PROBLEM SET #4 **AST243**

1. Temperature Jumps at Shocks

The Rankine-Hugoniot conditions can be used to show that for an ideal gas with it adiabatic index γ

$$\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} = \frac{(\gamma+1)M_1^2}{(\gamma-1)M_1^2+2} \tag{1}$$

and

$$\frac{p_2}{p_1} = \frac{2\gamma M_1^2}{\gamma+1} - \frac{\gamma-1}{\gamma+1} \tag{2}$$

where the subscript 1 refers to upstream and subscript 2 refers to downstream. Here $M_1 = u_1/c_1$ is the Mach number of the pre-shock material and velocities are given in the shock's reference frame.

(a) Show that for strong ($M_1 \gg 1$) shock waves

$$\frac{T_2}{T_1} = \frac{(\gamma-1)p_2}{(\gamma+1)p_1} \tag{3}$$

While the density ratio approaches a limiting value with increasing Mach number, the pressure and temperature ratio can be arbitrarily large.

Hint: use the ideal gas law $p = nkT$.

(b) Consider a shock passing into the interstellar medium of initial density $n_1 \sim 1$ cm^{-3} and temperature $T_1 = 10^4$K. What pre-shock speed (in the shock frame), u_1, is required to account for a post shock temperature of 10^7K? You can either do this to order of magnitude or assume $\gamma = 5/3$.

Hint: use the temperature ratio to find the pressure ratio and use that to find the pre-shock Mach number M_1.

(c) Assume that the preshock gas has zero velocity with respect to the observer and that the shock normal is parallel to the line of sight. What is the post shock velocity in the observer frame?

Hint: you can find u_2 using M_1. But u_2 is in the frame moving with the shock. The difference $u_2 - u_1 = v_2 - v_1$ where v_2, v_1 are in the observer's frame.

2. Blast wave estimates

A self-similar blast wave solution relates energy E, ambient density ρ, radius R, and time, t, with

$$R \sim \left(\frac{E}{\rho}\right)^{1/5} t^{2/5}. \tag{4}$$

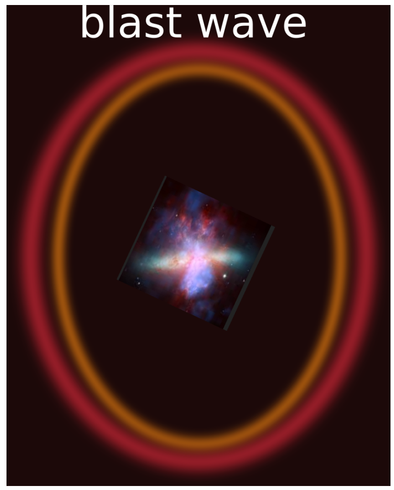

FIGURE 1. A hypothetical blast wave caused by a burst of star formation.

(a) Derive an approximate scaling relation for the blast wave's velocity V as a function of time t, energy E and ambient density ρ.

Hint: $V \sim \frac{dR}{dt} \sim R/t$.

(b) Assume that the blast wave is made by a constant energy injection rate $\dot{E}$ rather than a single explosion of energy E. Derive a scaling relation (like the above one) for the blast wave's radius R as a function of time t in terms of $\dot{E}$ and ambient density ρ.

Hint: this can be done with units.

(c) For the constant $\dot{E}$ driven blast wave, derive a scaling relation for the blast wave's velocity V in terms of the ambient density ρ and energy injection rate $\dot{E}$ and time.

(d) Show that

$$\dot{E} \sim \rho R^2 V^3. \tag{5}$$

3. A starburst fueled shell

In the center of a nearby elliptical galaxy you detect a spherical shell that has radius 500pc and is expanding at 200 km/s. You estimate the number density of the ambient

ISM (outside the blast wave) in the galaxy from its X-ray emission as $n \sim 10^{-3}\text{cm}^{-3}$. The shell is probably too big to be caused by a single supernova. You consider the possibility that many supernovae have contributed to its energetics.

(a) Assuming a constant $\dot{E}$, and the scaling relation derived in the previous problem (equation 13) relating $\dot{E}, V$ and ρ, how much energy per unit time ($\dot{E}$) in erg/s is needed to account for the shell?

(b) Assume that there is an on-going starburst at the center of the galaxy. About 1 per 100 stars born goes supernovae. Each supernovae has 10^{51} ergs but of this energy only about 1% might be injected into the galaxy ISM because energy is lost through radiation and emitting particles like neutrinos. Assume a constant star formation rate and that the average star has a mass of about 1 $M_\odot$.

Find a coefficient, X, such that

$$\text{SFR} = X\ M_\odot \text{yr}^{-1} \times \left(\frac{\dot{E}}{\text{erg s}^{-1}} \right) \tag{6}$$

where SFR is the star formation rate in solar masses per year.

(c) Approximately, what star formation rate in solar masses per year would be required to explain the shell in the context of a constant $\dot{E}$ blast wave model?

(d) Using its velocity and radius, estimate the age of the shell. This would give you an estimate for the duration of the starburst.

Hint: age $t \sim R/V$.

4. Components of the velocity gradient tensor

Consider the velocity gradient tensor $T_{ij} = \frac{\partial u_i}{\partial x_j}$ in Cartesian coordinates with $\mathbf{u}$ the velocity.

The tensor $\mathbf{T}$ is a 3×3 matrix that can be decomposed into the sum of three matrices. The first matrix in the sum is a diagonal matrix that is proportional to the identity matrix. The second matrix is symmetric and traceless. The third matrix is anti-symmetric and traceless.

(a) Construct and draw streamlines for a velocity field such that the trace of $\mathbf{T}$ is non-zero but the antisymmetric and trace-less symmetric parts are zero.

(b) Construct and draw streamlines for a velocity field such that the symmetric components of $\mathbf{T}$ are zero but the antisymmetric component is not.

(c) Construct and draw streamlines for a velocity field such that only the traceless symmetric component of $\mathbf{T}$ is non-zero.

Consider the traceless symmetric tensor $\sigma_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) - \frac{1}{3}u_{k,k}\delta_{ij}$ where $u_{k,k}$ is the trace of $\mathbf{T}$.

(d) Compute σ_{xy} for fluid flow near a fixed surface with velocity $\mathbf{u} = (ay, 0, 0)$ that approaches zero near a surface at $y = 0$.

(e) Compute σ_{xy} for rotation in a Keplerian disk with $\mathbf{u} = v_c(r)(-\frac{y}{r}, \frac{x}{r}, 0)$ and $v_c(r) = \sqrt{\frac{GM}{r}}$. Here the radius $r = \sqrt{x^2 + y^2 + z^2}$.

5. Are galaxy gas disks accretion disks?

Consider a gas rich spiral disk galaxy. A typical velocity dispersion for turbulent motions in the HI gas is of order $\sigma \sim 10$ km/s and a typical radial sizescale for the disk is of order $R \sim 10$ kpc. A typical circular rotation velocity would be $v_c \sim 200$ km/s.

(a) Using the velocity dispersion in the HI disk and hydrostatic equilibrium to estimate the scale height of the atomic gas component in parsec.

Hint: hydrostatic equilibrium gives $\frac{h}{r} \sim \frac{\sigma}{v_c}$ where h is the scale height at radius r and σ is the velocity dispersion and v_c is the circular velocity.

(b) Using the scale height and velocity dispersion in the HI disk, estimate a kinematic turbulent viscosity at radius r from their product.

Hint: A turbulent viscosity is commonly estimated as $\nu \sim \alpha h \sigma$ with uncertainty in the value described by dimensionless number α.

(c) Using the disk radius R of the galaxy, estimate an accretion timescale in years, or a timescale for accretion to take place across this radius.

Hint: The velocity of inflow due to accretion $v_r \sim r/\nu$. A timescale for accretion would be $t_\nu \sim r/v_r$.

(d) Compare an accretion timescale to the Hubble time. Is viscous accretion likely to be important in galactic gas disks?

It may be helpful to know that 1 km/s is approximately 1 pc/Myr is approximately 1 kpc/Gyr.

6. The temperature of a viscously heated accretion disk

Assume that an accretion disk is heated by viscous energy dissipation giving a heating rate per unit area (with units of power per unit area) of

$$\dot{q} = \frac{9}{4}\Sigma\Omega^2\nu \tag{7}$$

where $\Omega = \sqrt{GM/r^3}$ is the angular rotation rate at radius r and about a central mass M. Here ν is the viscosity and Σ is the disk's surface mass density. We assume that the generated heat is radiated from both sides of the disk with a black body spectrum with temperature T so

$$\dot{q} = 2\sigma_{SB}T^4 \tag{8}$$

where σ_{SB} is the Stefan-Boltzmann constant. Assume that the viscosity is that of a Shakura-Sunyaev disk with

$$\nu = \alpha c_s h \tag{9}$$

with sound speed

$$c_s = \sqrt{\frac{k_B T}{m}} \tag{10}$$

with m mass of a gas particle, and k_B the Boltzmann constant. The scale height h satisfies

$$h\Omega = c_s \tag{11}$$

due to hydrostatic equilibrium. Assume that the disk surface density obeys a power law

$$\Sigma = \Sigma_0 r^{-\beta} \tag{12}$$

with exponent $\beta > 0$.

(a) Assuming that the temperature relevant for the sound speed and the disk surface temperature are the same, and that radiation from a central source is not a significant contributor to the disk temperature, find the radial temperature profile. (Find $T(r)$)

(b) If the disk temperature is set by radiation from a central source with luminosity L, the disk temperature profile (ignoring optical depth) is

$$T = \left(\frac{L}{4\pi\sigma_{SB}}\right)^{\frac{1}{4}} r^{-\frac{1}{2}} \tag{13}$$

Which temperature profile drops faster, that due to radiation balance from a central source or that due to accretion?

7. The width of a shock

Consider Burger's equation, including viscosity

$$u_{,t} + uu_{,x} = \nu u_{,xx} \tag{14}$$

where we have used short hand for the partial derivatives and $u(x,t)$. This equation can illustrate phenomena associated with the Navier-Stokes equation in 1 dimension. Because of the viscous term, solutions should no longer exhibit sharp discontinuities. However we can consider a solution that approaches a constant value at large distances from a region where u is rapidly changing.

We consider a solution that has the form $f(x - st)$ of a traveling wave. Moving into a frame that moves with the traveling solution we have a time independent equation

$$uu_{,x} = \nu u_{,xx} \tag{15}$$

(a) Show that

$$u(x) = c \tanh \frac{cx}{2\nu} \tag{16}$$

is a solution to the time independent form of Burger's equation (with viscosity).

For this solution, $u \to c$ at large positive x and $u \to -c$ at large and negative x so the velocity jump is $\Delta u = 2c$.

(b) Estimate the width of a shock that has a velocity jump of Δu if the viscosity is ν. By width we mean the size of the region where the transition in velocity takes place.