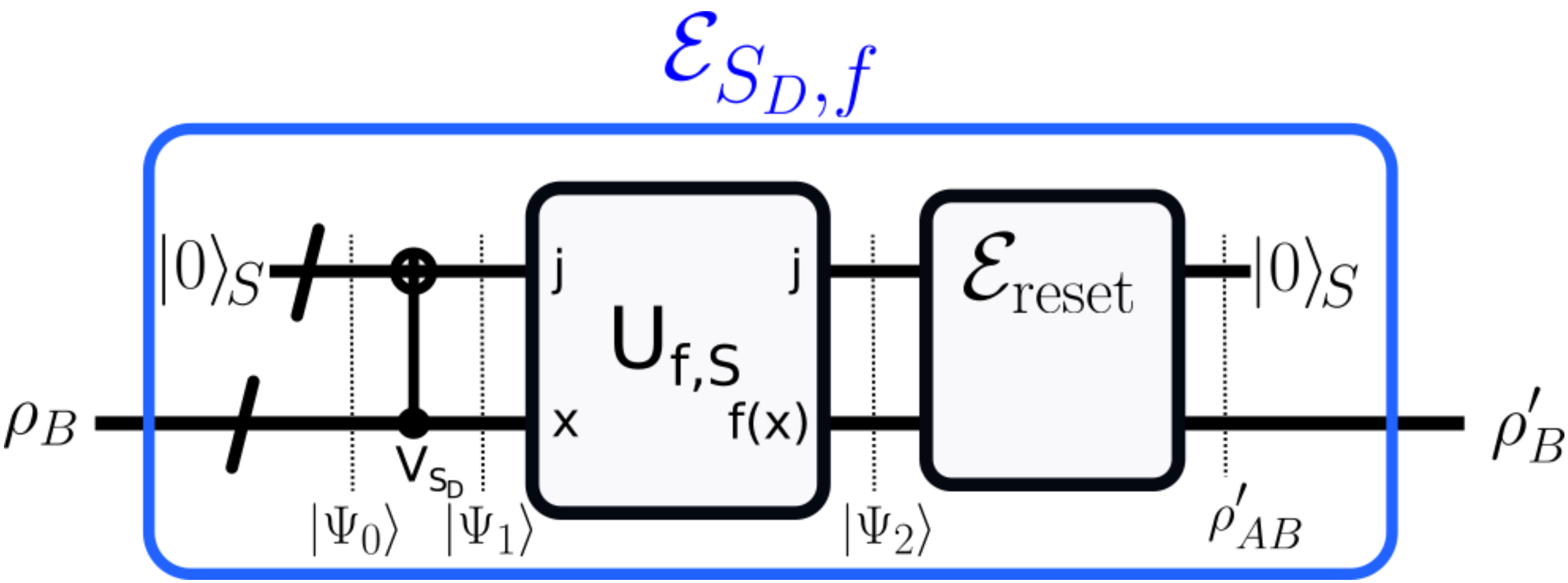




Error correction schemes are examples of channels constructed from functions on discrete sets



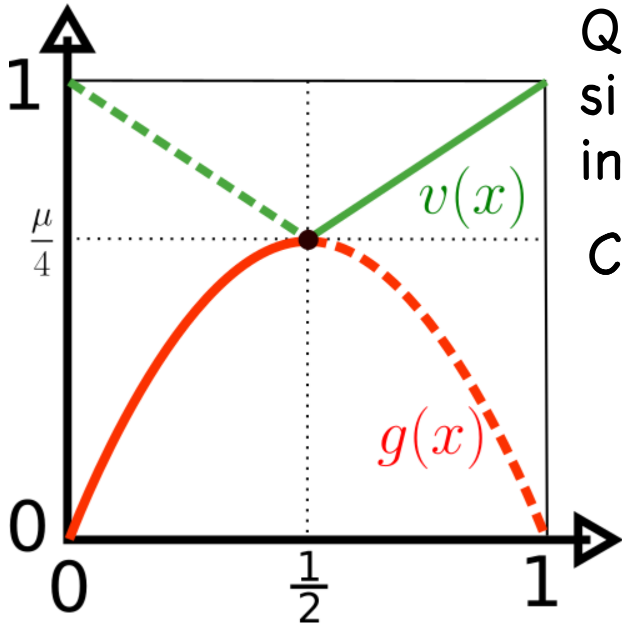
Disjoint sets gives subspaces. Subspaces retain their coherence and are not dephased.



What if register B is a continuous Hilbert space?

(not relevant for quantum computing but perhaps relevant for modeling an open quantum system)

A channel in the continuum limit (the logistic map)



Quantum system is the product space of a single qubit and a Hilbert space on the unit interval

half intervals are the disjoint sets

Control op

$$V_{S_D} = (|0\rangle_S\langle 0|_S + |1\rangle_S\langle 1|_S) \otimes \int_0^{\frac{1}{2}} dx\,|x\rangle_B\langle x|_B + (|0\rangle_S\langle 1|_S + |1\rangle_S\langle 0|_S) \otimes \int_{\frac{1}{2}}^{1} dx\,|x\rangle_B\langle x|_B$$

Two invertible functions
One from the solid line and the other from the dashed line

The channel is described by two Kraus ops

$$K_0 = \int_0^{\frac{1}{2}} dx\,|g(x)\rangle\langle x|$$

$$K_1 = \int_{\frac{1}{2}}^{1} dx\,|g(x)\rangle\langle x|$$

unitary op based on invertible functions

$$U_{f,S} : |0\rangle_S\,|x\rangle_B \to \begin{cases} |0\rangle_S\,|g(x)\rangle_B & \text{for } 0 < x \leq \frac{1}{2} \\ |0\rangle_S\,|v(x)\rangle_B & \text{for } x > \frac{1}{2} \end{cases}$$

$$|1\rangle_S\,|x\rangle_B \to \begin{cases} |1\rangle_S\,|v(x)\rangle_B & \text{for } 0 < x \leq \frac{1}{2} \\ |1\rangle_S\,|g(x)\rangle_B & \text{for } x > \frac{1}{2} \end{cases}$$

How much information is removed via the logistic map?

Answer: 1 bit per iteration

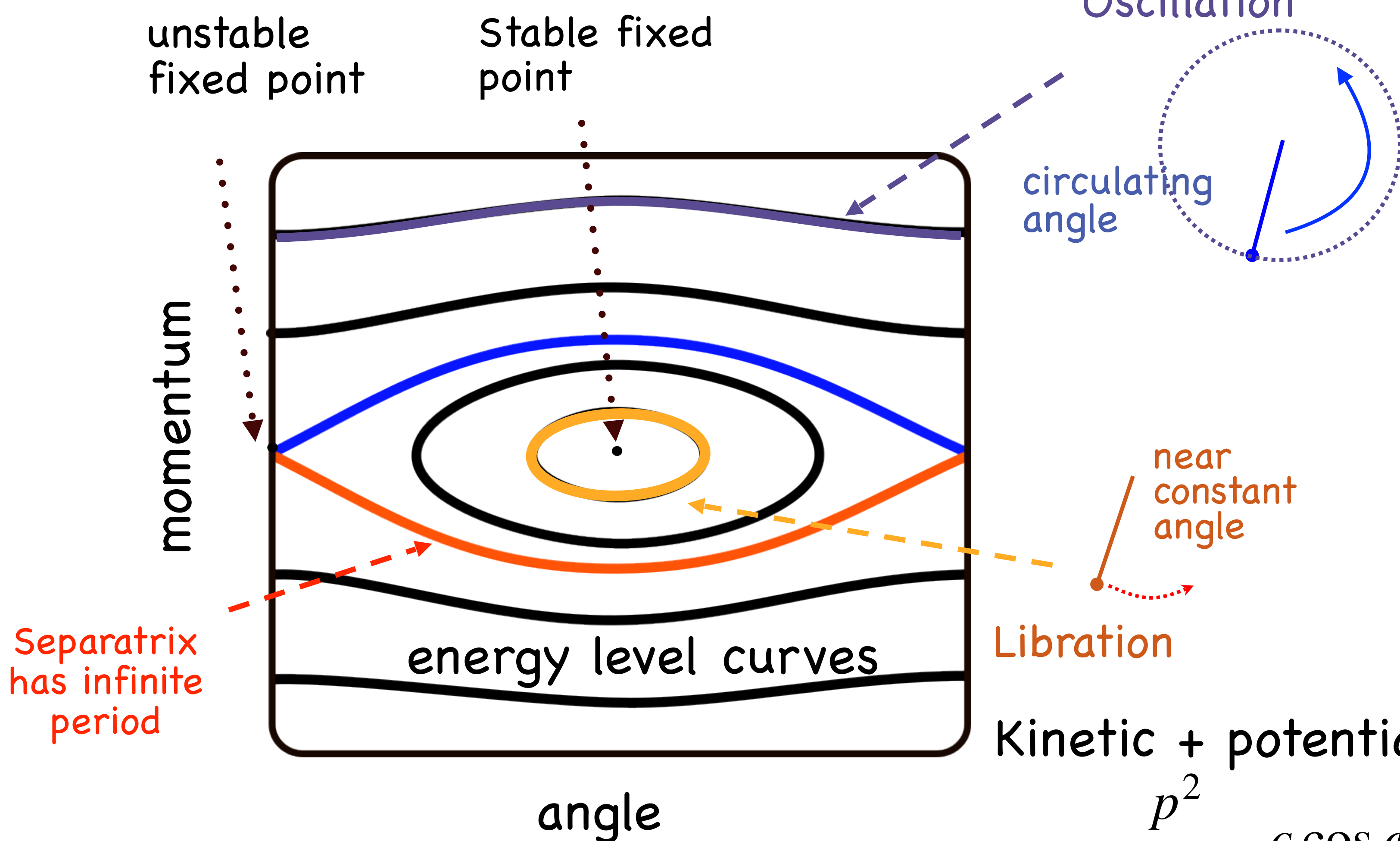
Summary

- We constructed a class of quantum iterators that are not unitary and can display phenomena of classical iterators like cycles and period doubling
- Even when generated from a function on a discrete set, quantum channels can display combinatorial complexity
- 2^{53} is a large number of states, so the number of possible hybrid iterators is large.
- Information loss induces complexity but also decoheres a quantum system or a quantum subspace. Coherence can be maintained in a subspace while some information is removed.
- Quantum computing algorithms live in a hybrid classical/quantum world. Opportunity for exploring classification of computational complexity in hybrids.

The classical pendulum

Base model for resonance and chaos

Trajectories in phase space



Kinetic + potential energy

$$\frac{p^2}{2} - \epsilon \cos \phi$$

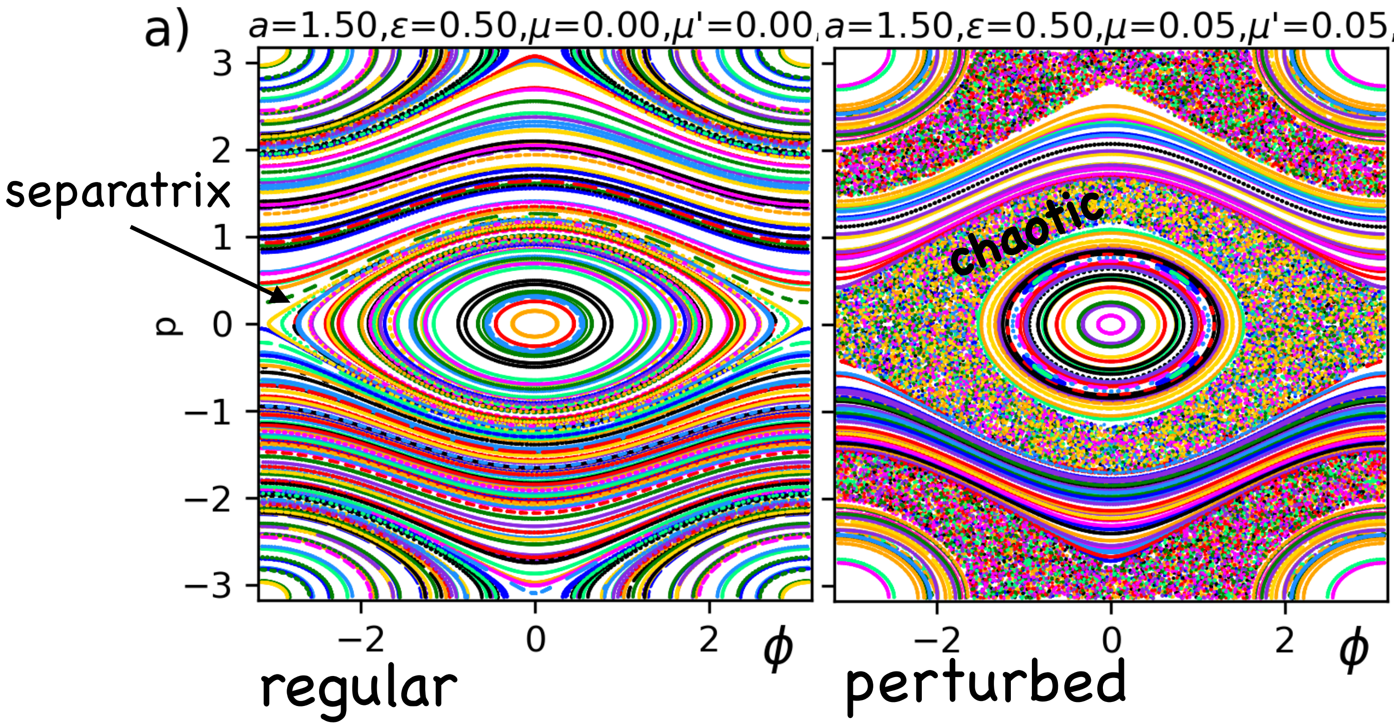
Chaos induced by periodic perturbation

$$H(\phi, p, \tau)_{\rm PH} = H_0(\phi, p) + H_1(\phi, \tau)$$
$$H_0(\phi, p) = a\left(1 - \cos p\right) - \epsilon \cos(\phi)$$
$$H_1(\phi, \tau) = -\mu \cos(\phi - \tau) - \mu' \cos(\phi + \tau)$$

$$(1 - \cos p) \sim \frac{p^2}{2}$$ for small p

Resembles kinetic energy

The perturbed Harper model is similar to the sinusoidally perturbed pendulum model, (e.g. Chirikov)



Classically integrated orbits. Plotting a point every perturbation period. These are surfaces of section. Each orbit is in a specific color.

hybrid phase space!

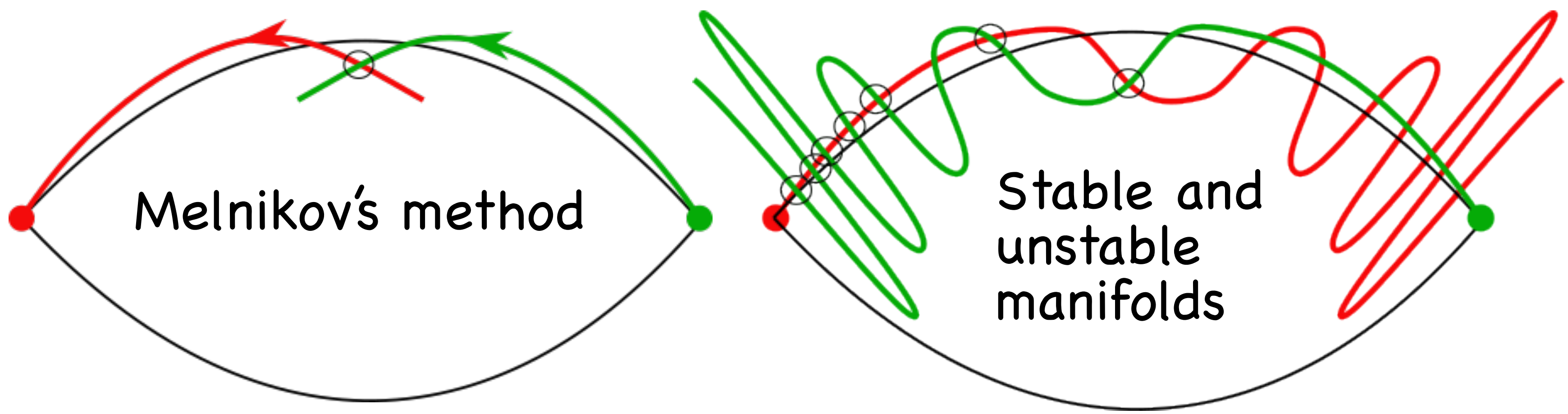
Chaos at the separatrix

Estimating the width of the chaotic zone

$$\Delta H \approx \int \frac{\partial H_1(\phi_s, p_s, t)}{\partial t} dt$$

Classical estimate for width of chaotic region formed at separatrix (e.g. Chirikov 79)

Integrate the perturbation along the separatrix orbit $\phi_s(t), p_s(t)$
For the perturbed pendulum, the result is the Melnikov-Arnold integral



What is the quantum equivalent?

Can we predict the width of an ergodic region in our quantum system?

Notions of **chaos**

Classical world

- Sensitivity to initial conditions, Exponential divergence of nearby orbits (Lyapunov)
- Recurrence, mixing, infinite number of periodic orbits, orbits fill phase space
- Behaves like a stochastic process

Quantum world

- Statistics of eigenvalues are described by a random matrix model
- Anderson localization
- In the classical limit, for almost all eigenstates, the expectation values of observables converges to their phase-space average (Shnirelman) (most eigenstates are spread out in phase space)

→ quantization

Bohigas-Giannoni-Schmit and Berry-Tabor Conjectures

Quantization of a classical model giving tunable and predictable chaotic region

phase space
$\phi, p \in [0,2\pi]$
on a torus

quantization →
Bohr-Sommerfeld
or via Weyl operator

operators $\hat{\phi}, \hat{p}$

related via discrete
Fourier transform

Why work on a torus?

When quantized, the quantum space is finite dimensional (easier to work with and relevant for quantum computing)

Classical Perturbed Harper Model

$$H(\phi, p, \tau)_{\text{PH}} = H_0(\phi, p) + H_1(\phi, \tau)$$
$$H_0(\phi, p) = a\left(1 - \cos p\right) - \epsilon \cos(\phi)$$
$$H_1(\phi, \tau) = -\mu \cos(\phi - \tau) - \mu' \cos(\phi + \tau)$$

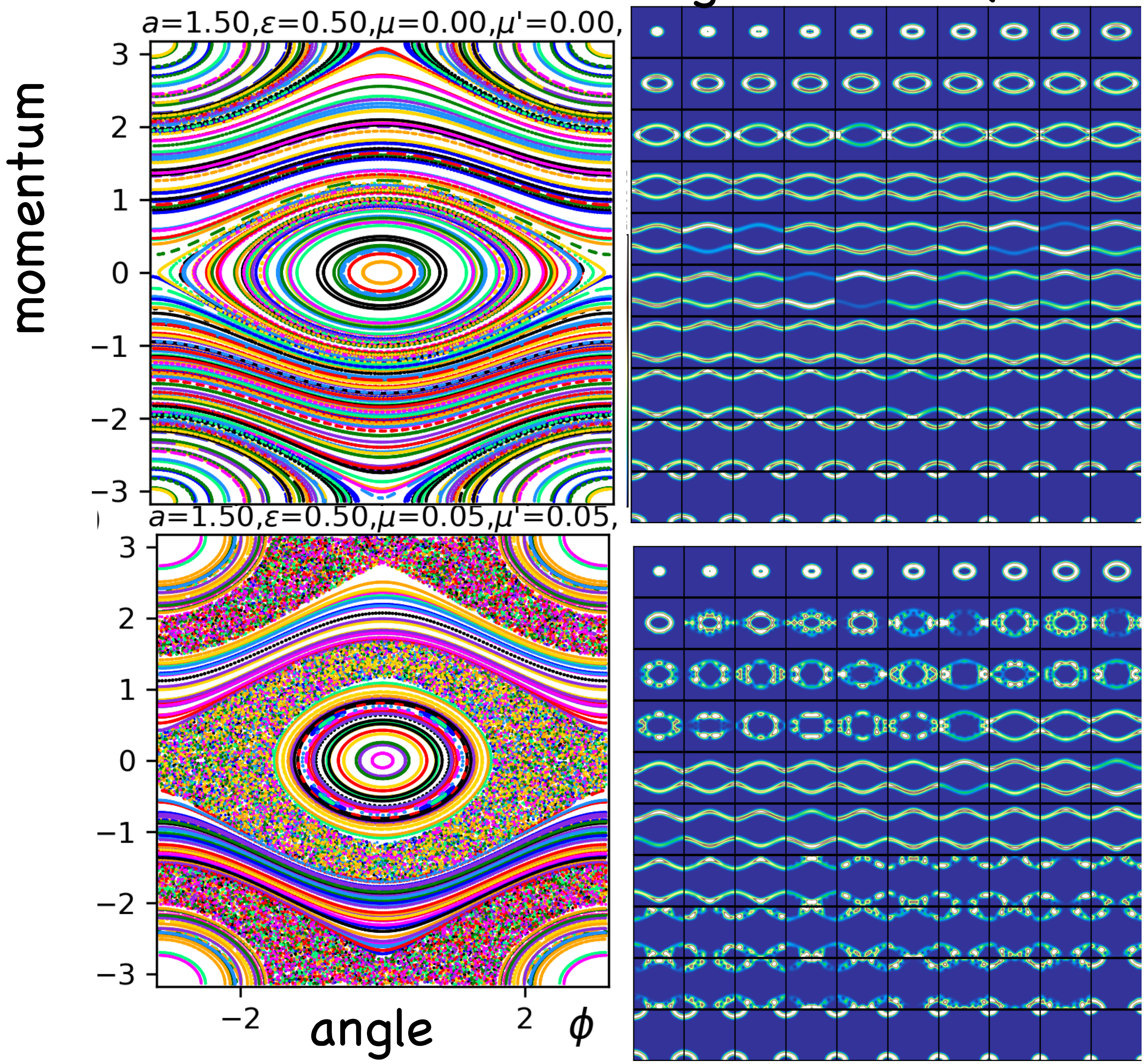
Quantum perturbed Harper model

$$\hat{h}(\tau)_{\text{PH}} = \hat{h}_0(\hat{p}, \hat{\phi}) + \hat{h}_1(\hat{\phi}, \tau)$$
$$\hat{h}_0(\hat{\phi}, \hat{p}) = a\left(1 - \cos \hat{p}\right) - \epsilon \cos \hat{\phi}$$
$$\hat{h}_1(\hat{\phi}, \tau) = -\mu \cos(\hat{\phi} - \tau) - \mu' \cos(\hat{\phi} + \tau)$$

periodically perturbed == Floquet $\quad \hat{U}_T = \mathcal{T} e^{-\frac{i}{\hbar}\int_0^T \hat{h}(t)dt}$

in phase space

Classical orbits Eigenstates (Husimi distributions) N=100



Each panel shows an eigenstate (of the propagator)

Husimi distribution giving probabilities in phase pace

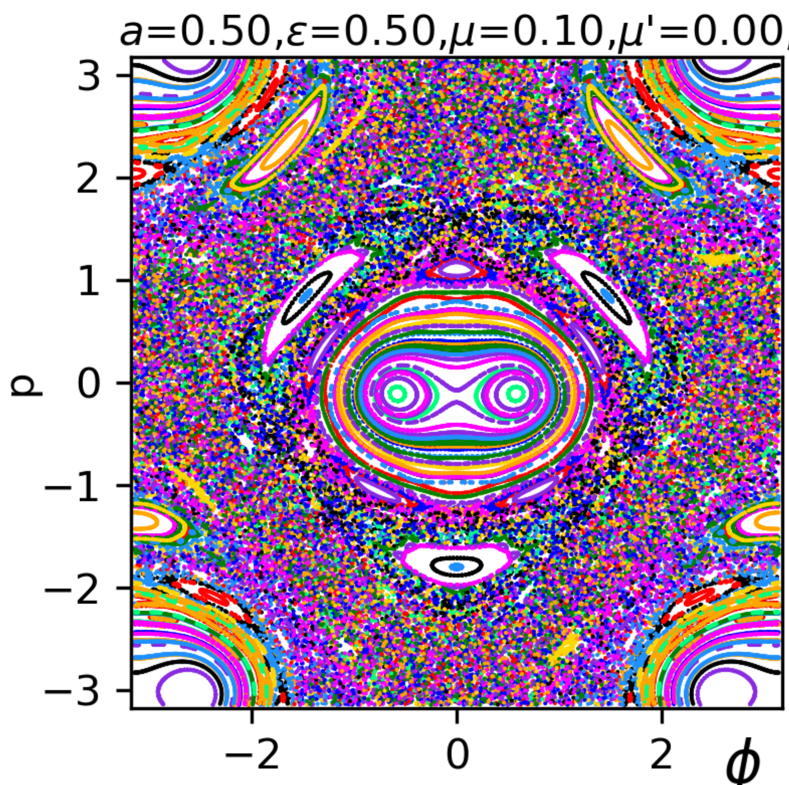
$$H_Q(k,l) = |\langle\psi|k,l\rangle|^2$$
$$k, l \in [0,....,N-1]$$

generated using discrete coherent states analogs

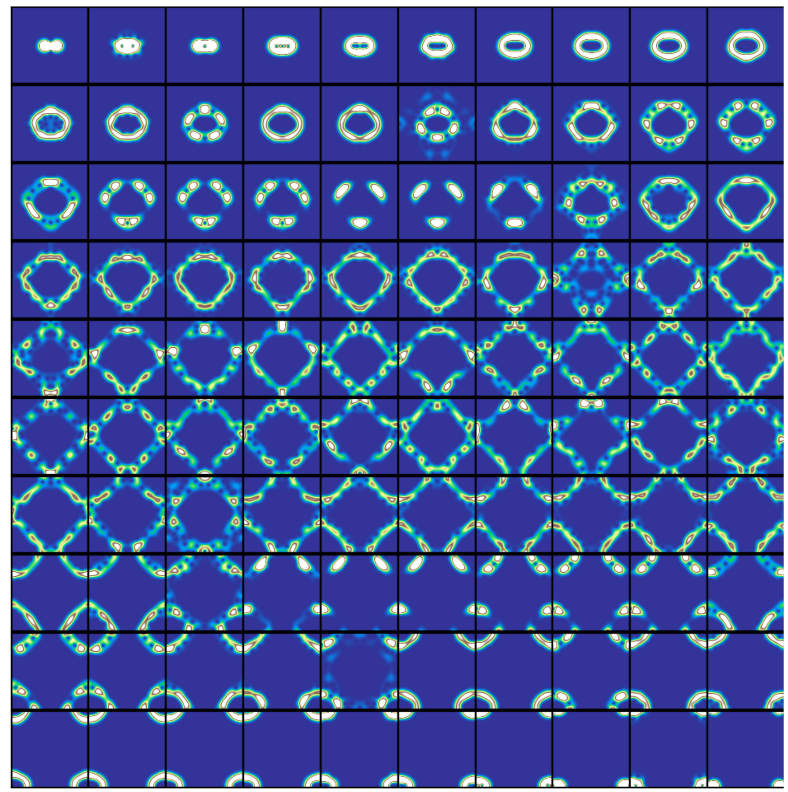
Ordering the eigenstates of the Floquet propagator

$$\mu_{j,h0} = \langle w_j | \hat{h}_0 | w_j \rangle$$

expectation value of unperturbed energy



Husimi functions for the eigenstates **in order** of $\mu_{j,h0}$

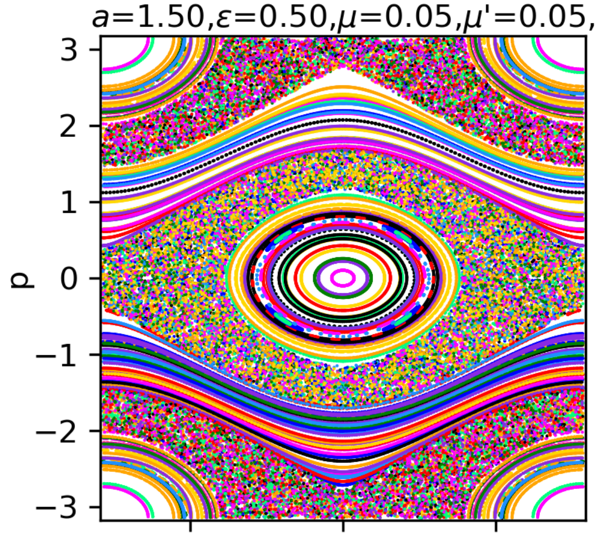


$$\sigma_{j,h0} = \sqrt{\langle w_j | \hat{h}_0^2 | w_j \rangle - \mu_{j,h0}^2}$$

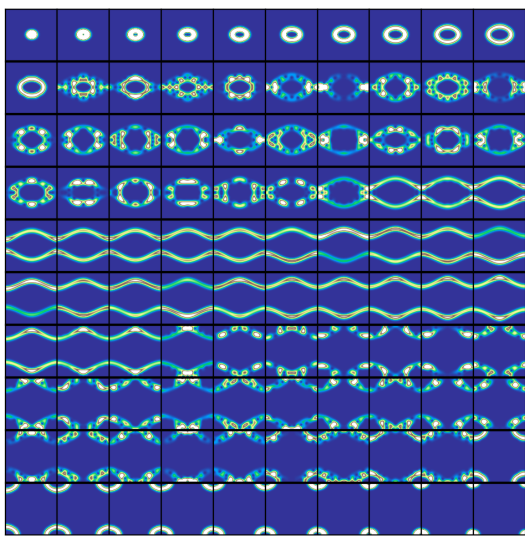
we also computed the dispersion

A measure of ergodicity

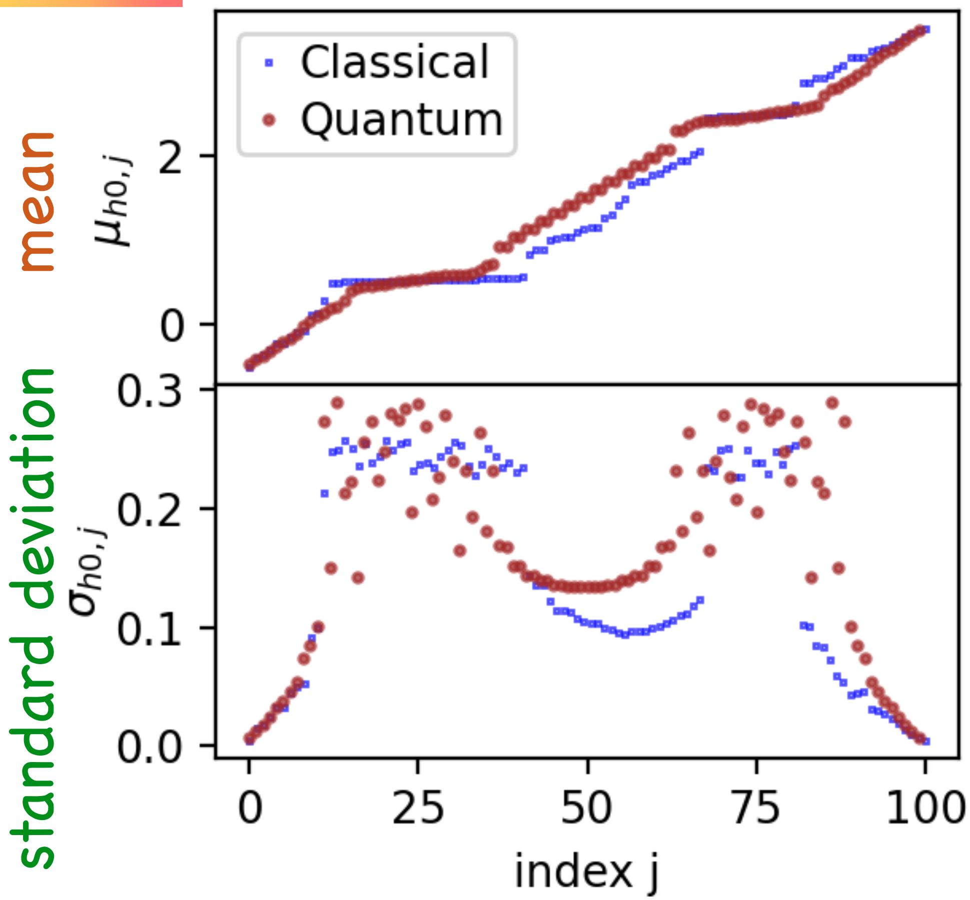
Classical orbits



Husimi functions

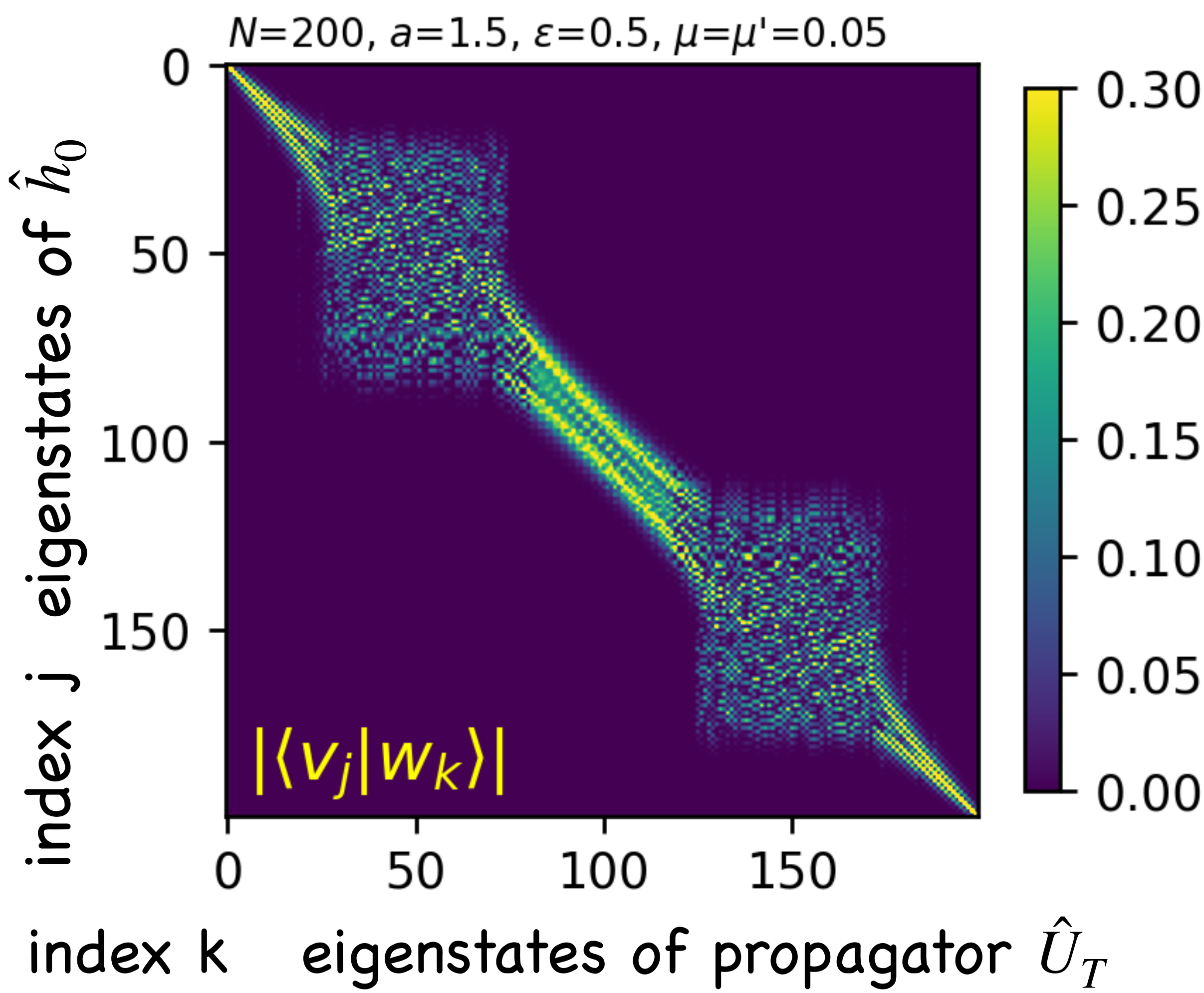


Energies



Expectation value and dispersion of unperturbed Hamiltonian operator for Floquet eigenstates is similar to those computed from classical orbits. The dispersion gives a measure of ergodicity

Another view of quantum ergodicity

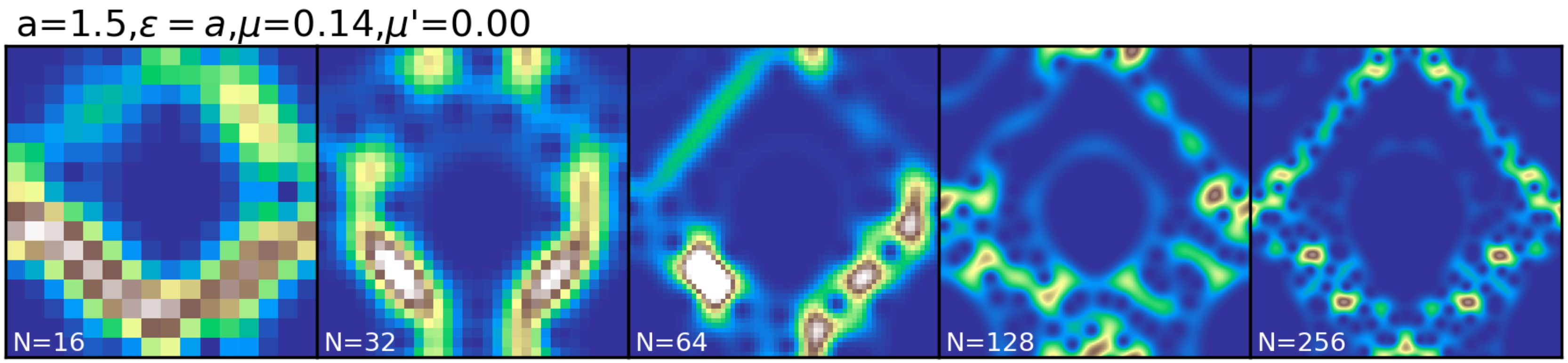


transition probabilities

initially localized information becomes mixed across the chaotic region

akin to thermalization

Another notion of quantum ergodicity



N=16 N=32 N=64 N=128 N=256

Shnirelman theorem (for certain maps) refers to the asymptotic properties of eigenfunctions of the Schrodinger operator in case of a classically chaotic system. For almost all eigenvalues the probability of finding the system in a vicinity of a given classical state becomes uniformly distributed along the surface of constant energy in phase space.

Chaotic zone width estimates

Classical

ϕ_s, p_s describe separatrix orbit of unperturbed system in phase space

$$\Delta H \approx \int_{-\infty}^{\infty} \frac{\partial H_1(\phi_s, p_s, t)}{\partial t} dt$$

Integrate perturbation on separatrix orbit

Quantum counterpart

Eigenstate of unperturbed system with energy nearest that of the separatrix is $|v_s\rangle$

$$\Delta H \sim \sqrt{\sum_{k \neq s} \left| \langle v_s | \frac{1}{T} \int_0^T dt\, e^{\frac{i}{\hbar}\hat{h}_0 t} \hat{h}_1(t) e^{-\frac{i}{\hbar}\hat{h}_0 t} | v_k \rangle \right|^2}$$

Average over perturbation and sum over eigenstates near the separatrix energy

How accurate are these estimates?
Answer: Great to order of magnitude.

Summary

- Harper model on the torus is a compact system for exploring resonant quantum/classical connections. Via perturbation we have hybrid chaotic/regular behavior in related classical and quantum systems
- Husimi functions look great! Classical orbits in phase space resemble quantum eigenstates!
- We have a measure of ergodicity in both classical/ quantum systems
- We derived a quantum counterpart to a predictive and widely applied classical formula for estimating the width of a chaotic region

Notions of adiabatic drift $H(t)$

Classical world

- Conservation of an action variable that encloses a constant volume in phase space.

What happens if phase space is divided by an orbit with an infinite period?

Near the separatrix dynamics cannot be adiabatic

Quantum world

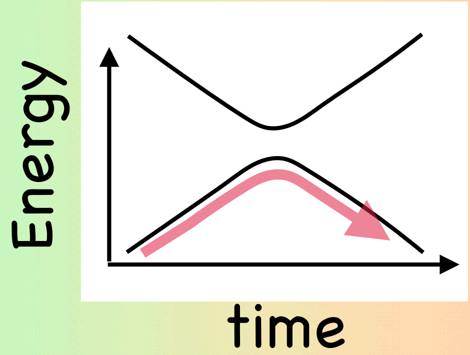
- A system initialized in an eigenstate remains in one
- Landau-Zener model for 2-level systems

Probably of a diabatic transition

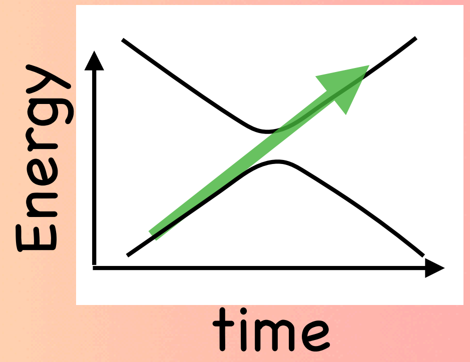
$$P_{diabatic} = e^{-2\pi\Gamma} \qquad \Gamma = \frac{\Delta E^2}{\hbar \dot{E}}$$

If $\Gamma \gg 1$ then the system remains in the same eigenstate

adiabatic



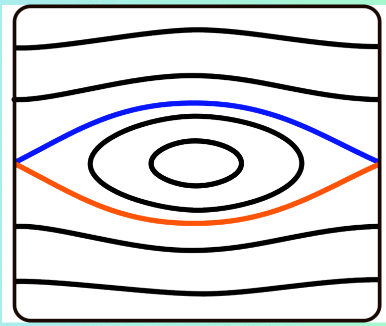
diabatic



Information loss via adiabatic drift $H(t)$

Classical world

- Theory of resonance capture

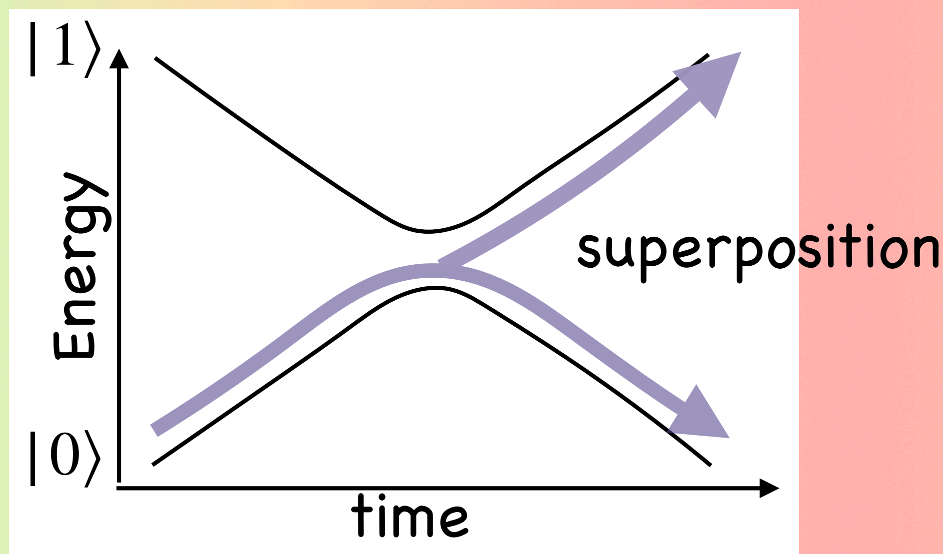


Phase when approaching separatrix orbit determines the outcome

Sensitivity to phase is replaced with a probability of resonance capture

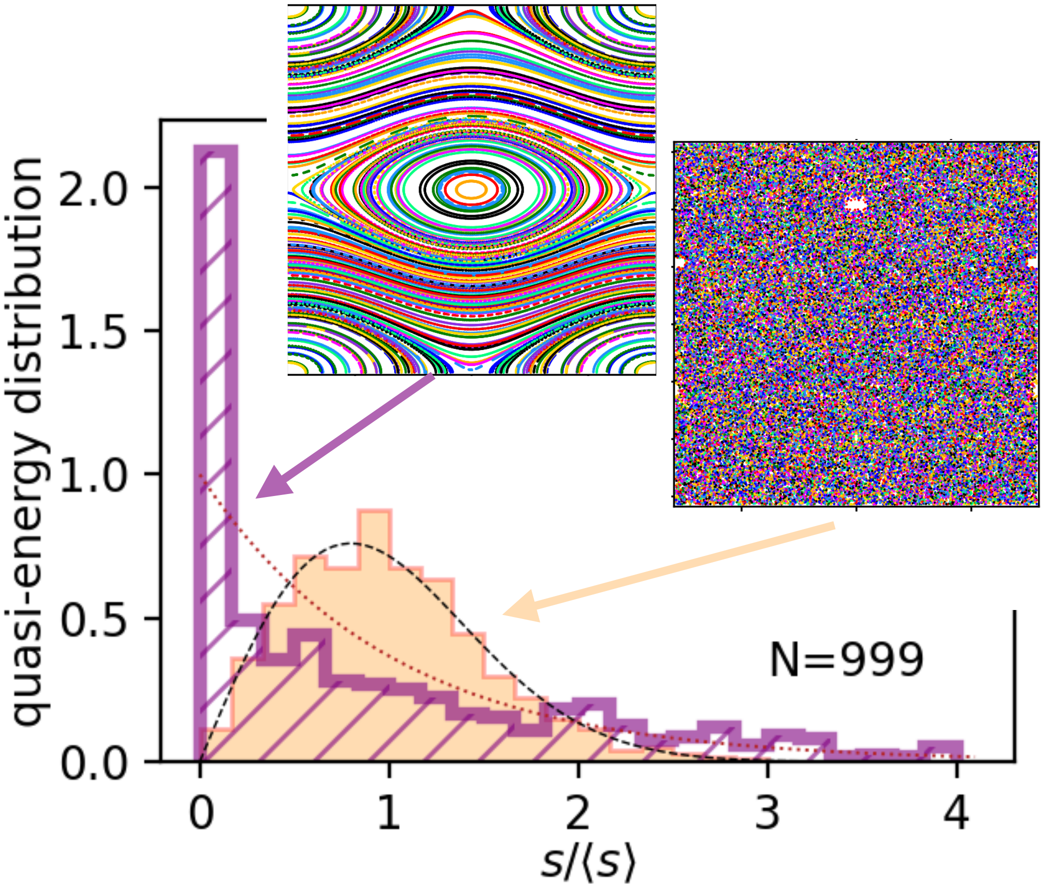
Quantum world

$$|0\rangle \longrightarrow \alpha|0\rangle+\beta|1\rangle$$



Transition into a superposition state with a sensitive relative phase

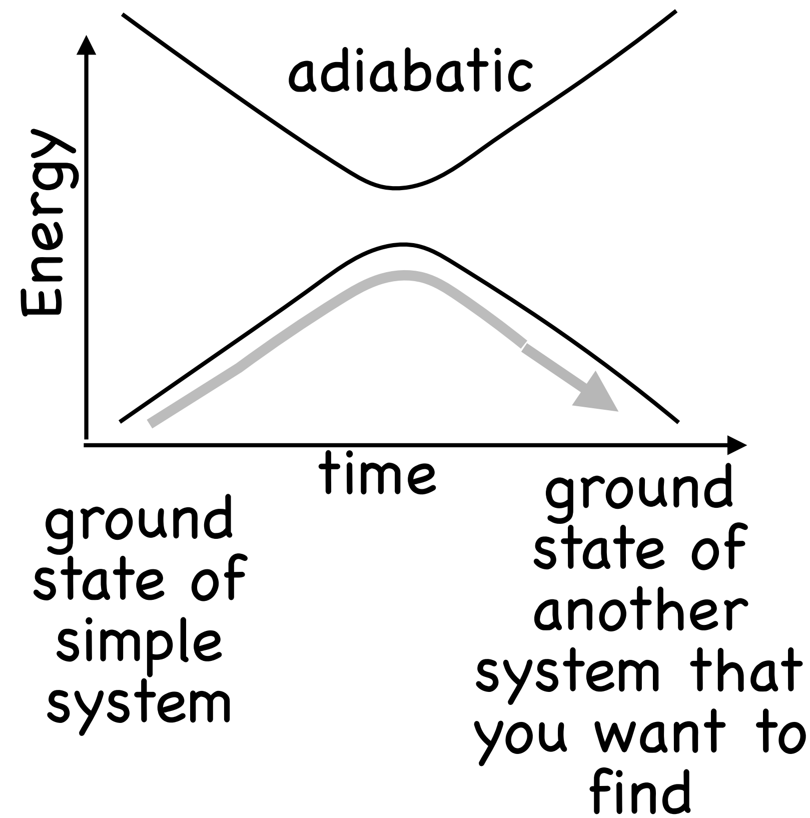
Statistics of quasi-energy spacings



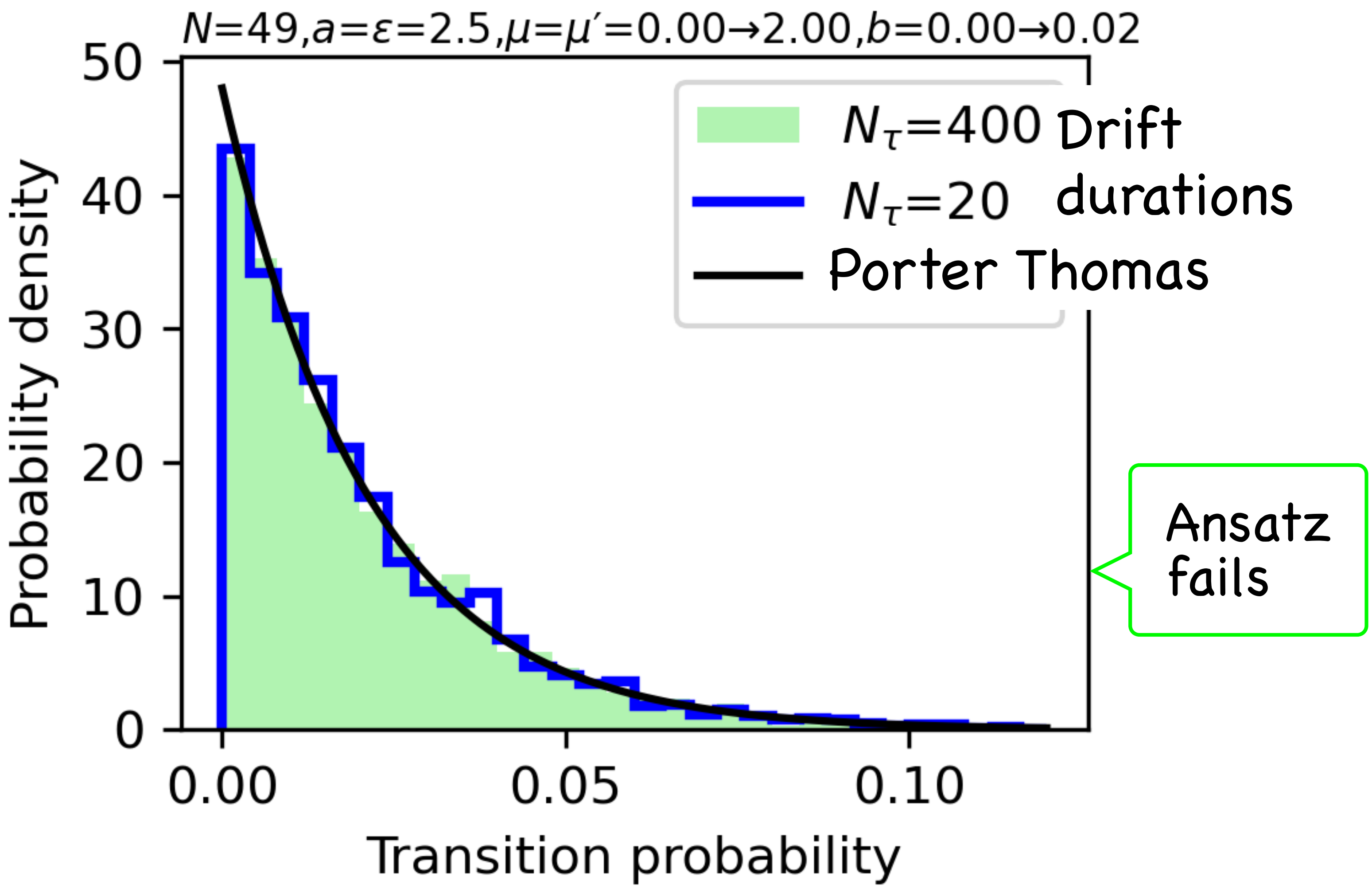
Regular (non perturbed) compared to a chaotic system

Wigner/Dyson statistics (spacings of a random matrix model).

Ansatz: If energy levels repel in chaotic systems, then they could aid in adiabatic computing algorithms



For a slowly drifting chaotic system, transition probabilities are randomly distributed and independent of drift rate

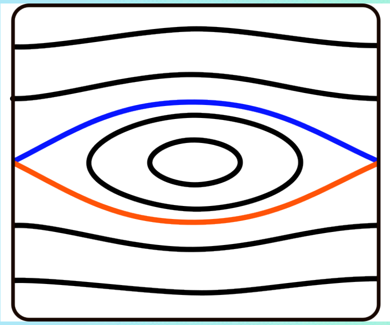


Notions of adiabatic drift $H(t)$

Classical world

- Theory of resonance capture

If the time to drift across resonance is long compared to libration period then



probability of capture into resonance

$$P_c = \frac{V_+}{V_+ - V_-}$$

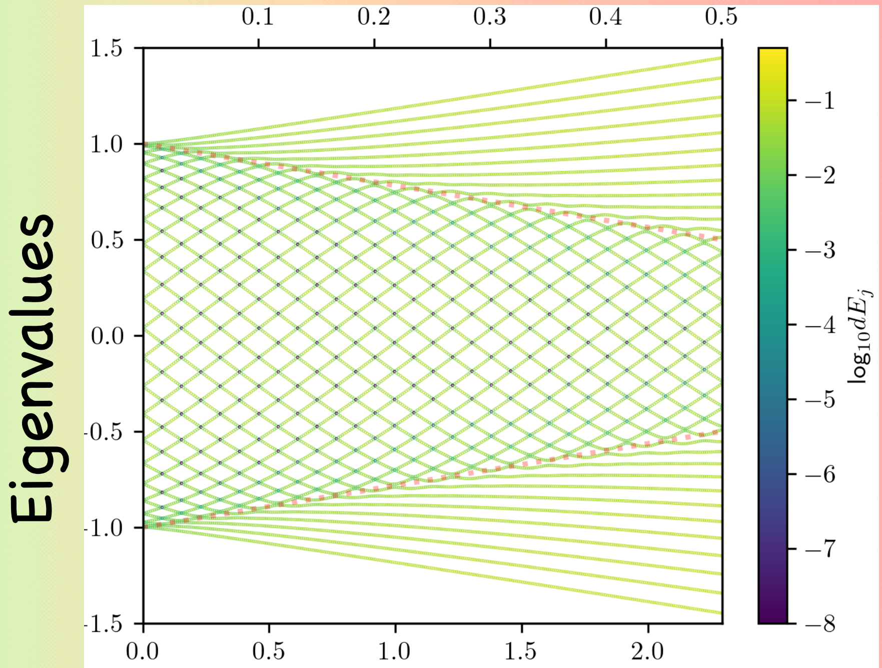
V_+ Rate of volume change of upper separatrix

V_- Rate of volume change of lower separatrix

$V_+ - V_-$ Growth rate of volume inside separatrix

Kruskal-Neishtadt-Henrard Theorem

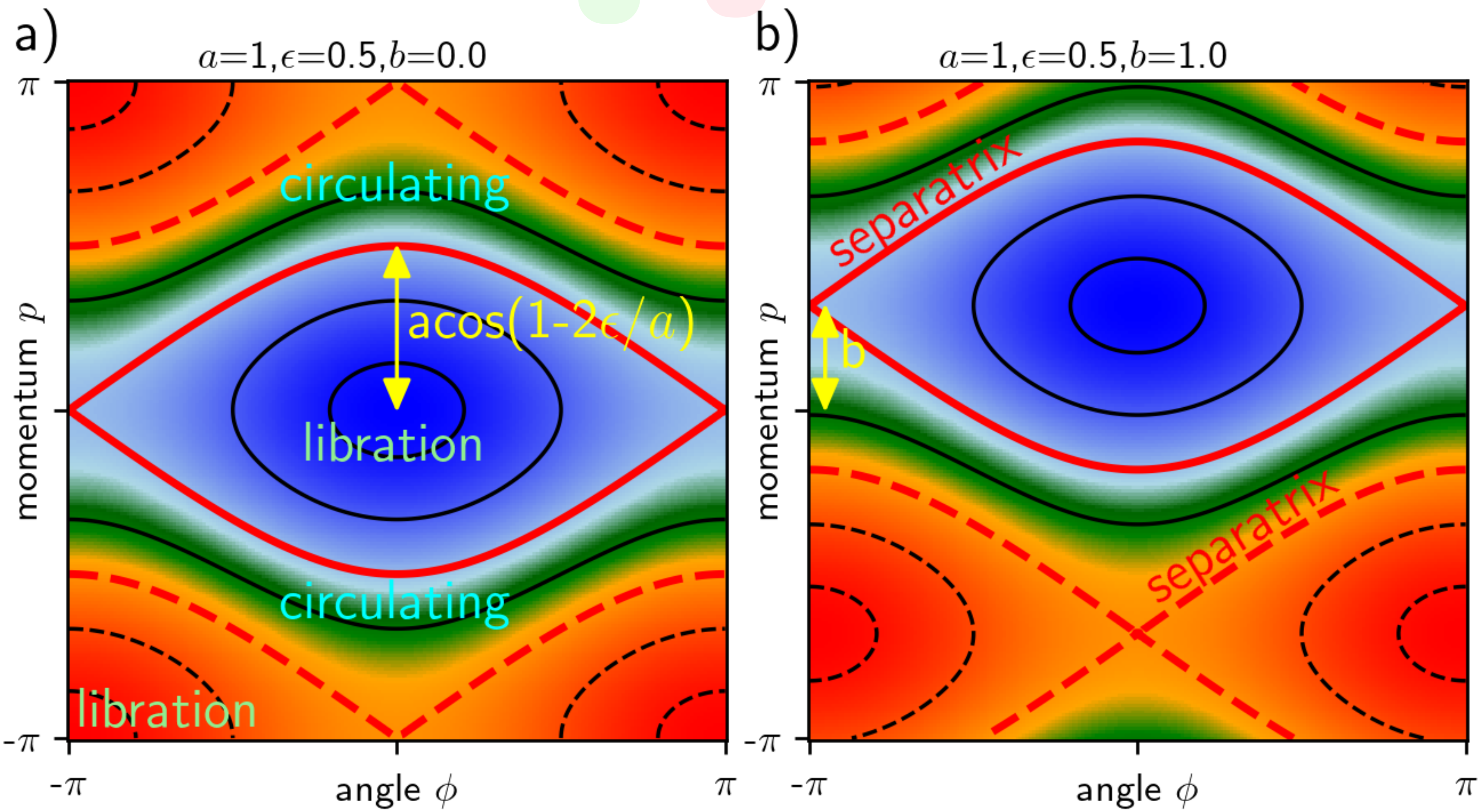
Quantum world



A lattice of resonance crossings. Probability of entering a different region in phase space can be estimated from a sum of probabilities for each crossing (Stadel et al. 2022)

Shifting the Classical Harper operator

$$H(p,\phi) = a\cos(p - b) + \epsilon\cos\phi$$

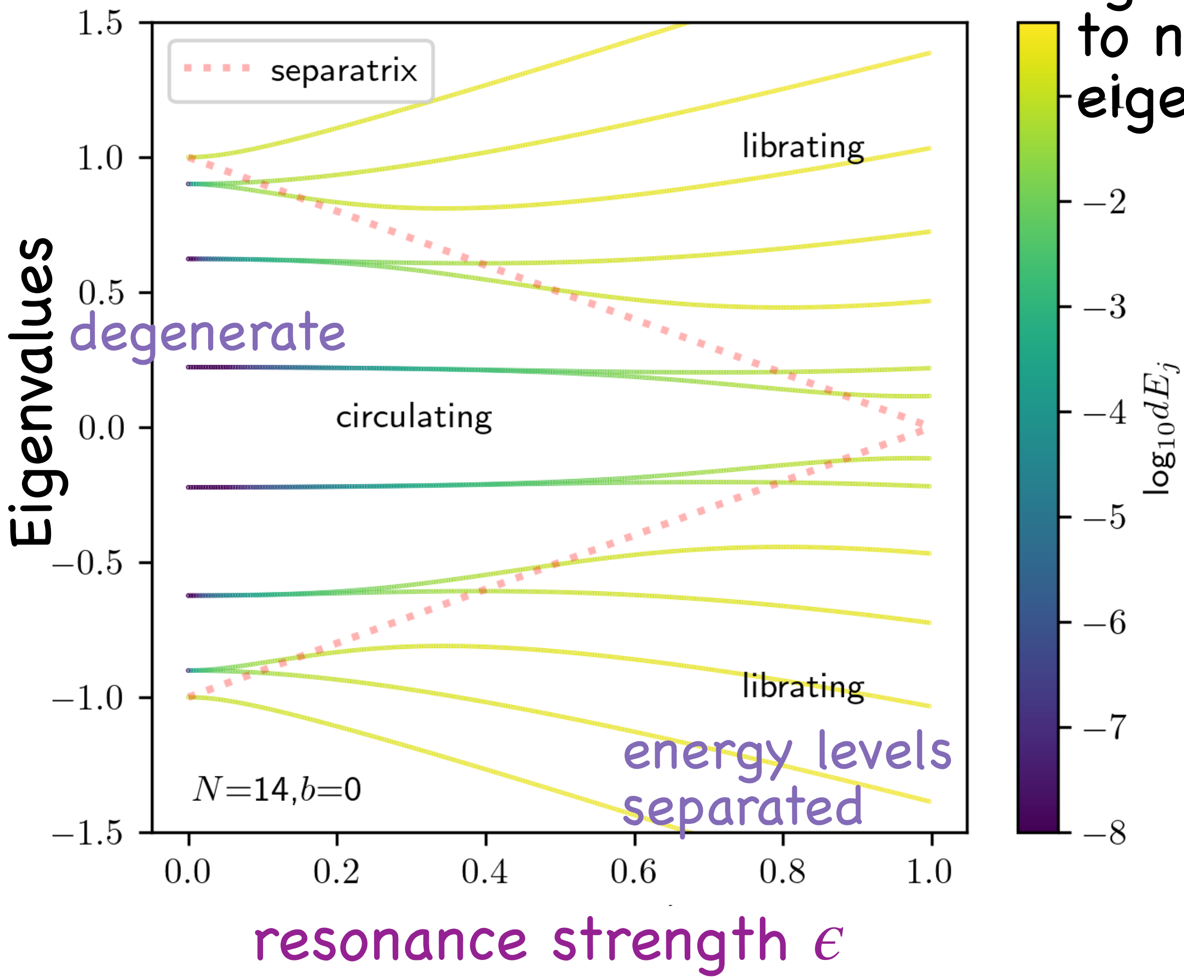


Spectrum of the Harper model

Vary resonance strength

$$\hat{h}(a, b, \epsilon) = a \cos(\hat{p} - b) + \epsilon \cos \hat{\phi}$$

No drift!



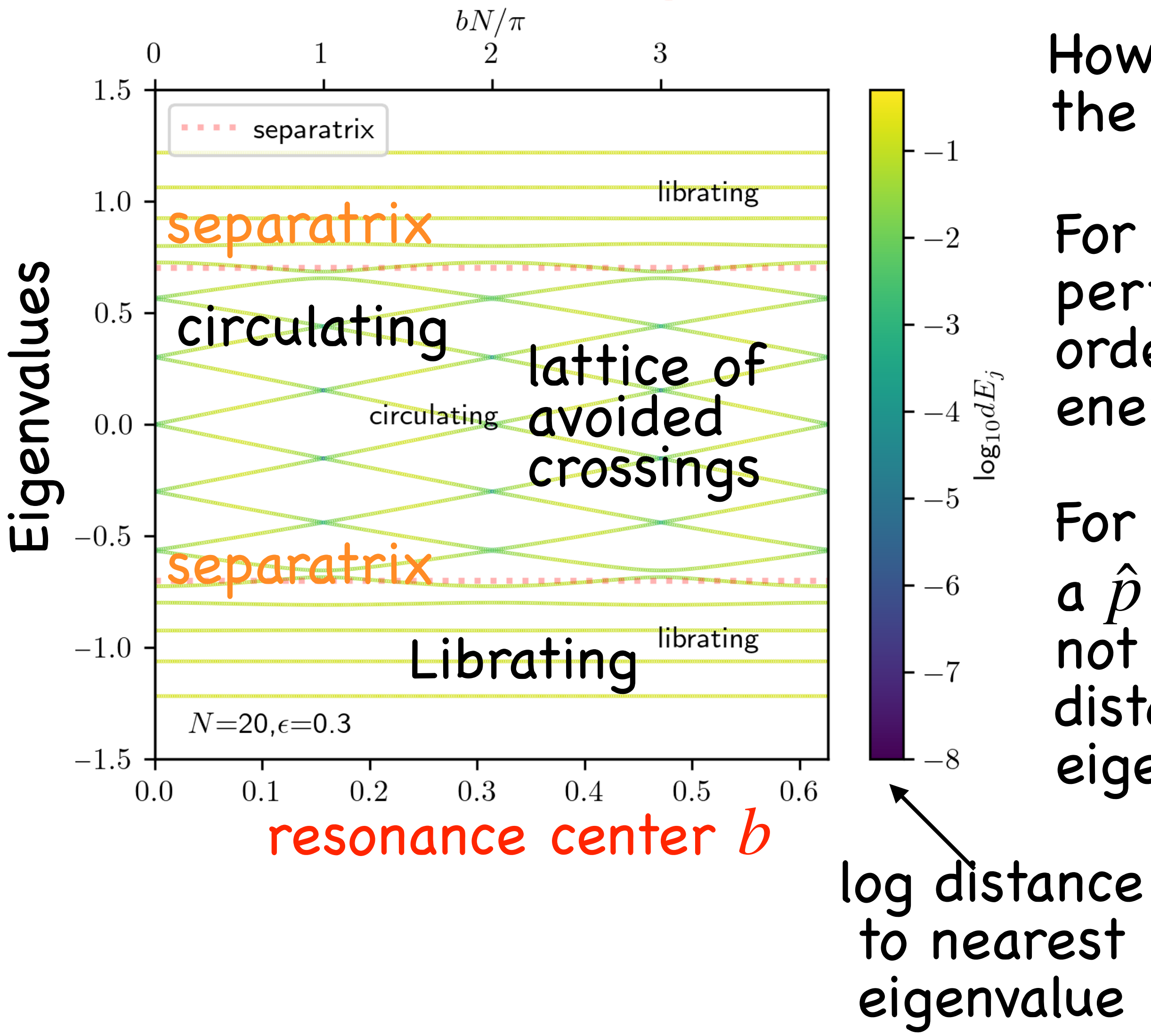
For $\epsilon = 0$, eigenvalues are in pairs of multiplicity 2 (degenerate!)

Eigenvalues are well separated in the librating region, but remain nearly degenerate in the circulating region

Spectrum of the Harper model

$$\hat{h}(a, b, \epsilon) = a\cos(\hat{p} - b) + \epsilon\cos\hat{\phi}$$

Vary resonance center



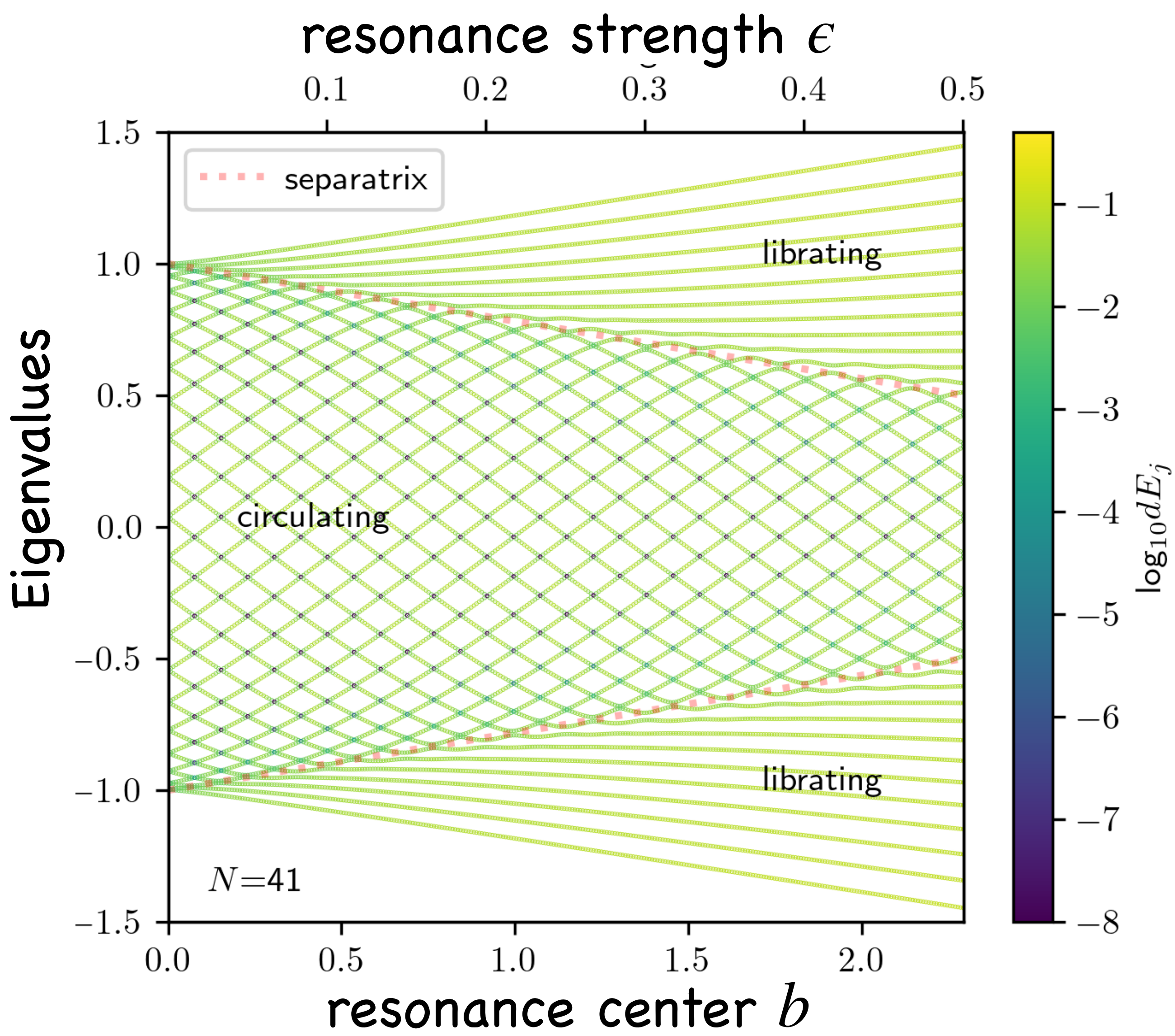
How to account for the lattice:

For a free rotor a $\hat{p}$ perturbation gives first order variations in energy levels

For a harmonic oscillator a $\hat{p}$ perturbation does not strongly affect the distance between eigenvalues

Resonance capture

The libration region increases in size

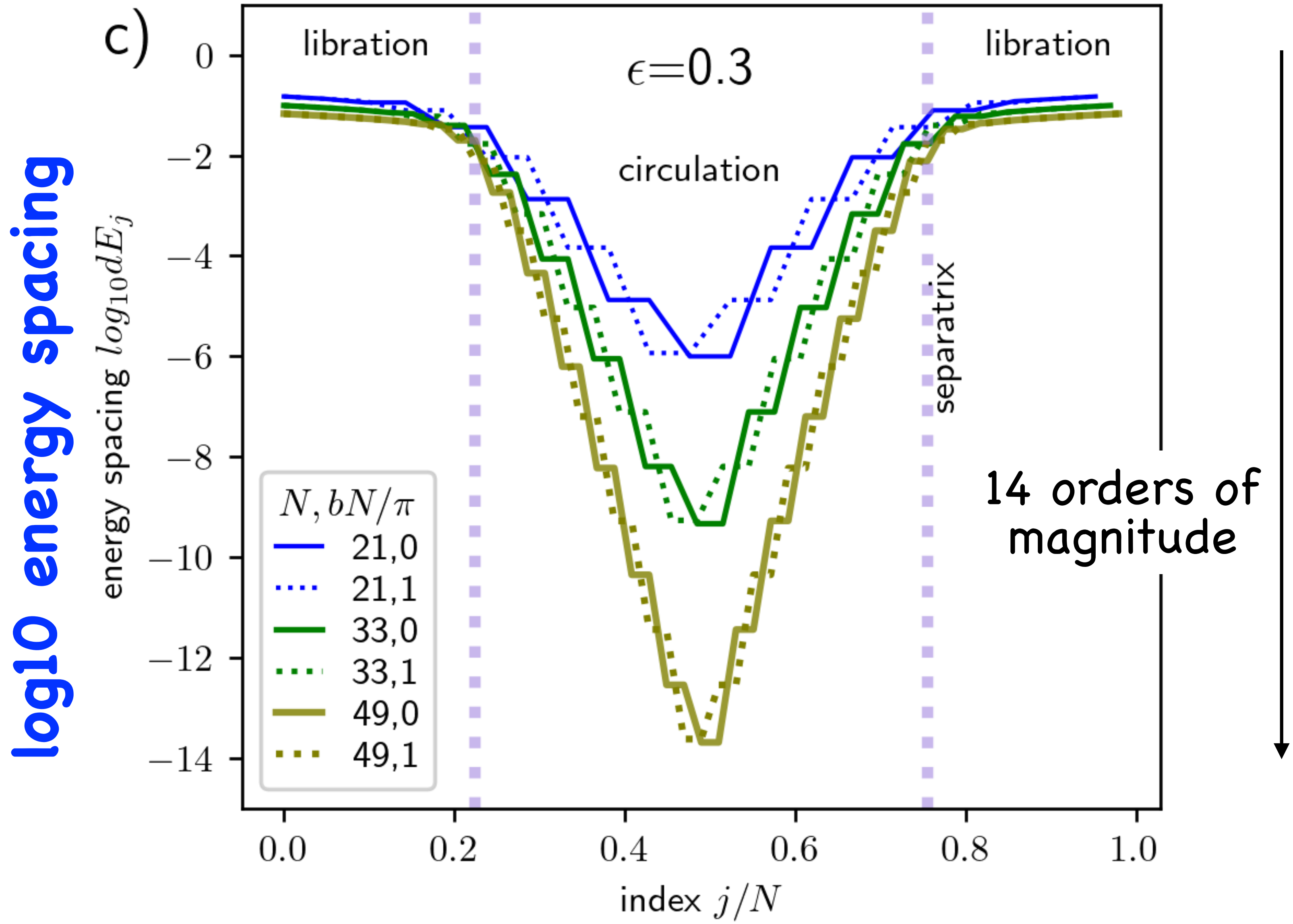


drift in both b and ϵ

Wide range of energy spacings! How close can they be?

Gap sizes

Gaps between nearly degenerate eigenstates range over many orders of magnitude!



Transition amplitudes between initial and final **eigenstates** with b varying $\hat{h}(a, b(t), \epsilon)$

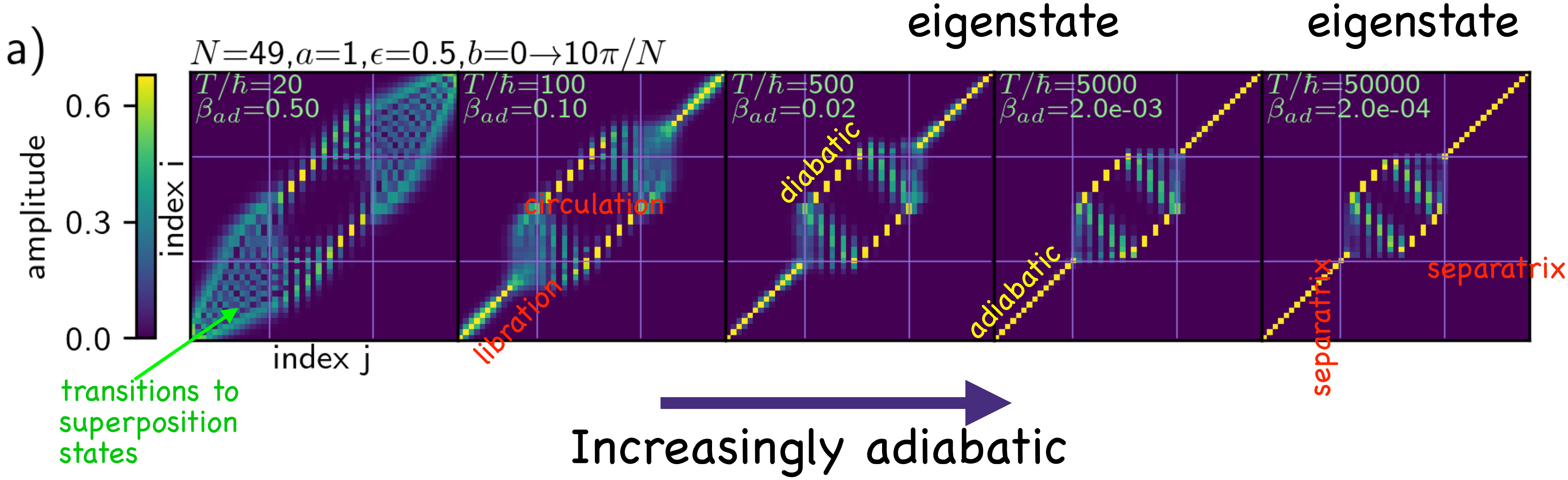
propagator

$$\hat{U}(t, t_0) = \mathcal{T} e^{-\frac{i}{\hbar}\int_{t_0}^{t} \hat{h}(t)dt}$$

transition amplitude:

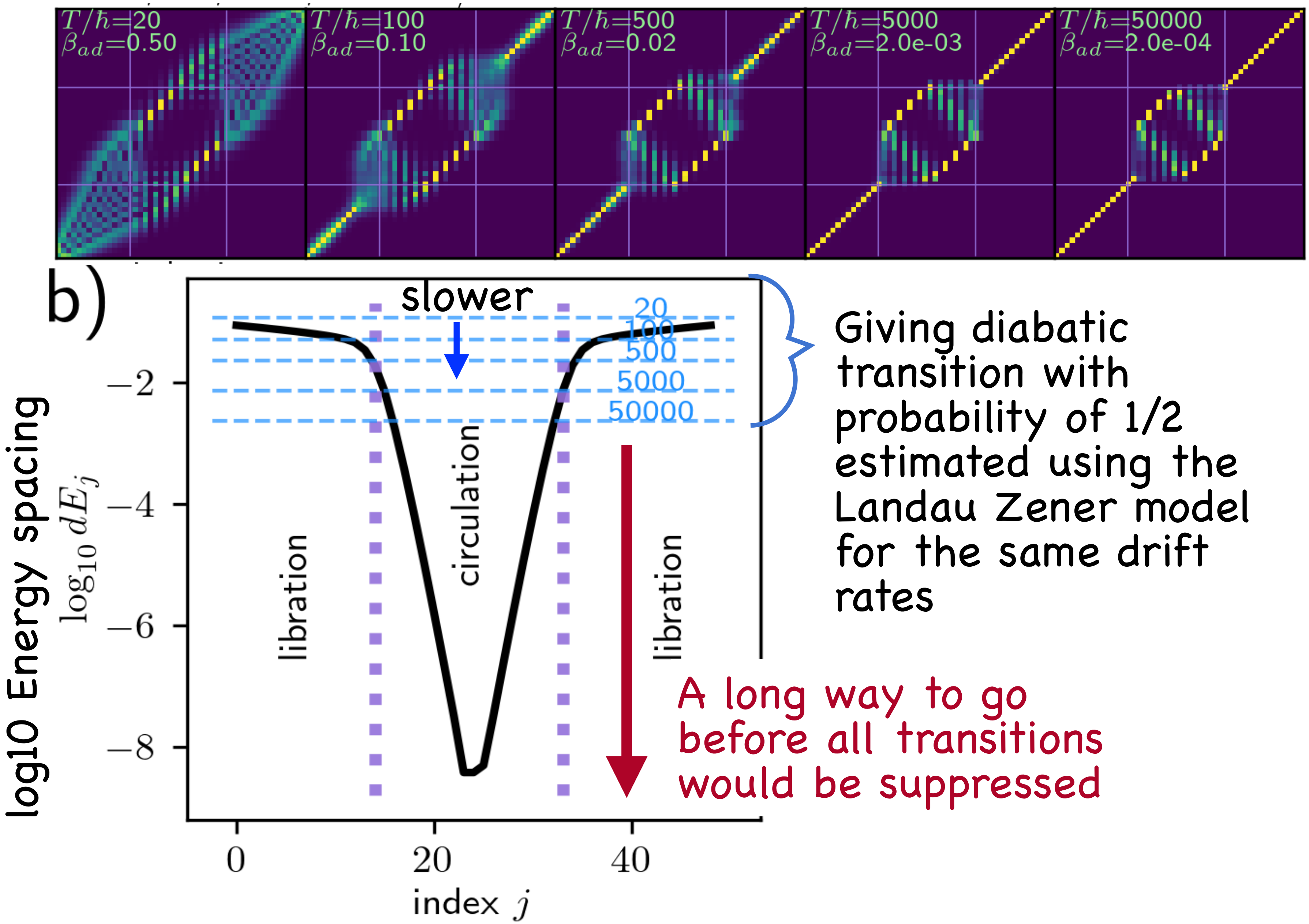
$$A_{ij} = \left| \langle w_j | \hat{U}(t_0, t_0 + T) | v_i \rangle \right|$$

final eigenstate — initial eigenstate



Transitions between eigenstates take place over a wide range of drift rates (here a factor of a few thousand) !

Reaching the adiabatic limit is essentially impossible

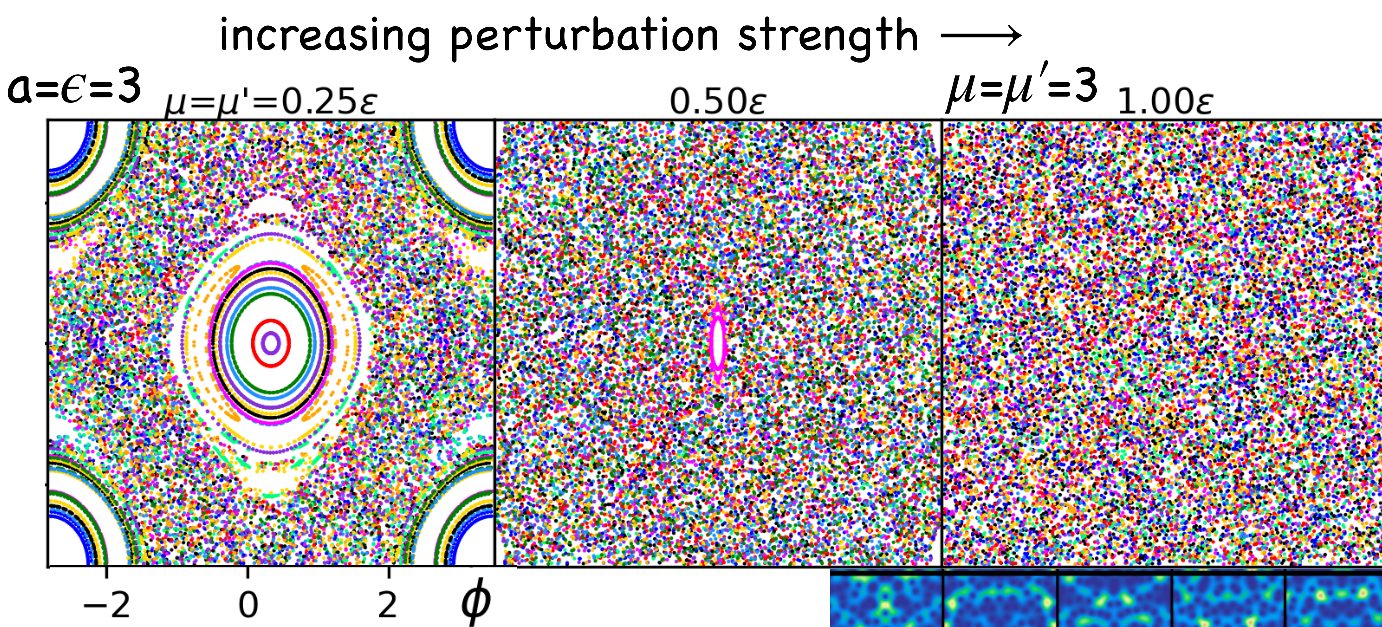


Summary

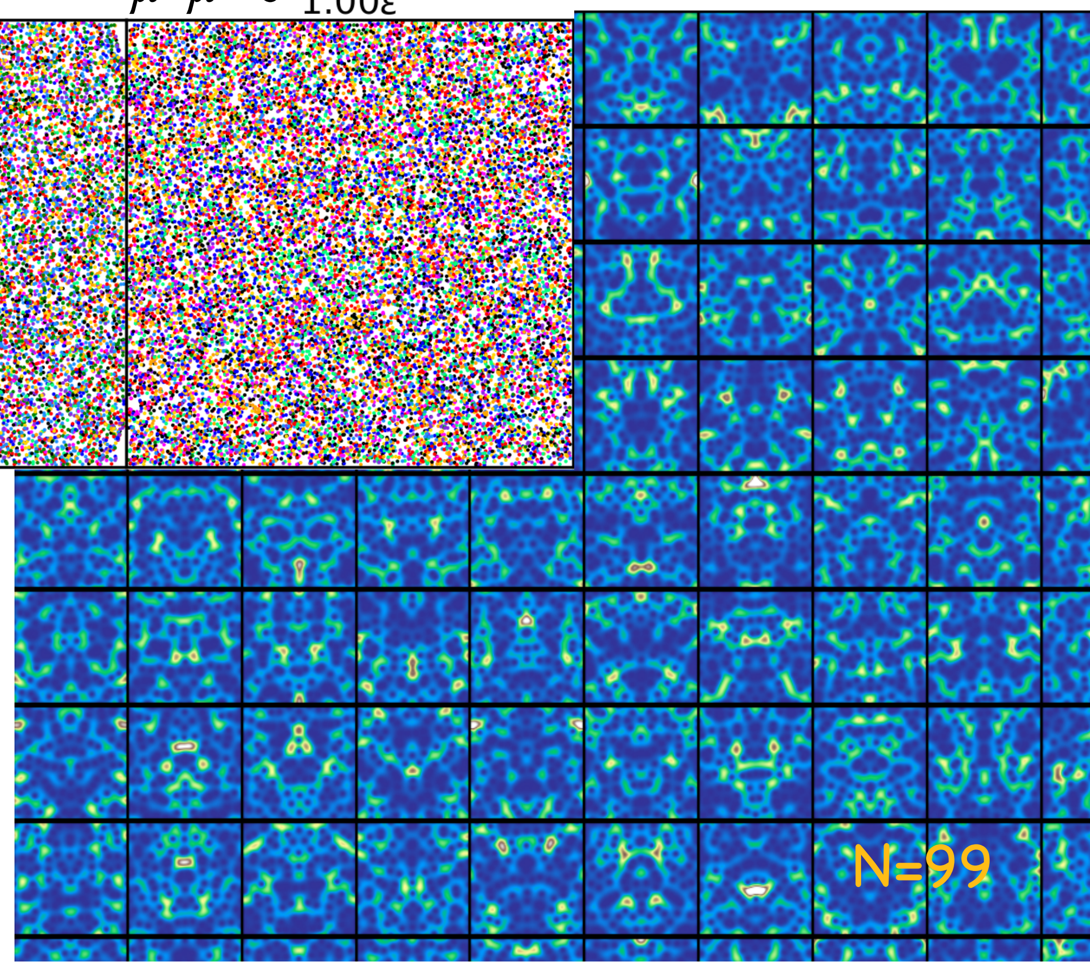
- Quantum systems that are related to resonant classical systems may exhibit a wide range in the spacing between nearby eigenvalues
- It is hard to estimate gap sizes
- Transitions to superposition states would be induced in the Harper model and in the quantized pendulum over an extremely wide range of possible drift rates
- It seems impossible to suppress all possible transitions
- Drifting systems provide way to control loss of quantum information via causing transitions into superposition states
- Drifting systems are a possible way to simulate thermalization

What next? Quantum sampling

Goal: Use controllable chaotic systems to generate random unitary operators with desirable probability distributions
Quantum Supremacy claim in 2019 based on quantum sampling



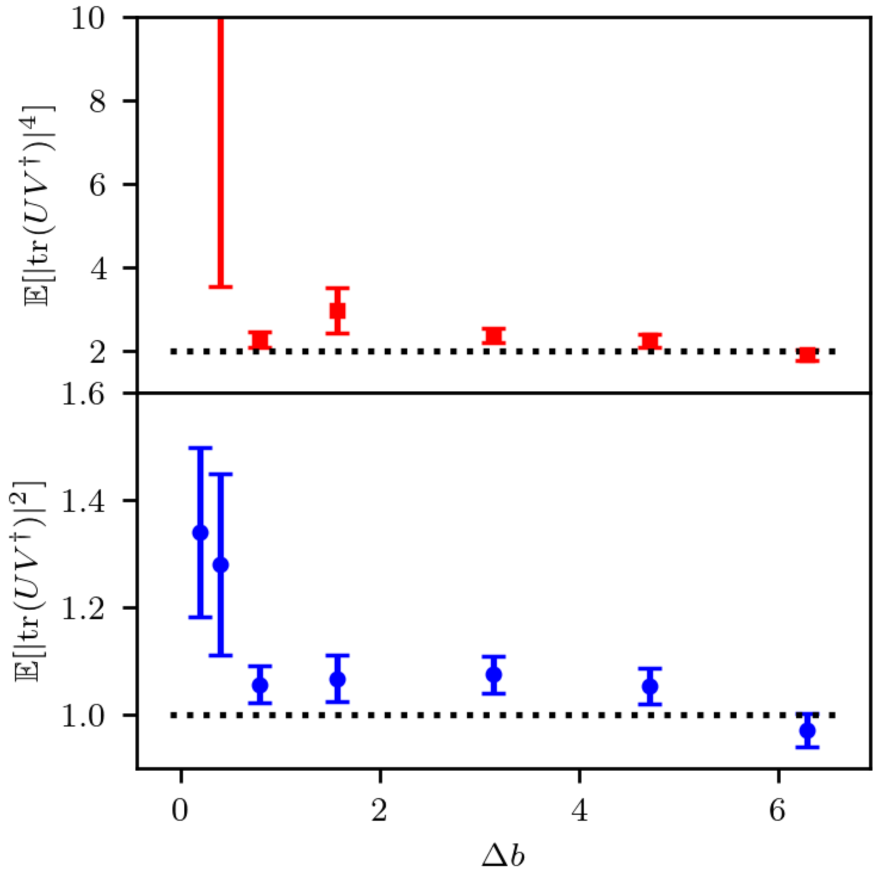
Most efficient chaos with perturbation frequency near the libration frequency
Less prone to resonant islands if perturbation is somewhat slower than libration frequency

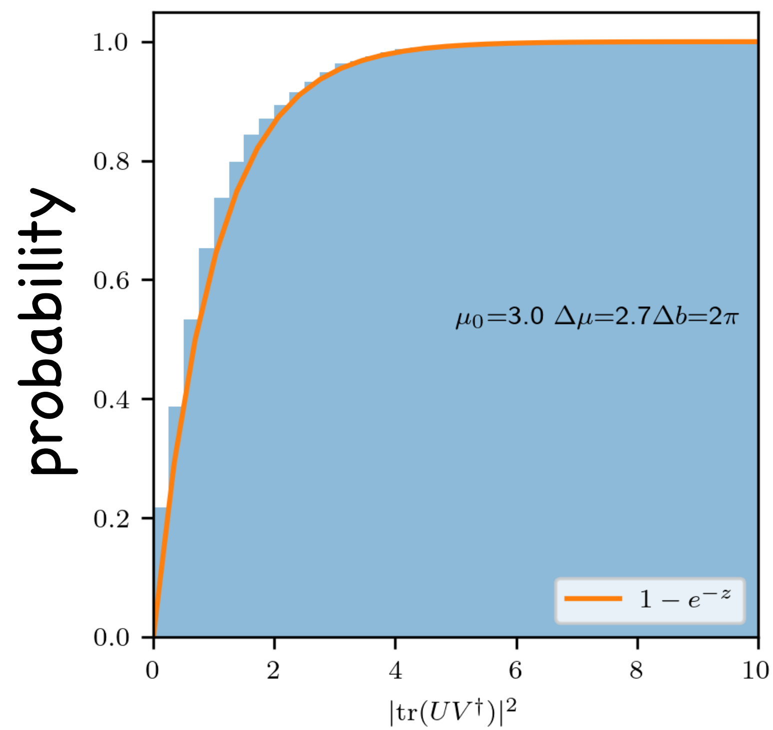


Approximate 2-designs

Approximating a distribution
of random unitaries

Within 1 Floquet period but
parameters are
chosen with a uniform
distribution





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