1. Eulerian and Lagrangian view points with Traffic flow

Consider traffic flow described by a density of cars per unit length of one side of a divided highway, ρ, (numbers of cars per kilometer) and velocity of cars, u, (kilometers per hour). Let x describe distance along a road. For $x < 0$ the speed limit is u_a and assume that the cars are driving at this speed limit. At $x = 0$ there is a sign letting the drivers know that there is a change in speed limit. They begin to decelerate to the new speed limit, u_b. At $x = L$ the cars have reached the new speed limit u_b.

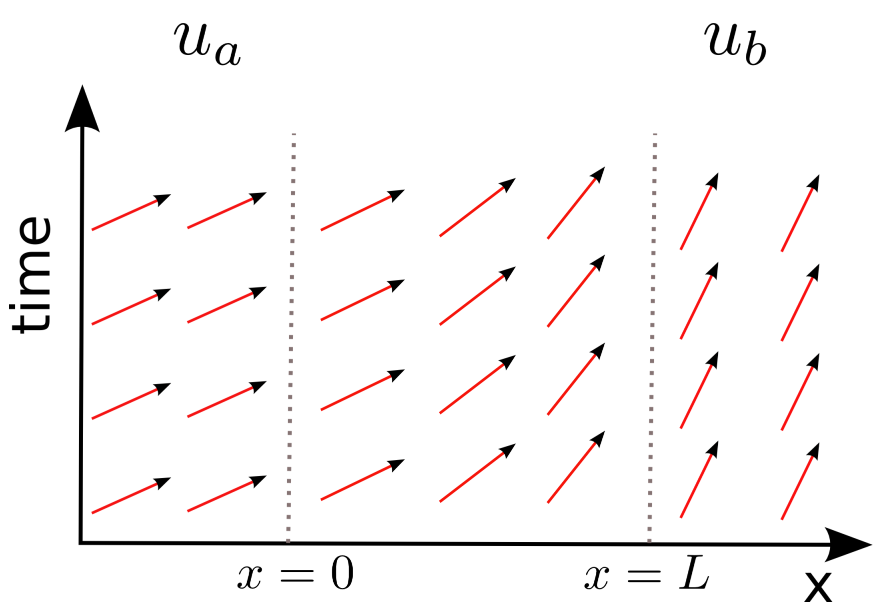

FIGURE 1. Note the x axis is space and the y axis is time. On this figure **an arrow pointing upwards** corresponds to a **low velocity**, and an arrow pointing nearly **horizontally** corresponds to a **high velocity**. Car trajectories $x(t)$ connect the arrows. Even though the velocity between $x = 0$ and L **decreases linearly** between u_a and u_b, the trajectory of a car in the flow, $x(t)$, are not linear. The flow shown here is for cars **decelerating** between $x = 0$ and L.

Assume that the car speed is described by

$$u(x) = \begin{cases} u_a \\ u_a + (u_b - u_a)\frac{x}{L} \\ u_b \end{cases} \qquad \text{for} \qquad \begin{cases} x \le 0 \\ 0 < x \le L \\ x > L \end{cases}$$

Here $u_a \neq u_b$, $u_a, u_b > 0$ and $L > 0$. For the cars to **decelerate**, $u_b < u_a$.

The Eulerian view point describes the flow in terms of $\rho(x)$, $u(x)$. This flow is steady-state so ρ and u are only functions of x (and not also of t). The Lagrangian viewpoint concerns the velocity and density from the view point of the driver, or moving with the flow.

(a) What is $\rho(x)$ if the density of cars is ρ_a for $x < 0$?

Hints: consider the mass flux ρu of cars.

(b) What is the position of a car as a function of time that is at $x = 0$ at time $t = 0$? Hint: $dx/dt = u(x)$, solve for $x(t)$. This indefinite integral may be useful; $\int \frac{f'(x)}{f(x)} dx = \ln |f(x)| + $ constant.

(c) What acceleration does a car in the traffic flow experience as a function of position x in the flow? Compute this using a Lagrangian derivative.

(d) Check that your answer in part b is consistent with that in part c.

(e) Find a conservation law that is obeyed by the system that is in the form of

$$\frac{\partial}{\partial t}(\text{density of something}) + \frac{\partial}{\partial x}(\text{a flux}) = 0. \tag{1}$$

2. Non-linearity of the Lagrangian derivative

Consider again the setting of traffic flow with the linear density of cars $\rho(x,t)$ and the velocity of cars $u(x,t)$ as seen from an outside observer. Assume that the velocity of the cars is a traveling wave with wave vector k and angular frequency ω

$$u(x,t) = u_0 \left(1 + A \sin(kx - \omega t)\right)$$

and with unit less amplitude $A < 1$ so that the cars are always traveling in one direction down the road. Unlike the previous problem, this is not a steady-state flow.

(a) Compute the instantaneous acceleration Du/Dt of a car at position x and time t.

(b) Qualitatively discuss why the acceleration contains a second order term proportional to A^2 that is not exactly $90°$ out of phase with the velocity.

3. Practicing Index Gymnastics

(a) Use summation notation and the Levi-Civita tensor permutation tensor to show that the following identity is true.

$$\epsilon_{ijk}\epsilon_{ijk} = 3! \tag{2}$$

(b) Use summation notation and the permutation tensor to show the vector identity is true

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$$

Hints: A cross product can be written in summation notation like this:

$$(\mathbf{a} \times \mathbf{b})_i = \epsilon_{ijk} a_j b_k. \tag{3}$$

A product of Levi-Civita tensors obeys this relation

$$\epsilon_{ijk}\epsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km} \tag{4}$$

4. Streamlines

We consider a flowing fluid. Assume that the density at time zero is constant, ρ_0, for the following flows.

(a) Draw streamlines for a velocity field with non-zero trace, $\mathbf{u} = (ax, by, 0)$ where a, b are constants and in Cartesian coordinates. Consider the case with $a, b > 0$. What happens if $a > 0, b < 0$?

Using an equation describing conservation of mass, how does the density change in time?

(b) Draw streamlines for the velocity field $\mathbf{u} = (-y, x, 0)$. How does the density change in time?

(c) Draw streamlines for the velocity field $\mathbf{u} = (ay, 0, 0)$ for a flow with a fixed surface at $y = 0$. The velocity at $y = 0$ is zero and is everywhere in the x direction. You might have such a velocity field near a river bank where the bank is at $y = 0$ and the river is flowing in the x direction. How does the density change in time?

(d) Compute the vorticity $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \mathbf{u}$ of the velocity fields in a–c. When the vorticity is non-zero we say the flow is rotational. Which of the above velocity fields are rotational?

5. Conservation Law form for conservation of momentum

Our relation for conservation of mass

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{u}) = 0$$

can be said to be in conservation law form. The term $\rho\mathbf{u}$ is the mass flux. However Euler's equation describing conservation of momentum

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla} \mathbf{u} = -\frac{1}{\rho} \boldsymbol{\nabla} p$$

is not in conservation law form. Conservation of momentum can be put in conservation law form using the stress tensor $\boldsymbol{\pi}$,

$$\frac{\partial (\rho \mathbf{u})}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{\pi} = 0$$

Use Euler's equation and that for conservation of mass to show that conservation of momentum can be put in conservation law form.

In Cartesian coordinates and using index notation the stress tensor is $\boldsymbol{\pi}_{ij} = p\delta_{ij} + \rho u_i u_j$.

6. Ram pressure and Drag

Consider a thin slightly stretchy wire with diameter d and linear mass density λ held at each end by a pole (like a power cable strung between two telephone poles). Because of gravity, the wire sags. Assume that a strong wind is blowing at a speed u with respect to the cable. The density of air we denote ρ_{air} and the gravitational acceleration, g.

(a) Using ram pressure, estimate the force on a small segment of the wire due to the wind.

(b) Compare the force from ram pressure to that due to gravity. Estimate how fast the wind needs to blow to overcome sagging due to gravity.

This problem need only be done to order of magnitude. Don't worry about the vector direction of the wind.

7. An incompressible flow in spherical coordinates - The Maxwell Z-model

The Maxwell Z-model is a popular model for an ejecta curtain caused by an impact. We work in spherical coordinates r, θ, ϕ. The velocity field has velocity components

$$
\begin{aligned}
u_r &= \frac{a(t)}{r^Z} \\
u_\theta &= \frac{a(t)}{r^Z}(Z-2)\frac{\sin\theta}{1+\cos\theta} \\
u_\phi &= 0
\end{aligned}
\tag{5}
$$

where Z is a positive real number. With $Z = 2$ the flow is radial. The function $a(t)$ is a rather unphysical but time dependent function that is intended to describe how the flow field varies in time. The impact point is at $r = 0$.

Here the azimuthal angle θ is also called the polar angle. The polar angle is conventionally defined with unit-vector $\hat{\mathbf{z}} = \cos\theta\hat{\mathbf{r}} - \sin\theta\hat{\boldsymbol{\theta}}$ pointing upward, where $\hat{\mathbf{r}}$ is a unit vector pointing in the direction of increasing r and $\hat{\boldsymbol{\theta}}$ is a unit vector pointing in the direction of increasing θ. To make the flow field consistent with upward flow and ejecta that is launched above a flat surface at $z = 0$, we take z to be depth, so $\theta = 0$ corresponds to a downward direction and $\theta = \pi/2$ is along the surface.

For an illustration of the streamlines of the flow described by the velocity field see Figure 2.

In spherical coordinates the divergence of vector with components A_r, A_θ, A_ϕ is

$$
\nabla \cdot \mathbf{A} = \frac{1}{r^2}\frac{\partial(r^2 A_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(A_\theta \sin\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}. \tag{6}
$$

Streamlines are described with the radius R_s where the streamline intersects the surface. Streamlines that cross the surface at radius R_s (from impact point) have radius as a function of azimuthal angle

$$
r(\theta, R_s) = R_s(1-\cos\theta)^{\frac{1}{Z-2}}. \tag{7}
$$

(a) Show that the flow field is incompressible.

Hint: compute $\nabla \cdot \mathbf{u}$ in spherical coordinates.

(b) Show that equation 7 for the streamlines is consistent with the velocity field. In other words show that a tangent to a streamline at a particular point is parallel to the velocity velocity at that point.

Hint: Invert equation 7 to give $R_s(r,\theta)$, the radius that a streamline at r, θ crosses the surface. Each streamline is described by a level curve of the function $R_s(r,\theta)$. Then show that ∇R_s is perpendicular to the velocity vector $\mathbf{u}$.

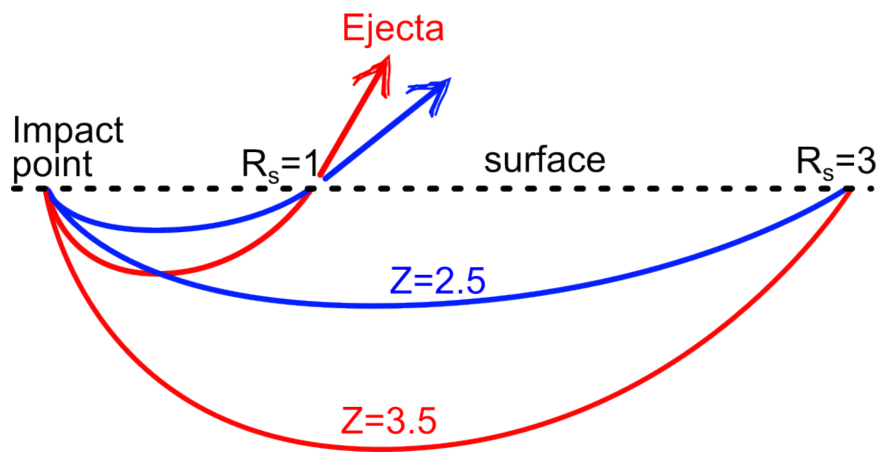

FIGURE 2. Streamlines of the Maxwell Z model. These curves are given by equation 7 for two different Z values (2.5 and 3.5) and two different R_s values (1 and 3). A nice property of the Maxwell Z model is that the angle of ejecta launch from the surface is directly related to the subsurface flow.

In spherical coordinates the gradient of a function $f(r, \theta, \phi)$ is

$$\boldsymbol{\nabla} f = \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\hat{\boldsymbol{\phi}}. \tag{8}$$