PROBLEM SET #2 **AST243**

1. Validity of continuum or fluid approximation in a hydrostatic planetary atmosphere

Consider the Euler equation

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla} \mathbf{u} = -\frac{1}{\rho} \boldsymbol{\nabla} p + \mathbf{g} \tag{1}$$

with an additional force from $\mathbf{g}$, the gravitational acceleration. In *hydrostatic equilibrium* we can assume that the velocity $\mathbf{u} = 0$ and remains that way.

(a) Assume a planet's atmosphere is isothermal and has temperature T and sound speed c_s. Show that a solution for the density as a function of height above the surface z is $\rho \propto \exp(-z/h)$ and find an expression for the scale height h in terms of the gas temperature.

(b) The velocity, v_K, of an object in circular orbit of radius r around a planet is

$$v_K = \sqrt{\frac{GM_p}{r}}$$

where M_p is the planet's mass and G the gravitational constant. Compare this to the sound speed of the gas. For what sound speed would the atmosphere scale height be the same order as the planet's radius, R_p? For what scale height (in units of the planet's radius) is the mean thermal velocity equal to or below the escape velocity? When the scale height is of order R_p or greater, a constant gravitational acceleration is a bad assumption.

(c) High above a planet's surface, the atmosphere becomes more and more rarified. Assume a density at the planet's surface of ρ_0, the atmosphere is comprised of molecules of mass m and a collision cross section for the molecules of σ. When the mean free path is greater than the scale height the fluid approximation fails. When the fluid approximate fails, equilibrium is no longer maintained by collisions. This height is called the *exobase*.

At what height above the planet's surface does a fluid approximation fail?

This problem is based on one posted by Eugene Chiang.

2. Reynolds number of headwinds on planetesimals

A planetesimal is embedded in the midplane of a circumstellar disk that is composed mostly of molecular hydrogen. Due to gas pressure in the disk, the gas in the disk orbits more slowly than the planetesimal and the planetesimal feels a headwind. With respect to the planetesimal, the speed of the headwind is about $u_{\rm wind} \sim 20$ m/s and independent of orbital radius. Our goal is to estimate the Reynolds number of the gas flow about the planetesimal.

A fairly conventional[1] circumstellar disk model has midplane density at orbital radius r

$$\rho(r) \sim 10^{-5} \text{ kg m}^{-3} \times \left(\frac{r}{\rm AU}\right)^{-\frac{11}{4}}. \tag{2}$$

The gas number density

$$n \sim \frac{\rho}{m} \tag{3}$$

where $m \sim 3.4 \times 10^{-27}$kg is the mass of a hydrogen molecule. The disk midplane temperature for the conventional disk has

$$T(r) \sim 170K \left(\frac{r}{\rm AU}\right)^{-\frac{1}{2}}. \tag{4}$$

We note that ρ, n, T are functions of r.

The mean free path in the gas is

$$\lambda_{\rm mfp} \sim (n\sigma)^{-1} \tag{5}$$

where $\sigma = 2 \times 10^{-19}$ m^2 is the collisional cross sectional area of a molecular hydrogen. The gas kinematic viscosity

$$\nu \sim \lambda_{\rm mfp} c_s \tag{6}$$

where the sound speed

$$c_s \sim \sqrt{\frac{k_B T}{m}} \tag{7}$$

and k_B is Boltzmann's constant. Note that $\lambda_{\rm mfp}, \nu, c_s$ are all functions of r.

The Reynolds number of the flow about a planetesimal of diameter D is

$$Re = \frac{D u_{\rm wind}}{\nu}. \tag{8}$$

The Reynolds number is a dimensionless number that is used to characterize the ratio of viscous stress to inertial force.

(a) Estimate the Reynold's number of the flow about the planetesimal for a $D = 1$ km diameter planetesimal at $r = 1$ AU.

[1] Slightly enhanced surface density compared to the minimum mass solar nebula, and with a radial temperature decay profile that is approximately set by radiation balance with the central star.

(b) How does the Reynolds number depend on orbital radius?

3. Potential flow

A irrotational flow has zero vorticity, $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \mathbf{u} = 0$ where $\mathbf{u}$ is the velocity field.

A potential function $\Phi(\mathbf{x})$ can be used to model an irrotational flow (one with no vorticity) with

$$\mathbf{u} = \boldsymbol{\nabla}\Phi$$

as the curl of a gradient vanishes; $\boldsymbol{\nabla} \times (\boldsymbol{\nabla}\Phi) = 0$.

An incompressible flow satisfies $\boldsymbol{\nabla} \cdot \mathbf{u} = 0$. Hence an incompressible potential flow satisfies Laplace's equation; $\boldsymbol{\nabla} \cdot \boldsymbol{\nabla}\Phi = \boldsymbol{\nabla}^2\Phi = 0$.

A solution of Laplace's equation can be described as *harmonic*, which is a powerful concept that is exploited for analytical models for 2-dimensional flows.

Near a surface, a boundary layer is likely to cause vorticity, so potential flow models do not describe real flows. However, they can be convenient as they can provide simple analytical models and they can be used to approximate irrotational flow distant from a boundary, even in astrophysics, where flows tend to be compressible and rotational.

In spherical coordinates, flow about a sphere of radius R can be modeled with the potential

$$\Phi(r, \theta, \phi) = U\left(r + \frac{1}{2}\frac{R^3}{r^2}\right)\cos\theta \tag{9}$$

where U is the velocity of the flow distant from the sphere. Here the origin is taken to be at the center of the sphere. In Cartesian coordinates the same potential function

$$\Phi(x, y, z) = Uz\left(1 + \frac{1}{2}\frac{R^3}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}\right) \tag{10}$$

(a) Show that the normal component of the velocity field vanishes at the surface of the sphere where $r = R$.

(b) Show that the velocity field derived from the potential at large distances from the object approaches $\lim_{r\to\infty} \mathbf{u} = U\hat{\mathbf{z}}$

Hints: ones of these problems is easier in Cartesian coordinates and the other easier in spherical coordinates.

The gradient in spherical coordinates is

$$\boldsymbol{\nabla}\Phi = \frac{\partial\Phi}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial\Phi}{\partial\theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial\Phi}{\partial\phi}\hat{\boldsymbol{\phi}}.$$

4. Divergence in spherical coordinates

Spherical coordinates (r, θ, ϕ) in terms of Cartesian coordinates (x, y, z) are

$$x = r \sin\theta \cos\phi$$
$$y = r \sin\theta \sin\phi$$
$$z = r \cos\theta.$$

Given x^j Cartesian coordinates as a function of y^j a different set of coordinates, the components of the metric tensor in the y coordinate system are

$$\underline{g}_{ij} = \sum_k \frac{\partial x^k}{\partial y^i} \frac{\partial x^k}{\partial y^j} \tag{11}$$

(a) Compute the metric tensor in spherical coordinates.

(b) Find a set of vectors $\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}$ that are pointed in the direction that their coordinates increases and are unit vectors.

(c) Using the relation for the covariant divergence

$$\boldsymbol{\nabla} \cdot \mathbf{A} = \sum_k \frac{1}{\sqrt{|g|}} \frac{\partial(\sqrt{|g|} A^k)}{\partial x^k} \tag{12}$$

show that the divergence of a vector with components A_r, A_θ, A_ϕ is

$$\nabla \cdot \mathbf{A} = \frac{1}{r^2} \frac{\partial(r^2 A_r)}{\partial r} + \frac{1}{r \sin\theta} \frac{\partial(A_\theta \sin\theta)}{\partial\theta} + \frac{1}{r \sin\theta} \frac{\partial A_\phi}{\partial\phi}. \tag{13}$$

Here $|g|$ is the determinant of the metric tensor.

5. Thermal equilibrium with a source term

Consider a spherical body of radius R with an internal heat source such as radioactive decay. We assume spherical symmetry so all equations will only depend on radius r. The temperature as a function of radius is described by the heat equation with a source term

$$\rho c_V \frac{\partial T}{\partial t} - \lambda \nabla^2 T = \rho \dot{Q}_{heat} \text{ for } 0 \le r \le R, \tag{14}$$

where ρ is density, c_V is specific heat, λ is thermal conductivity, $\dot{Q}_{heat}$ is heating rate per unit mass generated via an internal heat source, and $\rho\dot{Q}_{heat} > 0$ is the energy heating rate per unit volume. We neglect any radial dependence of ρ, λ and $\dot{Q}_{heat}$.

We assume a steady state giving

$$-\lambda \nabla^2 T = \rho \dot{Q}_{heat} \tag{15}$$

In spherical coordinates, and only considering radial variations in T,

$$-\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right) = \frac{\rho \dot{Q}_{heat}}{\lambda} \tag{16}$$

Consider $f(r)$ a radial function that satisfies Laplace's equation

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) = 0 \tag{17}$$

(a) Find a general form for a solution $f(r)$ to equation 17 (Laplace's equation with spherical symmetry).

Hint: integrate twice. The result should include two unknown constants.

(b) Find a general form for a solution $T(r)$ to equation 16 (with a heating source).

Note: If you add $f(r)$ (a solution to Laplace's equation) to any particular solution $T(r)$ to equation 16, you will have another solution to equation 16. The two types of solutions are called homogeneous and inhomogeneous.

(c) Adopt boundary conditions of zero flux at the origin

$$\left.\frac{dT(r)}{dr}\right|_{r=0} = 0 \tag{18}$$

Find the general solution for $T(r)$.

As you have used one boundary condition, the general result should now contain only 1 undefined constant.

(d) The heat flux

$$\mathbf{F} = -\lambda \nabla T \tag{19}$$

and at the surface

$$\mathbf{F} = -\lambda \left.\frac{\partial T(r)}{\partial r}\right|_{r=R} \hat{\mathbf{r}}. \tag{20}$$

Compute the heat flux at the surface.

(e) Ignoring external sources of heat, compute the surface temperature assuming that the heat flux at the surface is radiated thermally with a black body spectrum.

The energy flux at the surface due to emitted thermal radiation should be $\sigma_{SB} T^4$ where σ_{SB} is the Stefan Boltzmann constant.

This condition should give an additional constraint on the solution.

(f) See if you can update the general solution, which should no longer contain any unknowns and use it to compute the central temperature (at $r = 0$).