AST243 LECTURE NOTES PART 1

Contents

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1. Introduction to Ideal Fluid Dynamics

1.1. **Variables.** We take $\mathbf{x}$ to be a Cartesian coordinate in three dimensions t to be time. A **fluid** is characterized by a mean *velocity field* $\mathbf{u}(\mathbf{x}, t)$ and at least two integrated or thermodynamic quantities. Often these are the mass density, $\rho(\mathbf{x}, t)$ and pressure $p(\mathbf{x}, \mathbf{t})$. The energy density e, or temperature, T, can treated as an additional variable or can be related to ρ and p.

There can be additional degrees of freedom, such as the molecular or nuclear composition and the ionization state. There could also be radiation or a magnetic field.

1.2. **Eulerian and Lagrangian views.** We view the system from a fixed coordinate system and describe each variable as a function of $(\mathbf{x}, t)$. The partial time derivative

$$\frac{\partial}{\partial t}$$

describes how variables change in time from the point of view of a fixed point in space attached to a coordinate system or an inertial frame. Equivalently from the point of view of an external observer who is in a static inertial frame. This is the Eulerian viewpoint.

We could also describe the system from the view point of particles that are moving with the fluid. Suppose we have a scalar quantity like T (like temperature). We would like to predict what would cause a small change δT as our fluid element moves. Over a small change in time δt and with small changes in coordinates $\delta x, \delta y, \delta z$.

$$\delta T = \frac{\partial T}{\partial t}\delta t + \frac{\partial T}{\partial x}\delta x + \frac{\partial T}{\partial y}\delta y + \frac{\partial T}{\partial z}\delta z$$

We now divide by δt.

$$\frac{\delta T}{\delta t} = \frac{\partial T}{\partial t} + \frac{\partial T}{\partial x}\frac{\delta x}{\delta t} + \frac{\partial T}{\partial y}\frac{\delta y}{\delta t} + \frac{\partial T}{\partial z}\frac{\delta z}{\delta t} \tag{1}$$

If we chose $\delta x, \delta y, \delta t$ to be an element of the fluid that is moving along with the fluid then $\frac{\delta \mathbf{x}}{\delta t} = \mathbf{u}$ and we can write the above as

$$\frac{\delta T}{\delta t} = \frac{\partial T}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla} T$$

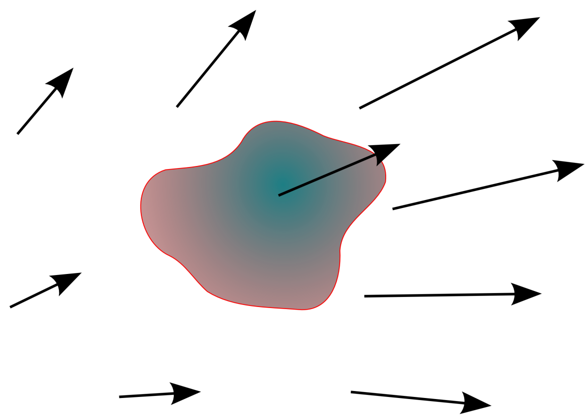

FIGURE 1. A fluid element moving within a larger flow.

If we consider derivatives from the point of view of particles moving with the fluid then we can describe changes with the Lagrangian time-derivative or

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla}. \tag{2}$$

Let us write this out in terms of components

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \sum_i u_i \frac{\partial}{\partial x_i}$$

as we had done in equation (124).

Another way to think about this is to consider a fluid element at $\mathbf{x}$ that has moved by $\mathbf{u}\delta t$ in a time δt. If we consider T for that fluid element we can write T as

$$T(\mathbf{x} + \mathbf{u}\delta t, t + \delta t)$$

so the change in T moving with the fluid element

$$\begin{aligned}
\frac{DT}{Dt} &= \lim_{\delta t \to 0} \left(\frac{T(\mathbf{x} + \mathbf{u}\delta t, t + \delta t) - T(\mathbf{x}, t)}{\delta t} \right) \\
&= \lim_{\delta t \to 0} \frac{1}{\delta t} \left[T + \frac{\partial T}{\partial x} u_x \delta t + \frac{\partial T}{\partial y} u_y \delta t + \frac{\partial T}{\partial z} u_z \delta t + \frac{\partial T}{\partial z} u_z \delta t - T \right] \\
&= \left[\frac{\partial}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla} \right] T. \qquad (3)
\end{aligned}$$

If we write equations from the view point of fluid elements that are moving we say we are using the Lagrangian view point.

Consider traffic flow. We can describe traffic flow in terms of density, ρ, (cars per unit length) and a velocity, u, the speed of cars on the road. If we describe ρ and u as a function of position on the road we are using the Eulerian view point. If we describe ρ and u in terms of those seen by individual drivers we say we are using the Lagrangian viewpoint.

Numerical methods that use fixed grids work in the Eulerian view point. Numerical methods that allow particles to move in the simulation and compute forces on these particles work in the Lagrangian viewpoint. Smooth Particle Hydrodynamics (SPH) codes use the Lagrangian viewpoint.

1.3. **An aside on notation.** In three dimensions the Cartesian coordinates are often denoted x, y, z. Sometimes we use an index instead to denote each coordinate; x_1, x_2, x_3 or x_i, x_j, x_k with $i, j, k \in \{1, 2, 3\}$.

Partial derivatives can be written in a variety of ways including

$$\frac{\partial u}{\partial x} \qquad u_{,x} \qquad u_x \qquad \partial_x u$$

1.4. **Streamlines and flow visualization.**

- **Streamlines** are integrated curves that have local tangent or **slope** equal to the instantaneous flow velocity, $\frac{d\mathbf{x}}{dt} = \mathbf{u}$. However these are only equivalent to particle paths if the flow is steady or stationary.
- **Paths** are trajectories of particles that move with the fluid.
- Flow can also be visualized by adding die or smoke continuously at particular fixed locations in the flow. These are called **streaks**.

Streamlines are the not the same as paths if the flow is unsteady.

Another example is that by Clarke and Carswell. Consider a non-steady state 2-dimensional flow with

$$\mathbf{u} = (1, 0) \qquad \text{for} \qquad t < 0 \tag{4}$$

$$= (0, 1) \qquad \text{for} \qquad t > 0 \tag{5}$$

so that velocities are moving to the right at $t < 0$ and moving upward for $t > 0$. At $t < 0$ streamlines are moving to the right. At $t > 0$ streamlines are moving upwards (see Figure 3).

However particle trajectories start moving to the right and then bend upwards. Particle trajectories are not streamlines. The location of the bend depends on where the particle was at $t = 0$. After $t = 0$ streaks have a bend and move upwards; see Figure 4.

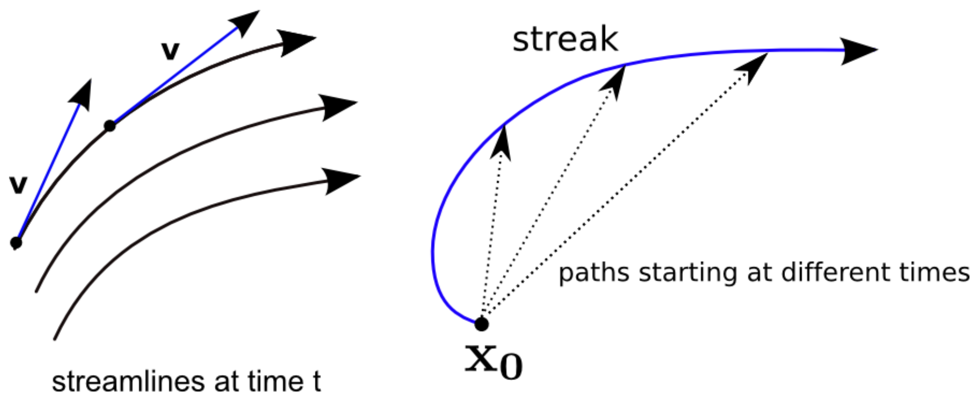

FIGURE 2. Streamlines, paths and streaks. A streamline is a curve with tangent aligned with the velocity vector. A path is the path that a particle moving with the fluid would take. A streak is what would be seen if die were continuously emitted from a particular fixed location.

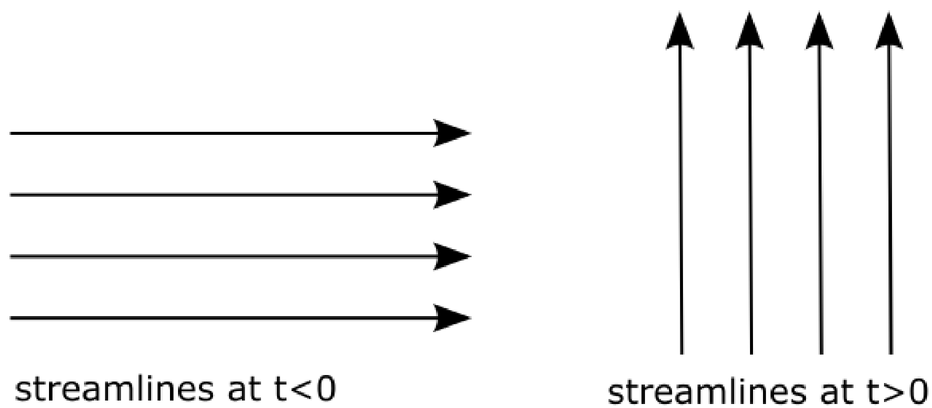

FIGURE 3. Streamlines for the two dimensional flow with $\mathbf{u} = (1, 0)$ for $t < 0$ and $\mathbf{u} = (0, 1)$ for $t > 0$.

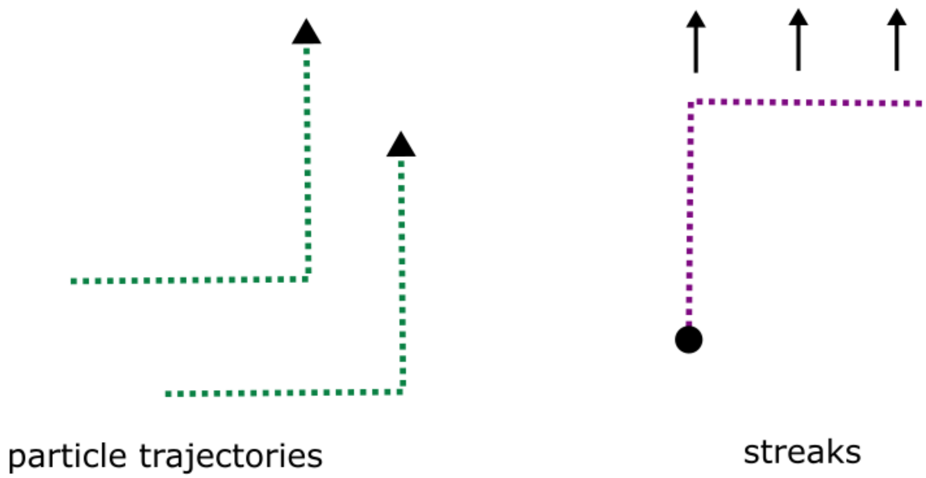

FIGURE 4. Paths and a streak for the flow described in equation 5. The paths depend on where the particle was at $t = 0$. The paths shown are at some time after $t = 0$. The bent streak (on right) keeps drifting upwards after the change in velocity takes place. It is also shown after the velocity change.

1.5. **Conservation of mass.** Consider a flow with density $\rho(\mathbf{x}, t)$ and velocity $\mathbf{u}(\mathbf{x}, t)$. Let us list units

$$\rho \qquad \frac{\text{mass}}{\text{distance}^3}$$
$$\mathbf{u} \qquad \frac{\text{distance}}{\text{time}}$$

The mass flux

$$\rho\mathbf{u} \qquad d\frac{\text{mass}}{\text{distance}^3} \times \frac{\text{distance}}{\text{time}} = \frac{\text{mass}}{\text{distance}^2 \text{ time}}$$

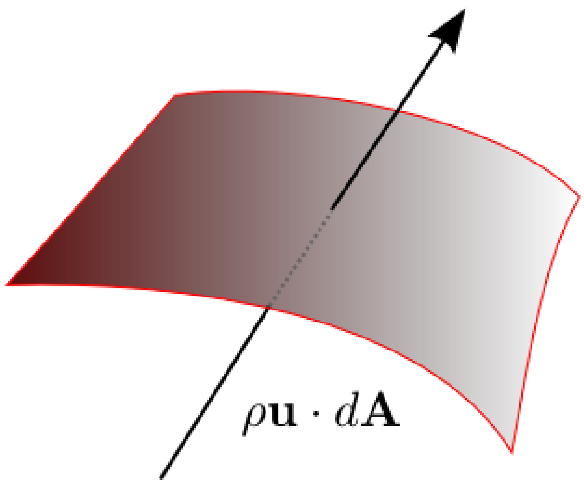

FIGURE 5. For a flow with density ρ and velocity $\mathbf{u}$ the mass flow through a surface oriented in direction $d\mathbf{A}$ and with area $|d\mathbf{A}|$ is $\rho\mathbf{u}\cdot d\mathbf{A}$.

We consider the change in density with an Eulerian view point. If we consider the mass flux $\rho\mathbf{u}$ moving through a closed surface S

$$\int_S \rho\mathbf{u} \cdot \mathbf{dA} = \int_V \boldsymbol{\nabla} \cdot (\rho\mathbf{u})\, dV$$

where I have used Gaus's law to write this as an integral over a volume V enclosed by the surface S. The mass flux through the surface S must be balanced by change of density in V or

$$\int_V \frac{\partial \rho}{\partial t}\, dV = -\int_V \boldsymbol{\nabla} \cdot (\rho\mathbf{u})\, dV$$

Equating the two expressions (locally)

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho\mathbf{u}) = 0 \tag{6}$$

This is the general form for a **conservation law** for a scalar quantity, which in this case is the mass density ρ.

Why is equation 6 equivalent to conservation of mass? The integral of all the mass in a particular domain V at a particular time

$$\text{total mass} = \int_V \rho(x,t) dV. \tag{7}$$

The time derivative of this

$$\begin{aligned} \frac{d}{dt}\text{total mass} &= \int_V \frac{\partial \rho}{\partial t} dV \\ &= \int_V \nabla \cdot (\rho \mathbf{u}) dV \\ &= \int_S (\rho \mathbf{u}) \cdot d\mathbf{A} \end{aligned}$$

If the surface S of the domain is impermeable then the flux through this surface must be zero and the right hand side vanishes. In an infinite domain, often one can assume that the density drops to zero and that too gives a zero right hand side. If the right hand side is zero then there is no change in the total integrated mass and mass is conserved.

We can write a conservation law in the form

$$\frac{\partial}{\partial t}(\text{quantity}) + \boldsymbol{\nabla} \cdot (\text{flux of quantity}) = 0. \tag{8}$$

Our example for conservation of mass involves the time derivative of a scalar quantity. However when we discuss momentum we can consider more general vector or tensor forms for conservation laws.

Some numerical algorithms require putting differential equations in conservation law form. The above equation does not involve a Lagrangian derivative and so is in the Eulerian view point.

Consider the form of the conservation of mass using the Lagrangian viewpoint. The mass flux of a fluid element moving with the fluid is reduced if there is mass leaving the fluid element boundary. The rate that mass leaves the fluid element depends on the velocity field and the density in the fluid element. Using Gaus's law again and integrating over a closed volume V bounded by surface S

$$\rho \int_S \mathbf{u} \cdot d\mathbf{A} = \rho \int_V \boldsymbol{\nabla} \cdot \mathbf{u}$$

Locally the mass leaving the fluid element would be $\rho \nabla \cdot \mathbf{u}$. The density variation in the fluid element is computed with the Lagrangian derivative, so our conservation law looks like

$$\frac{D\rho}{Dt} = -\rho \boldsymbol{\nabla} \cdot \mathbf{u}$$

The above equation is given in the Lagrangian viewpoint.

We can expand the above equation

$$\frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla} \rho + \rho \boldsymbol{\nabla} \cdot \mathbf{u} = 0$$

and we see that this is equivalent to our previous form in the Eulerian viewpoint for the conservation of mass.

An **incompressible** fluid is one that does not change density even if moving so that $\frac{D\rho}{Dt} = 0$ and so that $\nabla \cdot \mathbf{u} = 0$.

Now is a good time to look at the movie on Eulerian vs Lagrangian view points at NCFMF.

1.6. **Conservation of momentum.** What is the change of momentum for a moving fluid element? It is

$$\rho \frac{D\mathbf{u}}{Dt} = \rho \left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} \right)$$

Why, we may ask is the change of momentum for a moving fluid element not $D(\rho\mathbf{u})/Dt$ instead of $\rho D\mathbf{u}/Dt$? When the fluid element is accelerated (changes in velocity) it experiences a change in momentum. However when it expands and changes density it does not experience a change in momentum. Our expression $\rho D\mathbf{u}/Dt$ gives us the (*rate of change of momentum*) per unit volume whereas the expression $D(\rho\mathbf{u})/Dt$ would describe the rate of change of (*the momentum per unit volume*). Another way to think about this is to consider the fluid element as a fixed amount of mass, m. The rate of change of its momentum would be $m\frac{Du}{Dt}$ where $m = \rho V$ and V is the volume of our small fluid element. We want to equate the force on this mass to its mass times acceleration.

Consider the total force due to *pressure* acting on a fluid element. We integrate the pressure around the surface of the fluid element and transform to a volume integral

$$-\mathbf{F} = \oint_S p d\mathbf{A} = \int_V \boldsymbol{\nabla} p dV.$$

To see why this is true $\int_V \frac{dp}{dx}\, dx\, dy\, dz = \int_S p dy dz$.

The pressure exerts a force of $-\boldsymbol{\nabla} p dV$ on a fluid element of volume dV and the minus sign arises as the force is in the direction from high to low pressure, as shown in Figure 6. Taking into account only the pressure force on the fluid element we can write

$$\rho \frac{D\mathbf{u}}{Dt} = -\boldsymbol{\nabla} p$$

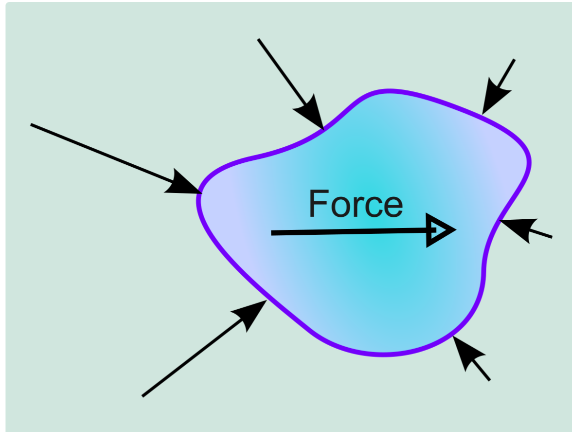

FIGURE 6. A pressure gradient exerts a force on a fluid element.

in the Lagrangian viewpoint. Expanding the derivative we find

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} = -\frac{1}{\rho}\boldsymbol{\nabla} p \tag{9}$$

which is known as Euler's equation and since it lacks a Lagrangian derivative it is in the Eulerian viewpoint.

On a point mass of mass m, gravity exerts a force $-m\boldsymbol{\nabla}\Phi$ where Φ is the gravitational potential. The force per unit volume exerted by gravity on a fluid element would be $-\rho\boldsymbol{\nabla}\Phi$. Our force balance equation including gravity

$$\rho\frac{D\mathbf{u}}{Dt} = -\boldsymbol{\nabla} p - \rho\boldsymbol{\nabla}\Phi. \tag{10}$$

Euler's equation including gravity (useful for many astrophysical settings)

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} = -\frac{1}{\rho}\boldsymbol{\nabla} p - \boldsymbol{\nabla}\Phi. \tag{11}$$

Let's be clear on what we mean by the above equation by writing out one of the components, the j-th component

$$\frac{\partial u_j}{\partial t} + \sum_i u_i \frac{\partial u_j}{\partial x_i} = -\frac{1}{\rho}\frac{\partial p}{\partial x_j} - \frac{\partial \Phi}{\partial x_j}$$

or more specifically the z component

$$\frac{\partial u_z}{\partial t} + u_x\frac{\partial u_z}{\partial x} + u_y\frac{\partial u_z}{\partial y} + u_z\frac{\partial u_z}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - \frac{\partial \Phi}{\partial z}$$

1.7. **Hydrostatic equilibrium.** If the fluid is not moving and is in equilibrium then $\frac{\partial \mathbf{u}}{\partial t} = 0$ and $\mathbf{u} = 0$. Euler's equation including gravity (Equation 11) becomes

$$\boldsymbol{\nabla} p = -\rho \boldsymbol{\nabla} \Phi$$

Using an equation of state relating pressure to density this can be solved to find a density or pressure profile as a function of radius in a star or as a function of height in a planetary atmosphere.

1.8. **When is hydrostatic equilibrium a good approximation?** A planetary atmosphere may be turbulent and so not perfectly static. We can ask when is it justified to drop the first two terms in Euler's equation? Let's do some dimensional analysis on Euler's equation. Recall Euler's equation including gravity,

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla} \mathbf{u} = -\frac{1}{\rho} \boldsymbol{\nabla} p - \boldsymbol{\nabla} \Phi$$

The term

$$\frac{\partial \mathbf{u}}{\partial t} \sim \frac{\delta u}{\delta t} \sim \frac{(\delta u)^2}{l}$$

where δu is the velocity of a typical turbulent motion of eddy size l that would last a time $\delta t \sim l/\delta u$. The term

$$\mathbf{u} \cdot \nabla \mathbf{u} \sim \frac{(\delta u)^2}{l}$$

is of the same order of magnitude. The pressure term

$$\frac{1}{\rho} \nabla p \sim \frac{c_s^2}{h}$$

where h is the atmosphere scale height and c_s the sound speed. The gravity term

$$|\nabla \Phi| = g$$

the acceleration due to gravity. We expect we can neglect the velocity terms in Euler's equation when they are small compared to the pressure and gravity terms. We find that hydrostatic equilibrium is a good approximation as long as

$$\frac{(\delta u)^2}{l} \ll g.$$

This is equivalent to saying accelerations in the eddies are smaller than that due to gravity. The other condition (accelerations in eddies smaller than that due to pressure) would be

$$\frac{(\delta u)^2}{l} \ll \frac{c_s^2}{h}$$

which for large eddies is equivalent to saying that eddies are subsonic.

1.9. **Failure of continuum fluid approximation.** Above we considered when velocity perturbations make hydrostatic equilibrium a bad approximation. Hydrostatic equilibrium is based on a continuum fluid approximation. The continuum fluid approximation is no longer valid on a lengthscale shorter than the mean free path of a particle or on a timescale shorter than the mean collision timescale. Consider the gas number density (particles per unit volume),

$$n = \frac{\rho}{\mu m_p}$$

where μ is the mean molecular weight and m_p the mass of a proton (we should be using an atomic mass unit but that is approximately the mass of a proton). If the gas is molecular then we should correct the above to take into account the mass of the average molecule. Given a collision cross section σ and a gas number density n, the mean collision timescale

$$\tau_{col} = (nv\sigma)^{-1}$$

where v is a mean velocity usually set by the temperature, and of order $v \sim c_s$. The mean free path is

$$\lambda \sim v\tau_{col} \sim (n\sigma)^{-1} \tag{12}$$

Since our fluid approximation fails on length scales shorter than the mean free path, our hydrostatic equilibrium approximation would fail when the atmospheric scale height $h < \lambda$.

To evaluate collision timescales or the mean free path an estimate for the cross section is required. For molecules typically

$$\sigma_{molecules} \sim 10^{-15}\text{cm}^2$$

(Note the Bohr radius is 5×10^{-9} cm.)

In astrophysics we often have an ionized gas, but Coulomb interactions have an infinite range. However we can define an effective radius, r_{eff}, for an encounter with

$$\frac{e^2}{r_{eff}} \sim k_B T$$

were e is the electron charge. Our cross section $\sigma \sim r_{eff}^2$. The previous equation and equation 12 gives us a mean free path

$$\lambda \sim n_e^{-1} e^{-4} (k_B T)^2$$

where n_e is the electron density. We can also generate a timescale for collisions τ_{col} using a thermal velocity

$$v_{thermal} \sim \sqrt{k_B T/m_e}$$

giving

$$\tau_{col} \sim \lambda / v_{thermal} \sim \frac{m_e^{1/2} (k_B T)^{3/2}}{n_e e^4} \sim 0.9\ T^{3/2} n_e^{-1} \text{ second.}$$

Note the interesting temperature dependence. We will see similar temperature dependence in astrophysical settings where collisions are important such as radiative cooling, ionization and recombination, and conduction.

1.10. **Isopotential and isodensity contours in hydrostatic equilibrium.** Consider our equation for hydrostatic equilibrium

$$\boldsymbol{\nabla} p = -\rho \boldsymbol{\nabla} \Phi. \tag{13}$$

Take the curl of both sides of the equation

$$0 = \boldsymbol{\nabla} \times (\rho \boldsymbol{\nabla} \Phi) = \boldsymbol{\nabla} \rho \times \boldsymbol{\nabla} \Phi + \rho \boldsymbol{\nabla} \times \boldsymbol{\nabla} \Phi.$$

As the curve of a gradient is zero, this is equivalent to

$$\boldsymbol{\nabla} \rho \times \boldsymbol{\nabla} \Phi = 0 \tag{14}$$

Taking the length of both sides of the equation

$$|\boldsymbol{\nabla} \rho| |\boldsymbol{\nabla} \Phi| \sin \theta = 0$$

where $\sin \theta$ is the angle between the two gradients. The angle θ must be 0 or π. This implies that isopotential contours are the same as isodensity contours (even underwater). Furthermore the equation $\nabla p = \rho \nabla \Phi$ implies that isopressure contours (isobars) are the same as isopotential contours. We could describe any of p, ρ, or Φ as a function of one of them, for example $dP/d\Phi = -\rho(\Phi)$.

1.11. **Isothermal atmosphere with constant gravitational acceleration.** For a small change in radius the gravitational potential is constant and the pressure at height z is the weight of the fluid above it

$$P(z) = g \int_z^\infty \rho(z) dz$$

Assuming we have an idea gas, pressure is related to density and temperature with the ideal gas law

$$p = \frac{\rho k_B T}{\mu m_p}$$

Here k_B is Boltzmann's constant, m_p the mass of the proton, p pressure, T temperature, and μ the mean atomic weight. (Here m_p loosely used as an atomic mass unit). If the atmosphere is isothermal then pressure is proportional to density;

$$p = A\rho$$

with constant A. Insert this into our equation for hydrostatic equilibrium (Euler's equation neglecting velocity)

$$\begin{aligned}
\frac{1}{\rho}\boldsymbol{\nabla} p &= -\boldsymbol{\nabla}\Phi \\
\frac{A}{\rho}\frac{\partial \rho}{\partial z} &= -g \\
\frac{\partial}{\partial z}\ln\rho &= -\frac{g}{A} \\
\ln\rho &= -\frac{gz}{A} + \text{constant}
\end{aligned}$$

and solution

$$\rho(z) = \rho_0 e^{-\frac{g(z-z_0)}{A}}$$

We call $h = A/g$ the exponential scale height of the atmosphere. Checking units: A is in units of velocity^2 and g in units of velocity^2/distance so the ratio gives a length. The constant A is approximately the sound speed in the gas. The scale heights of the atmospheres for the planets and planetesimals in our solar system can be quickly estimated from their temperatures and surface gravities.

1.12. **One dimensional examples.** Recall Euler's equation without gravity

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} = -\frac{1}{\rho}\boldsymbol{\nabla} p$$

Let's write one component out completely, the z component.

$$\frac{\partial u_z}{\partial t} + u_x\frac{\partial u_z}{\partial x} + u_y\frac{\partial u_z}{\partial y} + u_z\frac{\partial u_z}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z}$$

Consider a one dimensional problem where quantities only vary in the z direction. The above equation reduces to

$$\frac{\partial u_z}{\partial t} + u_z\frac{\partial u_z}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} \tag{15}$$

The left hand side is the change of the velocity of a fluid element. The fluid element slows down or speeds depending upon the pressure gradient it encounters. If the fluid element encounters an increase in pressure then it slows down and if it encounters a decrease in pressure then it speeds up.

Consider what the equation looks like if the pressure gradient is zero. In this case

$$\frac{\partial u_z}{\partial t} + u_z\frac{\partial u_z}{\partial z} = 0$$

which we can write short hand

$$u_{,t} + uu_{,z} = 0$$

The above equation is a non-linear partial differential equation known as the inviscid Burger's equation. Even in the most simplest case (zero pressure gradient and in one-dimension) we find a non-trivial non-linear equation.

Let's go back to equation 15 but now assume that the system is steady state or time independent so $\frac{\partial u}{\partial t} = 0$. In this case

$$uu_{,z} = -\frac{1}{\rho} p_{,z}$$

which we can write

$$\frac{\partial}{\partial z}\left(\frac{u^2}{2}\right) = -\frac{1}{\rho}\frac{\partial p}{\partial z}$$

This illustrates that changes in pressure lead to corresponding changes in velocity. We will discuss this again in the context of Bernoulli's equation.

Let's consider our equation for conservation of mass.

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{u} = 0$$

Now consider the above equation in one dimension

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial z}(\rho u_z) = \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial z} u_z + \frac{\partial u_z}{\partial z}\rho = 0 \tag{16}$$

If we consider a fluid element with density ρ, this implies that only when there is a velocity gradient does the density of a fluid element increase or decrease. If we take the above equation and consider the solution in steady state and divide by ρu the equation takes a useful form

$$\frac{\rho_{,z}}{\rho} + \frac{u_{,z}}{u} = 0 \tag{17}$$

or

$$\frac{d}{dz}\ln\rho + \frac{d}{dz}\ln u = 0 \tag{18}$$

We can also write

$$\frac{\partial}{\partial z}(\rho u) = 0 \tag{19}$$

so that a mass flux

$$\dot{M} = \rho u \tag{20}$$

is independent of z. The above two examples show common manipulations of our fluid equations in one dimension.

Often in astrophysics we will be working in spherical or cylindrical coordinates and assuming that quantities only depend on radius. In this case our equation for conservation of mass contains factors of radius. In spherical coordinates conservation of mass equation

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \rho u_r\right) = 0 \tag{21}$$

and I have dropped all terms that depend on ϕ and θ assuming spherical symmetry. When the system is steady state there is a conserved mass inflow or outflow rate that is independent of radius

$$\dot{M} = 4\pi r^2 \rho u_r \tag{22}$$

Here $\dot{M}$ is the mass loss rate for a spherically symmetric wind (when u_r is positive) or accretion flow (when u_r is negative). For example, this relation can be used to estimate density of the Solar wind as a function of radius from the Sun.

In cylindrical coordinates

$$\frac{\partial \rho}{\partial t} + \frac{1}{R}\frac{\partial}{\partial R}\left(R \rho u_R\right) = 0 \tag{23}$$

and in steady state a conserved mass inflow or outflow rate that is independent of radius

$$\dot{M} = 2\pi R u_R \rho \tag{24}$$

Here $\dot{M}$ could be the accretion rate, through an accretion disk, for a young stellar object or active galaxy galaxy.

1.13. **Source terms.** Consider our law for conservation of mass

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{u}) = 0$$

Suppose there is some way external mass is added to the system. For example we can consider a two-dimensional system, such as a sand pile and describe it with a surface density. Mass can be added to the system by pouring sand at a rate $\frac{\partial \rho(\mathbf{x},t)_e}{\partial t}$. We would modify our conservation law

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{u}) = \frac{\partial \rho_e}{\partial t}$$

with the term on the right called the source term.

Another example is the concentration of a radioactive element that is decaying. In this case we would write

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{u}) = -\frac{\rho}{t_{decay}}$$

where t_{decay} is an exponential timescale for decay. In the absence of any motion

$$\frac{\partial \rho}{\partial t} = -\frac{\rho}{t_{decay}}$$

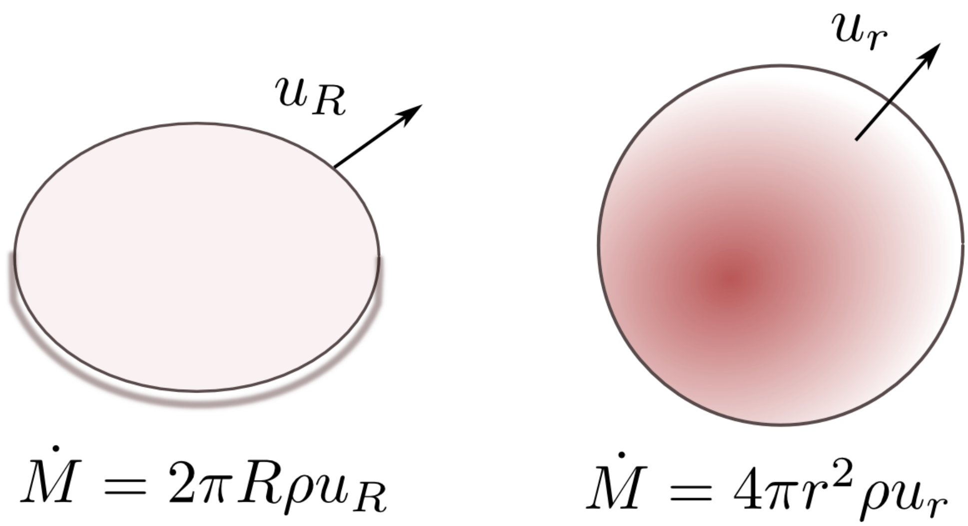

FIGURE 7. On the left we show the mass flow rate through an azimuthally symmetric disk radius of R. This is an accretion rate through a disk. The radial flow velocity is u_R. If $\dot{M}$ is not a function of radius or time then the flow is steady (time independent). On the right we show a wind mass loss rate (if the radial flow velocity $u_r > 0$) through a radius r in a spherically symmetric flow. It could also be an accretion rate (with $u_r < 0$). Again if $\dot{M}$ is not a function of time or radius then the flow is steady.

with solution

$$\rho(t) = \rho(0) e^{-\frac{t}{t_{decay}}}$$

as expected for an unstable radioactive nuclide.

In many astrophysical situations source terms are added to conservation laws.

1.14. **The stress tensor.** Consider the mass per second hitting a surface $\hat{\mathbf{s}}$ that is oriented in the j direction. The bulk flow through the surface is ρu_j. The momentum flux through the surface per second in the i direction is then $\rho u_i u_j$. This is the same thing as the i-th component of the force on the surface.

The stress tensor $\boldsymbol{\pi}$ is a tensor with two indexes and each index can be one of three coordinates. The component π_{ij} gives the force in direction i on unit surface with normal in direction j. The force in direction given by $\hat{\mathbf{n}}$

$$\mathbf{F} = \boldsymbol{\pi} \cdot \mathbf{n} \qquad \text{or} \qquad F_i = \sum_j \pi_{ij} \hat{n}_j$$

where $\hat{n}_j$ are the components of the vector $\hat{\mathbf{n}}$.

The contribution to the stress tensor from bulk motion is $\rho u_i u_j$ and is sometimes called **ram pressure**. This is sometimes written

$$\rho \mathbf{u} \otimes \mathbf{u} \qquad \text{with components} \qquad \rho u_i u_j$$

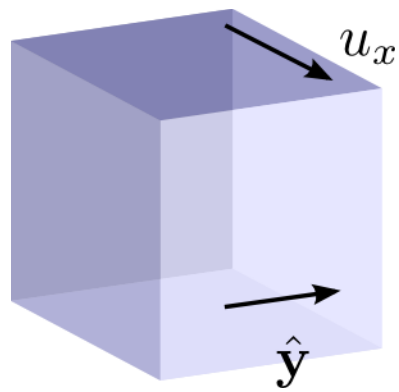

FIGURE 8. The flux of the x component of the momentum density through the y-oriented surface is equal to $\rho u_x u_y$. The x-component of the force per unit area on the y oriented surface is also $\rho u_x u_y$.

The force on a surface due to pressure is in the direction normal to the surface. Consequently the contribution of the pressure to the stress tensor is $p\delta_{ij}$ where δ_{ij} is the Kronecker delta and equal to 1 if $i = j$ and zero otherwise.

Altogether the stress tensor from ram pressure and pressure

$$\boldsymbol{\pi} = p\mathbf{g} + \rho \mathbf{u} \otimes \mathbf{u} \tag{25}$$

with metric tensor $\mathbf{g}$ with components δ_{ij} for a non-relativistic flat space time. The components of the stress tensor are

$$\pi_{ij} = p\delta_{ij} + \rho u_i u_j$$

What is the momentum flux $\mathbf{F}$ through a surface oriented in the $\hat{\mathbf{n}}$ direction?

$$F_i = \sum_j \pi_{ij} n_j = \pi_{ij} n_j$$

in summation notation or

$$\mathbf{F} = \boldsymbol{\pi} \cdot \hat{\mathbf{n}}$$

And all components individually

$$\begin{aligned} F_x &= \pi_{xx} n_x + \pi_{xy} n_y + \pi_{xz} n_z \\ F_y &= \pi_{yx} n_x + \pi_{yy} n_y + \pi_{yz} n_z \\ F_z &= \pi_{zx} n_x + \pi_{zy} n_y + \pi_{zz} n_z \end{aligned}$$

What is the force per unit area $\mathbf{F}$ on a surface with normal $\hat{\mathbf{n}}$? Identical to the above expression.

Consider flow through a pipe with velocity $\mathbf{u} = u_0\hat{\mathbf{x}}$ in the x direction. The stress tensor is

$$\boldsymbol{\pi} = \begin{pmatrix} p + \rho u_0^2 & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{pmatrix}$$

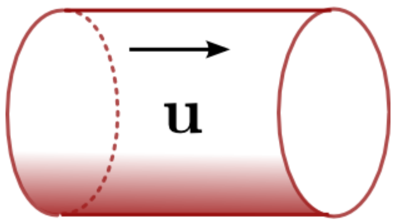

FIGURE 9. Flow through a pipe. The pressure on the sides is p but the pressure on on end (including ram pressure) is $p + \rho u^2$.

Unlike ram pressure, thermal pressure only acts in the direction perpendicular to a surface. But any surface will feel a force due to pressure. In contrast, a surface that is perpendicular to the fluid motion will not feel ram pressure. A surface would only feel ram pressure if its normal contains a component aligned with the fluid velocity.

As momentum is a vector, each component is conserved, and each component obeys a conservation law. Remember a conservation law can be written like that for conservation of mass $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$. Instead of ρ we will consider conservation of each component of momentum density $\rho\mathbf{u}$. The stress tensor also describes the momentum flux through the surfaces of a fluid element. So the stress tensor will enter the divergence term.

First let us consider conservation of the x component of momentum per unit volume ρu_x. The flux of this momentum component would be $\mathbf{F} = (\pi_{xx}, \pi_{yx}, \pi_{zx})$. The conservation law for this component can be written

$$\frac{\partial \rho u_x}{\partial t} + \frac{\partial \pi_{xx}}{\partial x} + \frac{\partial \pi_{yx}}{\partial y} + \frac{\partial \pi_{zx}}{\partial z} = 0$$

We can describe conservation of all three momentum components all together with

$$\frac{\partial (\rho \mathbf{u})}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{\pi} = 0$$

where $\boldsymbol{\nabla} \cdot \boldsymbol{\pi}$ has j-th component $= \sum_i \frac{\partial}{\partial x_i} \pi_{ij}$. The above equation is in conservation law form but the conserved quantity is a vector $\rho\mathbf{u}$ and the flux of the vector is a tensor $\boldsymbol{\pi}$.

In your problem set you will show that the above equation is consistent with Euler's equation.

In the above equation we have neglected gravity. However the force due to gravity can be included as a source term

$$\frac{\partial(\rho\mathbf{u})}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{\pi} = -\rho\nabla\Phi \tag{26}$$

Why is gravity a source term? When considering a fluid element, the force due to pressure is exerted across a boundary. The stress tensor describes changes in the momentum vector in direct directions. In contrast gravity is a force that is proportional to the total mass of the volume element.

To make it clear what the notation means in equation 26 we will write out the j-th component of this equation

$$\frac{\partial(\rho u_j)}{\partial t} + \sum_i \frac{\partial \pi_{ij}}{\partial x_i} = -\rho\frac{\partial\Phi}{\partial x_j}$$

We use equation 25 for $\boldsymbol{\pi}$ to evaluate the second term term on the left using summation notation (so dropping the $\sum_i$)

$$\frac{\partial \pi_{ij}}{\partial x_i} = \frac{\partial(p\delta_{ij})}{\partial x_i} + \frac{\partial(\rho u_i u_j)}{\partial x_i} = \frac{\partial p}{\partial x_j} + \frac{\partial(\rho u_i u_j)}{\partial x_i}$$

More specifically let us write out the z component for equation 26

$$\frac{\partial(\rho u_z)}{\partial t} + \frac{\partial p}{\partial z} + \frac{\partial(\rho u_x u_z)}{\partial x} + \frac{\partial(\rho u_y u_z)}{\partial y} + \frac{\partial(\rho u_z^2)}{\partial z} = -\rho\frac{\partial\Phi}{\partial z}.$$

1.15. **What is a tensor?** We have been discussing the stress tensor $\boldsymbol{\pi}$. It's an object with two indices π_{ij} where each index refers to a specific coordinate. We have been calling it a *tensor*. A *tensor* is a multi-index generalization of a vector. A vector $\mathbf{v}$ can be written as $\mathbf{v} = v^i\hat{\mathbf{e}}_i$ (using summation notation) where $\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2 .... \hat{\mathbf{e}}_n$ is an orthonormal coordinate basis. If we have a matrix $\mathbf{R}$ that transfers to a different coordinate basis $\hat{\mathbf{f}}_i = R_i^j\hat{\mathbf{e}}_j$ then we can write

$$\mathbf{v} = v^i\hat{\mathbf{e}}_i = v^i(R^{-1})_i^j R_j^k\hat{\mathbf{e}}_k = w^i\hat{\mathbf{f}}_i$$

The vector in the new basis is $w^i\hat{\mathbf{f}}_i$. The components of the vector transform as

$$w^i = (R^{-1})_j^i v^j$$

A tensor is a generalization of this using more than one index. It is important that the components of the tensor can transform with the coordinate system. A tensor with n indices on the top and m indices on the bottom

$$T^{i_1,i_2....i_n}_{j_1,j_2....j_m}[\mathbf{e}]$$

transforms as

$$T^{i_1,i_2....i_n}_{j_1,j_2....j_m}[\mathbf{f}] = (R^{-1})^{i_1}_{k_1}(R^{-1})^{i_2}_{k_2}....(R^{-1})^{i_n}_{k_n}T^{k_1,k_2....k_n}_{l_1,l_2..,l_m}R^{l_1}_{j_1}R^{l_2}_{j_2}...R^{l_m}_{j_m} \tag{27}$$

The components of the stress tensor we should have written as π^{ij} with upper indices so that we are precise that the tensor can be written as

$$\boldsymbol{\pi} = \pi^{ij}\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j$$

The moment of inertia of a solid body is also an example of a tensor with two indices.

1.15.1. *Comparison of conservation of mass and momentum equations.* To review let us compare the different forms we have for conservation of mass and momentum (but here neglecting gravity).

$$\frac{D\rho}{Dt} = -\rho\boldsymbol{\nabla}\cdot\mathbf{u} \qquad \frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\mathbf{u}) = 0$$
$$\rho\frac{D\mathbf{u}}{Dt} = -\boldsymbol{\nabla}p \qquad \frac{\partial\mathbf{u}}{\partial t} + (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} = -\tfrac{1}{\rho}\boldsymbol{\nabla}p$$

Above on the left using Lagrangian derivatives and on the right expanded in the Eulerian viewpoint. Below given in conservation law form

$$\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\mathbf{u}) = 0$$
$$\frac{\partial(\rho\mathbf{u})}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{\pi} = 0$$

We have a total of 4 equations (one for mass, and one for each component of momentum). However we have 5 free variables, $\rho, p, \mathbf{u}$ and we have counted each component of velocity. Using an equation of state we can relate $p(\rho)$ giving us a complete set (four variables and four equations). Or we can consider a third equation that takes into account energy.

With the including of gravity the Lagrangian form of the momentum equation gains $-\rho\boldsymbol{\nabla}\Phi$ on the right hand side, and the Eulerian form gains $-\boldsymbol{\nabla}\Phi$ on the right hand side. The conservation law form gains a *source term* $-\rho\boldsymbol{\nabla}\Phi$. It is called a source term because it cannot be lumped into the divergence term.

1.16. **An aside on units.** The theoretical astrophysics community (including cosmologists) tends to work in cgs (centimeters, grams, second). This puts magnetic fields in Gaus which is convenient for estimating magnetic pressure. Almost everybody else, including the geophysics community, tend to work in mks (meters, kilograms, second). In mks pressures are in conveniently in Pa. Particle physicists like energy in eV (electron volts). Conversion between unit conventions is harder in MHD than in other settings.

2. Some math tools

2.1. Summation notation, the Levi-Civita/permutation tensor and some index gymnastics. We will often be manipulating vectors in this class, so it will be useful to speed up calculations with summation notation and using a permutation tensor ϵ_{ijk}. The tensor ϵ_{ijk} is known as the Levi-Civita tensor or the permutation tensor and is

$$\epsilon_{ijk} \equiv \begin{cases} 1 & & ijk \text{ is an even permutation} \\ -1 & \text{for} & ijk \text{ is an odd permuation} \\ 0 & & \text{two or more of } ijk \text{ are equal} \end{cases} \tag{28}$$

Each of ijk is a coordinate like xyz. For example $\epsilon_{xyz} = 1$ however $\epsilon_{xzy} = -1$. Consider the cross product of two vectors

$$\mathbf{C} = \mathbf{A} \times \mathbf{B} \tag{29}$$

The $i$ith component of $\mathbf{C}$ can be written

$$C_i = \epsilon_{ijk} A_j B_k \tag{30}$$

and we have implicitly summed over all indices that appear twice in the equation. Here the indices j, k appear twice in the equation and so are summed over all coordinates, xyz. The Levi-Civita tensor is zero unless all indices differ. Each C_i can be written as a sum of two terms one with positive sign corresponding to the even permutation and the other with a negative sign corresponding to the odd permutation.

Using summation notation

$$\nabla \cdot \mathbf{A} = \frac{\partial A_i}{\partial x_i} \tag{31}$$

and we have implicitly summed over i. Here x_x represents x and x_y gives y and x_z gives z. Alternatively one can let ijk range from 1-3 and let $A_1 = A_x$, $A_2 = A_y$, $A_3 = A_z$, $\frac{\partial}{\partial x_1} = \frac{\partial}{\partial x}$, $\frac{\partial}{\partial x_2} = \frac{\partial}{\partial y}$, ... etc.

The expression δ_{ij} is the Kronecker delta which is a discrete version of the delta function.

$$\delta_{ij} \equiv \begin{cases} 1 & \text{for} & i = j \\ 0 & & i \neq j \end{cases} \tag{32}$$

When there are expressions with the Levi-Civita tensor it is often useful to use the relation

$$\epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl} \tag{33}$$

Once you notice the patterns in the indices it is not hard to remember the relation.

In computations, it is useful to remember that indices in the Levi-Civita tensor can be rotated

$$\epsilon_{ijk} = \epsilon_{jki} = \epsilon_{kij}$$

while maintaining the same sign in the permeation. Any two indices can be flipped

$$\epsilon_{ijk} = -\epsilon_{jik}$$

and reversing the sign.

An example we now derive a vector identity that we will use later on.

Example: Using summation notation, prove the vector identity

$$(\mathbf{u} \cdot \nabla)\mathbf{u} = \nabla \left(\frac{u^2}{2} \right) - \mathbf{u} \times (\nabla \times \mathbf{u}). \tag{34}$$

The i-th component for the expression on the right

$$\begin{aligned} u_j \frac{\partial u_j}{\partial x_i} - \epsilon_{ijk} u_j \epsilon_{klm} \frac{\partial u_m}{\partial x_l} &= u_j \frac{\partial u_j}{\partial x_i} - \epsilon_{kij} \epsilon_{klm} u_j \frac{\partial u_m}{\partial x_l} \\ &= u_j \frac{\partial u_j}{\partial x_i} - (\delta_{il}\delta_{jm} - \delta_{jl}\delta_{im}) u_j \frac{\partial u_m}{\partial x_l} \\ &= u_j \frac{\partial u_i}{\partial x_i} - u_j \frac{\partial u_j}{\partial x_i} + u_j \frac{\partial u_i}{\partial x_j} \\ &= u_j \frac{\partial u_i}{\partial x_j}. \end{aligned}$$

This is the i-th component for the expression on the left $(\mathbf{u} \cdot \nabla)\mathbf{u}$. We have proved the identity.

2.2. **Coordinate transformations.** The goal of this section is to introduce how conservation laws can be transformed into non-Cartesian coordinate systems, such as cylindrical or spherical coordinates.

In Cartesian coordinates in three dimensions, x, y, z, unit vectors in the direction of increasing x, y or z can be written as

$$\begin{aligned} \hat{\mathbf{x}} &= \hat{\mathbf{e}}_x \\ \hat{\mathbf{y}} &= \hat{\mathbf{e}}_y \\ \hat{\mathbf{z}} &= \hat{\mathbf{e}}_z. \end{aligned} \tag{35}$$

The gradient of a function $f(\mathbf{x})$ at a particular point is $\boldsymbol{\nabla} f = \left(\frac{\partial}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial y} \right)$. If we want to find how much the function f varies in the x direction we compute $df = \frac{\partial f}{\partial x} dx$.

We can think of $\frac{\partial}{\partial x}$ as a operator that is related to the direction that the x coordinate increases. That means we could associate

$$\hat{\mathbf{x}} = \frac{\partial}{\partial x}. \tag{36}$$

Let's change notation so that $x = x_1, y = x_2, z = x_3$. This gives

$$\hat{\mathbf{x}}_i = \hat{\mathbf{e}}_i = \frac{\partial}{\partial x^i} \tag{37}$$

for index $i \in \{1, 2, 3\}$.

In equation 37 we have associated a derivative with respect to a coordinate with a direction. Using Cartesian coordinates, we can define a vector as an operator

$$\mathbf{v} = \sum_i v^i \frac{\partial}{\partial x^i}. \tag{38}$$

If we have a function f, and operate on it with a vector $\mathbf{v}$, then we obtain the gradient of the function along the direction given by the vector $\mathbf{v}$. In other words

$$\sum_i v^i \frac{\partial f}{\partial x^i} = \mathbf{v} \cdot \boldsymbol{\nabla} f. \tag{39}$$

We could define the x, y, z components of a vector based on how the vector operators on the three functions $f^1(\mathbf{x}) = x$, $f^2(\mathbf{x}) = y$, and $f^3(\mathbf{x}) = z$. For example, suppose $\mathbf{v}$ gives a value of 3 if when it operates on the function x you get a value of 3. That means the x component of the vector $v^x = 3$.

In the context of differential geometry, vectors belong to the tangent space at a point on a manifold. If you consider a point on a curve across the manifold, the tangent vector to the curve at that point is a vector in the tangent space of that point of the manifold. For many astrophysical applications, hydrodynamics need not be done on curved spaces. However coordinate transformations are useful as it is common to work in spherical or cylindrical coordinates.

In polar coordinates we can write a vector in the same form as equation 38

$$\mathbf{v} = v^\theta \frac{\partial}{\partial \theta} + v^r \frac{\partial}{\partial r}. \tag{40}$$

The derivatives $\frac{\partial}{\partial \theta}$ and $\frac{\partial}{\partial r}$ are a **basis** for vectors but they may not be unit vectors. Basis vectors derived from a new set of coordinates also need not be orthogonal. To say what we mean by a unit vector or by orthogonal we need a dot product (also called an inner product).

With two vectors in Cartesian coordinates (and in two dimensions)

$$\mathbf{v} = v^x \frac{\partial}{\partial x} + v^y \frac{\partial}{\partial x}$$
$$\mathbf{w} = w^x \frac{\partial}{\partial x} + w^y \frac{\partial}{\partial x}. \tag{41}$$

the inner product operators on these two vectors

$$\langle \mathbf{v}|\mathbf{w}\rangle = \sum_i v^i w^i = v^x w^x + v^y w^y. \tag{42}$$

An inner product gives a notion of distance.

However, suppose we write the two vectors in the basis $\frac{\partial}{\partial \theta}$ and $\frac{\partial}{\partial r}$;

$$\mathbf{v} = v^\theta \frac{\partial}{\partial \theta} + v^r \frac{\partial}{\partial r}$$
$$\mathbf{w} = w^\theta \frac{\partial}{\partial \theta} + w^r \frac{\partial}{\partial r}. \tag{43}$$

We find that

$$\langle v|w\rangle = v^x w^x + v^y w^y$$
$$\neq v^\theta w^\theta + v^r w^r. \tag{44}$$

So that the inner product is independent of coordinate system we modify it so that it depends upon a tensor known as the **metric** tensor. In any set of coordinates we define

$$\langle \mathbf{v}|\mathbf{w}\rangle = \sum_{ij} g_{ij} v^i w^j \tag{45}$$

with $\mathbf{g}$, the metric tensor. Here v^i and w^j are the components of the vector in a particular coordinate system and g_{ij} are the components of the metric tensor in that same coordinate system. Cartesian coordinates have an inner product consistent with

$$g_{ij} = \delta_{ij}. \tag{46}$$

The metric tensor transforms as a tensor which means if we transform to another coordinate system, its components would also be transformed by the coordinate transformation (but twice as there are two indices).

Suppse x_1, x_2, x_3 is one coordinate system and y_1, y_2, y_3 is a different coordinate system. To be a good coordinate transformation the transformation (functions $y_1(x_1, x_2, x_3)$ and similarly for y_2, y_3) should be invertible almost everywhere. This would give a transformation from $x_1, x_2, x_3 \to y_1, y_2, y_3$. An example of a coordinate

transformation that is not invertible everywhere would be polar coordinates in two dimensions because at the origin, the azimuthal angle is not meaningful.
Example: A coordinate transformation in two dimensions from $x, y \to r, \theta$ is

$$r(x,y) = x^2 + y^2 \qquad \theta(x,y) = \text{arctan2}(y,x) \tag{47}$$

where *arctan2* is the arctangent but with result in $[0, 2\pi]$ and dependent on the quadrant of x, y. Also we can specify $r = 0$ if $x = y = 0$. The inverse transformation is

$$x(r,\theta) = r\cos\theta \qquad y(r,\theta) = r\sin\theta. \tag{48}$$

A vector

$$\mathbf{v} = \sum_j v^j \frac{\partial}{\partial x^j} = \sum_j v^j \frac{\partial y^k}{\partial x^j}\frac{\partial}{\partial y^k} \tag{49}$$

via the chain rule. To make it clear which coordinate system we are working in, we write components of a vector in the y coordinate system with an underline

$$\mathbf{v} = \sum_k \underline{v}^k \frac{\partial}{\partial y^k}. \tag{50}$$

Using the underline 49 implies that the components of the vector in the y coordinate system are

$$\underline{v}^k = \sum_j v^j \frac{\partial y^k}{\partial x^j}. \tag{51}$$

We transfer the vectors within a dot product into a different coordinate system. In a Cartesian coordinate system

$$\langle \mathbf{v}|\mathbf{w}\rangle = \sum_j v^k w^k \tag{52}$$

Using a coordinate transformation $(x_1, x_2, x_3 \to y_1, y_2, y_3)$

$$\begin{aligned} v^k &= \sum_j \underline{v}^j \frac{\partial x^k}{\partial y^j} \\ w^k &= \sum_m \underline{w}^m \frac{\partial x^k}{\partial y^m}. \end{aligned} \tag{53}$$

We insert these into equation 52 for the inner product

$$\begin{aligned}\langle \mathbf{v}|\mathbf{w}\rangle &= \sum_{jmk} \underline{v}^j \frac{\partial x^k}{\partial y^j} \underline{w}^m \frac{\partial x^k}{\partial y^m} \\ &= \sum_{jm} \underline{v}^j \underline{w}^m \underline{g}_{jm}\end{aligned} \tag{54}$$

with

$$\underline{g}_{jm} = \sum_k \frac{\partial x^k}{\partial y^j} \frac{\partial x^k}{\partial y^m}. \tag{55}$$

With this choice of metric tensor the inner product of the two vectors is independent of coordinate system. This illustrates how the components of the metric tensor transform with a coordinate transformation where x_1, x_2, x_3 is Cartesian but the new coordinate system y_1, y_2, y_3 is not.

Example: We compute the metric tensor for 2-dimensional polar coordinates

$$\begin{aligned}\underline{g}_{rr} &= \left(\frac{\partial x}{\partial r}\right)^2 + \left(\frac{\partial y}{\partial r}\right)^2 = \cos^2\theta + \sin^2\theta = 1 \\ \underline{g}_{r\theta} &= \frac{\partial x}{\partial r}\frac{\partial x}{\partial \theta} + \frac{\partial y}{\partial r}\frac{\partial y}{\partial \theta} = -\cos\theta r \sin\theta + \sin\theta r\cos\theta = 0 \\ \underline{g}_{\theta\theta} &= \left(\frac{\partial x}{\partial \theta}\right)^2 + \left(\frac{\partial y}{\partial \theta}\right)^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2\end{aligned} \tag{56}$$

The metric tensor for 2d polar coordinates can be written as a matrix

$$\underline{\mathbf{g}} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}. \tag{57}$$

Happily this metric tensor is diagonal! That means that the two vectors $\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial \theta}$ are orthogonal.

With the metric tensor we can measure the length of the vectors $\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial \theta}$. The length of a vector

$$|\mathbf{v}| = \sqrt{\langle \mathbf{v}|\mathbf{v}\rangle} \tag{58}$$

The vector $\frac{\partial}{\partial r}$ has length 1 because $\underline{g}_{rr} = 1$. The vector $\frac{\partial}{\partial \theta}$ has length r because $\underline{g}_{\theta\theta} = r^2$.

We construct **unit** vectors in 2d polar coordinates

$$\hat{\boldsymbol{\theta}} = \frac{1}{r}\frac{\partial}{\partial \theta}$$
$$\hat{\mathbf{r}} = \frac{\partial}{\partial \theta}. \tag{59}$$

The factor $1/r = 1/\sqrt{g_{\theta\theta}}$ is used to normalize the θ component.

2.3. **How to find a gradient and other quantities in non-Cartesian coordinates.** How does the gradient of a function transform? In a Cartesian coordinate system we know that

$$\boldsymbol{\nabla} f = \sum_i \frac{\partial f}{\partial x^i}\frac{\partial}{\partial x^i}. \tag{60}$$

This is a vector so we can compute

$$\langle \mathbf{w} | \boldsymbol{\nabla} f \rangle \tag{61}$$

for any vector $\mathbf{w}$. This inner product should be independent of our coordinate system. We transform $\mathbf{w}$

$$\mathbf{w} = \sum_j w^j \frac{\partial}{\partial x^j} = \sum_j \underline{w}^j \frac{\partial}{\partial y^j} \tag{62}$$

with

$$\underline{w}^j = \sum_k \frac{\partial y^j}{\partial x^k} w^k. \tag{63}$$

$$\langle \mathbf{w} | \boldsymbol{\nabla} f \rangle = \sum_k w^k \frac{\partial f}{\partial x^k} = \sum_{jk} \underline{w}^j \underline{(\nabla f)}^j g_{jk}.$$

The metric tensor has two factors of the coordinate transformation jacobian in it and one of them inverts the transformation on $\mathbf{w}$, the other inverts the transformation on the gradient operator. The result is that the gradient operator does not involve any factors of the metric tensor

$$\boldsymbol{\nabla} f = \sum_j \frac{\partial f}{\partial y^j}\frac{\partial}{\partial y^j} \tag{64}$$

Example: We compute the gradient of a function in 2d polar coordinates $f(r, \theta)$.

$$\boldsymbol{\nabla} f = \frac{\partial f}{\partial r}\frac{\partial}{\partial r} + \frac{\partial f}{\partial \theta}\frac{\partial}{\partial \theta} = \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\boldsymbol{\theta}} \tag{65}$$

where we have used equation 59 for the unit vectors.

Suppose we want to calculate the advective term in Euler's equation

$$(\mathbf{u} \cdot \nabla)\mathbf{u} \tag{66}$$

but in polar coordinates.

We first need to understand how to compute the gradient of a vector component

$$\frac{\partial}{\partial x^j} u^k \tag{67}$$

in such a way to give a coordinate independent tensor. Unfortunately when doing a coordinate transformation u^k gains derivatives of the coordinate functions which themselves are functions of coordinates. This then causes a problem when taking the derivative. To fix this problem and gain a derivative that gives a coordinate independent result, we replace the derivative with a corrected derivative known as the **covariant derivative**. The j-th component of the the covariant derivative of the i-th component of vector $\mathbf{v}$ is

$$\nabla_j v^i \equiv \frac{\partial v^i}{\partial x^j} + \sum_k \Gamma^i_{jk} v^k. \tag{68}$$

Here Γ^i_{jk} are connection coefficients (also called Christofel symbols) which involve derivatives of the metric tensor

$$\Gamma^i_{jk} = \sum_a \frac{1}{2} g^{ia} \left(\frac{\partial g_{ja}}{\partial x^k} + \frac{\partial g_{ak}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^a} \right). \tag{69}$$

We are not going to derive equation 68 but we will use it to compute the advective term in polar coordinates.

2.4. **The advective term in polar coordinates.** We assume that the velocity $\mathbf{u}$ is written in terms of the orthonormal basis in polar coordinates

$$\mathbf{u} = u^r \frac{\partial}{\partial r} + \frac{u^\theta}{r} \frac{\partial}{\partial \theta}. \tag{70}$$

The gradient operator does not involve any factors of the metric tensor

$$\mathbf{u} \cdot \boldsymbol{\nabla} = u^r \hat{\mathbf{r}} + u^\theta \hat{\boldsymbol{\theta}} = u^r \frac{\partial}{\partial r} + \frac{u^\theta}{r} \frac{\partial}{\partial \theta}. \tag{71}$$

Ignoring the terms that contain connection coefficients for the moment, we compute the two components separately

$$(\mathbf{u} \cdot \nabla) u^r = u^r \frac{\partial u^r}{\partial r} + \frac{u^\theta}{r} \frac{\partial u^r}{\partial \theta} \tag{72}$$

$$(\mathbf{u} \cdot \nabla) \left(\frac{u^\theta}{r} \right) = -\frac{u^r u^\theta}{r^2} + \frac{u^r}{r} \frac{\partial u^\theta}{\partial r} + \frac{u^\theta}{r} \frac{1}{r} \frac{\partial u_\theta}{\partial \theta}. \tag{73}$$

Now we compute connection coefficients. For 2d polar coordinates the metric is diagonal and only the term $g_{\theta\theta} = r^2$ is not 1 or 0.

$$\begin{aligned}
\Gamma^i_{jk} &= \frac{1}{2} g^{ia} \left(\frac{\partial g_{ja}}{\partial x^k} + \frac{\partial g_{ak}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^a} \right) \\
&= \frac{1}{2} g^{ia} \left(2r\delta_{j,\theta}\delta_{a,\theta}\delta_{k,r} + 2r\delta_{a,\theta}\delta_{k,\theta}\delta_{j,r} - 2r\delta_{j,\theta}\delta_{k,\theta}\delta_{a,r} \right) \\
\Gamma^r_{jk} &= r \left(\delta_{j,\theta}\delta_{r,\theta}\delta_{k,r} + \delta_{r,\theta}\delta_{k,\theta}\delta_{j,r} - \delta_{j,\theta}\delta_{k,\theta}\delta_{r,r} \right) \\
&= -r\delta_{j,\theta}\delta_{k,\theta} \\
\Gamma^\theta_{jk} &= \frac{1}{r} \left(\delta_{j,\theta}\delta_{\theta,\theta}\delta_{k,r} + \delta_{\theta,\theta}\delta_{k,\theta}\delta_{j,r} - \delta_{j,\theta}\delta_{k,\theta}\delta_{\theta,r} \right) \\
&= +\frac{1}{r} \left(\delta_{j,\theta}\delta_{k,r} + \delta_{j,r}\delta_{k,\theta} \right)
\end{aligned}$$

We used $g^{rr} = 1, g^{\theta\theta} = \frac{1}{r^2}$. The only non-zero connection coefficients are

$$\Gamma^r_{\theta\theta} = -r \tag{74}$$

$$\Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \frac{1}{r}. \tag{75}$$

To the r component of $(\mathbf{u} \cdot \nabla)\mathbf{u}$ the term arising from connection coefficients is

$$u_r \left(\Gamma^r_{rr} u^r + \Gamma^r_{r\theta} \frac{u^\theta}{r} \right) + \frac{u_\theta}{r} \left(\Gamma^r_{\theta r} \frac{u^\theta}{r} + \Gamma^r_{\theta\theta} \frac{u^\theta}{r} \right) = -\frac{u_\theta^2}{r} \tag{76}$$

To the θ component of $(\mathbf{u} \cdot \nabla)\mathbf{u}$ the term arising from connection coefficients is

$$u_r \left(\Gamma^\theta_{rr} u^r + \Gamma^\theta_{r\theta} \frac{u^\theta}{r} \right) + \frac{u_\theta}{r} \left(\Gamma^\theta_{\theta r} u^r + + \Gamma^\theta_{\theta\theta} \frac{u^\theta}{r} \right) = \frac{2u_r u_\theta}{r^2} \tag{77}$$

Adding equations 73, 76, 77 together (so that we compute the advective term with a covariant derivative) and writing the result in terms of the unit vectors $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$;

$$\begin{aligned}
(\mathbf{u} \cdot \nabla)\mathbf{u} = &\left(u^r \frac{\partial u^r}{\partial r} + \frac{u^\theta}{r} \frac{\partial u^r}{\partial \theta} - \frac{u_\theta^2}{r} \right) \hat{\mathbf{r}} \\
&+ \left(u^r \frac{\partial u^\theta}{\partial r} + \frac{u^\theta}{r} \frac{\partial u^\theta}{\partial \theta} + \frac{u^r u^\theta}{r} \right) \hat{\boldsymbol{\theta}}.
\end{aligned} \tag{78}$$

This is correct! We notice that there are two additional terms.

Key points for working with derivatives in different coordinate systems:

(1) Keep straight the difference between unit vectors and vectors in the basis $\frac{\partial}{\partial y^i}$. Tensors tend to be written in terms of the $\frac{\partial}{\partial y^i}$ (non-orthonormal) basis rather than a set of orthogonal unit vectors. In contrast the Navier Stokes equation

involves vectors that are usually written in terms of components with respect to a orthonormal basis.

(2) In fluid equations, gradients of vector components should be computed with a covariant derivative. This derivative can be computed with connection coefficients which are derived from the metric associated with the desired coordinate system.

(3) Computing the connection coefficients can be tedious (and prone to error) which is why we wind up frequently looking up the form of the Naviers Stokes equation in cylindrical coordinates.

(4) Equation 68 for the covariant derivative gives a tensor. To derive it we could again try to make sure that a dot product (of some sort) gives an number independent of coordinate system. Equivalently exploit the fact that the metric itself should not vary $\nabla_i g_{jk} = 0$.

A note about whether indices are up or down. For many classical non-relativistic flat space settings we don't need to keep track of whether indices are up or down. However, in general relativity a lower index refers to something in the tangent space and an upper index refers to something in the co-tangent space. The metric lets you convert a lower index to an upper one and vice versa. Summation notation tends to involve the sum of an upper index with a lower index.

2.5. **The divergence.** Conservation laws contain the divergence of a flux vector $\boldsymbol{\nabla} \cdot \mathbf{F}$. In Cartesian coordinates

$$\boldsymbol{\nabla} \cdot \mathbf{F} = \sum_i \frac{\partial F^i}{\partial x^i}. \tag{79}$$

Equation 68 for the covariant derivative can be summed to compute the divergence in equation 79 but with covariant derivatives

$$\boldsymbol{\nabla} \cdot \mathbf{F} = \sum_i \nabla_i F^i = \sum_i \frac{\partial F^i}{\partial x^i} + \sum_{ij} \Gamma^i_{ij} F^j. \tag{80}$$

With some manipulation it is possible to show that equation 80 is equal to

$$\boldsymbol{\nabla} \cdot \mathbf{F} = \sum_i \frac{1}{\sqrt{|g|}} \frac{\partial(\sqrt{|g|} F^i)}{\partial x^i} \tag{81}$$

where $|g|$ is the determinant of the metric tensor which is also equal to the square of the volume element.

Example: Find the covariant condition for incompressibility $\nabla \cdot \mathbf{u} = 0$ in polar coordinates.

We again write $\mathbf{u} = u^r \frac{\partial}{\partial r} + u^\theta \frac{\partial}{\partial \theta}$. Applying equation 80

$$\nabla \cdot \mathbf{u} = \frac{\partial u^r}{\partial r} + \frac{1}{r}\frac{\partial u^\theta}{\partial \theta} + \Gamma^\theta_{\theta r} u^r \tag{82}$$

$$= \frac{\partial u^r}{\partial r} + \frac{1}{r}\frac{\partial u^\theta}{\partial \theta} + \frac{u^r}{r} \tag{83}$$

which we recognize as the divergence in polar coordinates.

Using equation 81 we first compute the determinant of the metric tensor. Using equation 57 we find $|g| = r^2$. Using equation 81

$$\nabla \cdot \mathbf{u} = \frac{1}{r}\left(\frac{\partial (r u^r)}{\partial r} + \frac{\partial (r u^\theta / r)}{\partial \theta}\right)$$

$$= \frac{\partial u^r}{\partial r} + \frac{u^r}{r} + \frac{1}{r}\frac{\partial u^\theta}{\partial \theta}$$

which agrees with our previous expression, as expected.

2.6. **The rest.** With additional efforts (!) we could create tensor expressions for the Laplacian, cross products and curls and these cover most hydrodynamic equations in both Euclidean space and curved space time!

3. Conservation of Energy

3.1. **Heat Flux, Adiabatically.** The first law of thermodynamics is an expression of conservation of energy and is

$$dQ = de + pdV \tag{84}$$

where dQ is the quantity of heat absorbed per unit mass of fluid from its surroundings, pdV is the work done by the unit mass of fluid if its volume changes by dV and de is the change in the internal energy content per unit mass of the fluid. This law neglects viscous, dissipative, and radiative processes. It is equivalent to assuming that the entropy per unit mass of fluid does not change.

In the absence of non-adiabatic heating and cooling (for example by radiation) entropy is conserved in a fluid element and we can write

$$\frac{DS}{Dt} = 0$$

where S is the specific entropy or the entropy per unit mass. This condition can be violated (in for example shocks). However it is satisfied in many situations and so is often assumed. This condition can be used to relate T, S to p, ρ and so construct an *equation of state*. This description is equivalent to saying $dQ = 0$ for a small parcel of fluid.

In terms of thermodynamic quantities

$$dQ = TdS = de + pdV = de - p\frac{d\rho}{\rho^2} \tag{85}$$

where e is the internal energy per unit mass. This means that the quantity ρe is the internal energy *per unit volume*. In the above equation I have used V as the volume per unit mass so that $V = 1/\rho$ and

$$dV = -d\rho/\rho^2. \tag{86}$$

For a fluid element we can modify equation 85 to describe the rate of change of heat

$$\frac{DQ}{Dt} = T\frac{DS}{Dt} = \frac{De}{Dt} - \frac{p}{\rho^2}\frac{D\rho}{Dt}. \tag{87}$$

Recall the ideal gas law

$$p = \frac{\rho}{\mu m_p} k_B T \tag{88}$$

Here k_B is Boltzmann's constant, m_p the mass of the proton, p pressure, T temperature, and μ the mean atomic weight. (Here m_p loosely used as an atomic mass unit). The mean molecular weight μ is equal to 1.0 if the gas is composed of neutral (unionized) atomic hydrogen. Let us consider changes in T, ρ and p.

$$dp = \frac{d\rho}{\mu m_p} k_B T + \frac{\rho}{\mu m_p} k_B dT$$

Solving for dT

$$dT = \frac{\mu m_p}{k_B}\left[\frac{dp}{\rho} - \frac{p d\rho}{\rho^2}\right] \tag{89}$$

Recall Equation 85

$$TdS = de - p\frac{d\rho}{\rho^2}$$

Setting $dS = 0$

$$de = p\frac{d\rho}{\rho^2}$$

We replace de with $c_V dT$ using the specific heat c_V, so that

$$\frac{p}{\rho^2} d\rho = de = c_V dT.$$

Now replace dT using the equation 89, giving us

$$c_V \frac{\mu}{\mathcal{R}}\left[\frac{dp}{\rho} - \frac{p d\rho}{\rho^2}\right] = \frac{p d\rho}{\rho^2}$$

and I have used $\mathcal{R} = \frac{k_B}{\mu m_p}$. With some manipulation:

$$\frac{dP}{d\rho} = \frac{p}{\rho}\left(\frac{c_V + \mathcal{R}}{c_V}\right) = \frac{c_P}{c_V}\frac{p}{\rho} = \frac{\gamma p}{\rho} \tag{90}$$

where $\gamma = c_P/c_V$ is the ratio of specific heats and $c_P = c_V + \frac{k_B}{\mu m_p}$. Often you see this written as

$$\frac{\mathcal{R}}{\mu} = c_P - c_V$$

where $\mathcal{R}$ is the ideal gas constant and μ the mean molecular mass. I have been using $\mathcal{R} \sim k_B/m_p$ (which is not quite correct, I should be using an atomic mass unit instead of m_p) because it allows us to check our units quickly. Another way to write equation 90 is

$$\gamma = \left(\frac{\partial \ln p}{\partial \ln \rho}\right)_S$$

which we can recognize as appropriate for an ideal gas.

Consider equation 90 again and regrouping

$$\begin{aligned}\frac{dp}{p} &= \gamma\frac{d\rho}{\rho} \\ d\ln p &= \gamma \ln \rho\end{aligned}$$

implying a scaling

$$p \propto \rho^\gamma$$

With some manipulation it is also possible to show the scalings

$$\begin{aligned}p &\propto T^{\frac{\gamma}{\gamma-1}} \\ \rho &\propto T^{\frac{1}{\gamma-1}}\end{aligned}$$

3.2. **Heat flux with conductivity and heating or cooling.** The above equation (87) gives the heat change for a particular volume element of a given mass (as e and S are energy and entropy per unit mass). To describe the rate of change per unit volume we multiple by density. The rate of change of heat (units energy per unit volume per unit time) is

$$\rho T \frac{DS}{Dt}$$

This quantity should be equal to the divergence of the heat flux and the rate of change of internal energy $\dot{Q}$ per unit mass.

$$\rho T \frac{DS}{Dt} = -\boldsymbol{\nabla} \cdot \mathbf{h} - \rho \dot{Q}_{cool} \tag{91}$$

where $\mathbf{h}$ is the heat flux (units energy per unit area per unit time). Heat flux could be due to conduction, convection or radiation (optically thick limit). The sign of

$\dot{Q}_{cool}$ is such that it is a cooling rate. Internal energy loss or gain could be due to heating (such as radioactive decay) or cooling (in the optically thin limit so that photons escape).

Heat flux is often described by a conductivity equation

$$\mathbf{h} = -\lambda \nabla T \tag{92}$$

where λ is the thermal conductivity. Conductivity of heat can occur via a variety of processes such as thermal conduction, turbulence and convection or radiation transport. In some of these cases the above form for the heat conduction can be used. If we set

$$TdS = de = c_V dT$$

depending on the specific heat c_V (here we have ignored the energy change due to work and that is like assuming that density does not change so $d\rho = 0$). In the absence of cooling, ($\dot{Q}_{cool} = 0$), equation 91 becomes

$$\rho c_V \frac{\partial T}{\partial t} = -\boldsymbol{\nabla} \cdot \mathbf{h}$$

and using equation 92 we find

$$\rho c_V \frac{\partial T}{\partial t} = \lambda \boldsymbol{\nabla}^2 T \tag{93}$$

This is a **diffusion equation for temperature**. This equation can be used to describe temperature variations in asteroids, planets or frying pans. It is often useful to consider the units for equation 93

$$\frac{\text{Temperature}}{\text{time}} = \frac{\text{Temperature}}{\text{distance}^2} \frac{\lambda}{\rho c_V}$$

giving a **diffusion coefficient**

$$D \equiv \frac{\lambda}{\rho c_V} \qquad \text{with units} \qquad \left(\frac{\text{distance}^2}{\text{time}}\right)$$

Cooling timescales for planets or planetesimals, or skin depths on spinning asteroids can be estimated from the diffusion coefficient, computed from the density, heat conductivity and specific heat.

Equations for conservation of mass, momentum and thermal energy are sufficient to describe evolution of a system. Below we will develop a third conservation law directly describing energy.

3.3. **Pressure with a radiation field.** Often important in astrophysical applications is the important case of an ideal gas together with a radiation field that is a black body. In this case

$$p = \frac{k_B \rho T}{\mu m_p} + \frac{4\sigma_{SB} T^4}{3c} \tag{94}$$

Here c the speed of light and σ_{SB} the Stefan-Boltzmann constant. The second term in the above equation is due to radiation pressure and is important in high mass stars, near neutron stars and black holes and in the early universe.

Recall that the energy flux of emission from the surface of a black body only depends on a single quantity, the temperature T. This energy flux is $\sigma_{SB}T^4$ so the total luminosity of a spherical black body of radius R is $L = 4\pi R^2 \sigma_{SB} T^4$. Energy flux is in units of energy/area/time. Pressure is in units of energy density. Consequently something that has units of pressure would be the energy flux divided by a velocity, and since we are considering radiation pressure, the appropriate velocity would be the speed of light. This account for the term on the right hand side of equation 94 except for the factor of 4/3. I will take a moment to explain why there is a factor of 4/3 in the above equation.

Consider an area element dA with normal $\hat{\mathbf{n}}$. We would like to know the energy and momentum flux through this element. We chose a coordinate system with angles θ, ϕ oriented with respect to $\hat{\mathbf{n}}$. The energy of radiation crossing through the area dA element that has direction within solid angle $d\Omega$ of the normal $\hat{\mathbf{n}}$ of the area element per unit time (or in a time interval dt) and in frequency range $d\nu$ is

$$dE = I_\nu dA \; dt \; d\Omega \; d\nu. \tag{95}$$

Here I_ν is the specific intensity or brightness (units ergs s^{-1} cm^{-2} ster^{-1} Hz^{-1}). This is an energy flux (energy per unit area and time) that is also per unit frequency range (color) and takes into account the directions (so is also per unit steradian).

The net flux in different direction (not perpendicular to the area element) is reduced by $\cos\theta$ because the radiation field can have rays in all directions. Here θ is the angle between the normal $\hat{n}$ and the direction element $d\Omega$. The net flux can be integrated

$$F_\nu = \int I_\nu \cos\theta d\Omega \tag{96}$$

The units of F_ν are ergs s^{-1} cm^{-2} Hz^{-1} so this is an energy flux (energy per unit time and area) that is also per unit frequency range for the photons. This makes sense as integrating over all 4π should give a net flux of zero if the radiation field is isotropic.

The momentum flux along a ray at angle θ is dF_ν/c. To make sure we have the component of momentum perpendicular to dA we must multiple by another factor

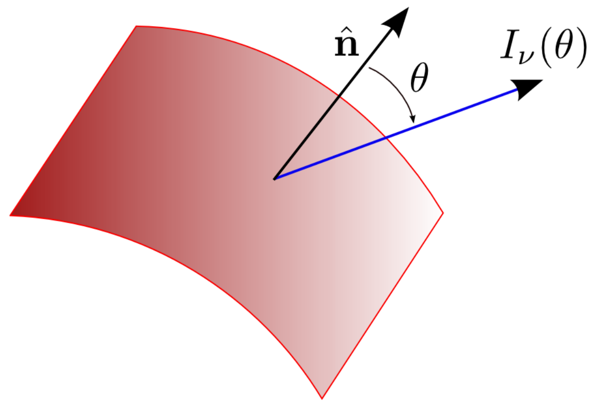

FIGURE 10. The energy flux through a surface depends on an integral of I_ν over angle. The i-th component of momentum contributed by a photon depends on the angle between the photon's momentum vector and the i direction.

of $\cos\theta$. The momentum flux (component perpendicular to our area element)

$$p_\nu = \frac{1}{c}\int I_\nu \cos^2\theta d\Omega = \frac{1}{c}\int_0^{2\pi} d\phi \int_{-1}^{1} I_\nu \mu^2 d\mu$$

where $\mu = \cos\theta$. If the radiation field is isotropic then I_ν does not depend on θ and

$$p_\nu = \frac{2\pi}{c}\int_{-1}^{1} I_\nu \mu^2 d\mu = \frac{4\pi}{3c} I_\nu$$

Now consider the surface of a star that is a black body. Integrating over ν and over 2π in directions we know that the net flux is

$$F = \sigma_B T^4 = \int_0^1 2\pi I \mu d\mu = \pi I$$

where $I = \int I_\nu d\nu$. The above equation implies that $I = \sigma_B T^4/\pi$. We can now estimate the momentum flux in radiation assuming isotropic black body radiation

$$p_r = \frac{4}{3c}\sigma_B T^4$$

This momentum flux is the flux of momentum through an area element with component perpendicular to the area element. Consequently it is part of the stress tensor π_{ii} but only contributes to a term that has two indexes the same (on the diagonal). This is why it can be considered as a pressure.

3.4. **Conservation of energy.** Recall that from the first law of thermodynamics the internal energy per unit mass

$$\frac{De}{Dt} = \frac{DW}{Dt} + \frac{DQ}{DT} \tag{97}$$

where $\frac{DW}{Dt}$ is the work done per unit mass and $\frac{DQ}{DT}$ is the energy/heat gained per unit mass. We can rewrite the work

$$\frac{DW}{Dt} = -p\frac{D(1/\rho)}{Dt} = \frac{p}{\rho^2}\frac{D\rho}{Dt} \tag{98}$$

giving us

$$\frac{De}{Dt} = \frac{p}{\rho^2}\frac{D\rho}{Dt} - \dot{Q}_{cool}$$

where we have defined $\dot{Q}_{cool}$ as a cooling rate, or the heat lost per unit mass and this gives us the minus sign in the above equation.

Define a **total energy** per unit volume E

$$E \equiv \rho\left(\frac{u^2}{2} + \Phi + e\right). \tag{99}$$

Each term is recognizable as the kinetic energy per unit volume, the potential energy and the internal energy. For a fluid element expanding out the directional derivative

$$\frac{DE}{Dt} = \frac{u^2}{2}\frac{D\rho}{Dt} + \rho\mathbf{u}\cdot\frac{D\mathbf{u}}{Dt} + \rho\frac{D\Phi}{Dt} + \rho\frac{De}{Dt} + (\Phi + e)\frac{D\rho}{Dt}. \tag{100}$$

Our heat equation (equation 85) gives

$$TdS = de - \frac{p}{\rho^2}d\rho$$

If I set TdS equal to a cooling rate $\dot{Q}$ then we can write our conservation of energy equation (equation 100) as

$$\frac{DE}{Dt} = \left[\frac{E}{\rho} + \frac{p}{\rho}\right]\frac{D\rho}{Dt} - \rho\dot{Q} + \rho\mathbf{u}\cdot\frac{D\mathbf{u}}{Dt} + \rho\frac{D\Phi}{Dt}$$

Using some Lagrangian derivatives

$$
\begin{aligned}
\frac{E}{\rho}\frac{D\rho}{Dt} &= -\frac{E}{\rho}\rho\boldsymbol{\nabla}\cdot\mathbf{u} \qquad \text{continuity/cons. mass} \\
\rho\mathbf{u}\cdot\frac{D\mathbf{u}}{Dt} &= \rho\mathbf{u}\cdot\left[\frac{\partial\mathbf{u}}{\partial t} + \mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u}\right] \qquad \text{Lagrang deriv.} \\
&= \mathbf{u}\cdot(-\boldsymbol{\nabla}p - \rho\boldsymbol{\Phi}) \qquad \text{momentum cons.} \\
\rho\frac{D\Phi}{Dt} &= \rho\frac{\partial\Phi}{\partial t} + \rho\mathbf{u}\cdot\boldsymbol{\nabla}\Phi \qquad \text{Lagrange deriv.} \\
\frac{p}{\rho}\frac{D\rho}{Dt} &= -\frac{p}{\rho}\rho\boldsymbol{\nabla}\cdot\mathbf{u} = -p\boldsymbol{\nabla}\cdot\mathbf{u} \qquad \text{continuity}
\end{aligned}
$$

By combining the above with our equations for conservation of momentum, conservation of mass and our definition for the directional derivative we can write conservation of energy in the following form

$$
\frac{\partial E}{\partial t} + \nabla\cdot[(E+p)\mathbf{u}] = -\rho\dot{Q}_{cool} + \rho\frac{\partial\Phi}{\partial t} \tag{101}
$$

where $\dot{Q}_{cool}$ is the heating cooling function, energy lost or gained per unit mass due to heating and cooling with $\dot{Q}_{cool} > 0$ when cooling. The terms in the above relation should make sense physically. The divergence term involving $p\mathbf{u}$ arrises from pdV work.

Note we have taken care to keep in a term from heating and cooling, commonly important in astrophysical settings. Here we have not taken into account dissipative processes such as viscosity that would give a heat term dependent on velocity gradients. We have also not taken into account heat conductivity.

3.5. **A polytropic gas and equations of state.** A *barytropic* fluid is one where we can form a relation $p(\rho)$. In this case we need not use an additional energy equation (along with that for momentum and mass) but can determine the dynamics with the equations for conservation of momentum and mass alone alone with an equation of state. Baritropic flow is more general than isentropric flow and includes isothermal flow as well as a polytropic approximation.

A *polytropic gas* is an ideal gas with constant c_P, c_V, γ, and μ. The polytropic index n (not necessarily an integer) is defined by $\gamma = 1 + 1/n$ and

$$
p \propto \rho^{1+1/n} \tag{102}
$$

Equipartition of energy with N degrees of freedom gives

$$
e = \frac{k_B T}{\mu m_p}\frac{N}{2} = c_V T \tag{103}
$$

and so $c_V = \frac{k_B}{\mu m_p}\frac{N}{2}$. Using the relations $c_P = c_V + k_B/(\mu m_p)$ and $\gamma = c_P/c_V$ we can show that $\gamma = 1 + 2/N$. A popular value for γ is 5/3 corresponding to $N = 3$ degrees of freedom.

We can also show that the specific internal energy of a polytropic gas is

$$e = \frac{p}{(\gamma - 1)\rho} \tag{104}$$

If the gas cools rapidly above a particular temperature then an *isothermal* equation of state can be adopted. This is often used for global simulations of molecular clouds or the interstellar medium in galaxies.

$$p = c_s^2 \rho \tag{105}$$

where c_s is the sound speed.

If the gas is polytropic and isentropic then

$$p = K\rho^\gamma \tag{106}$$

with constant K. The isentropic ideal gas has sound speed $\gamma P/\rho$. Note that an isothermal fluid is also one with $\gamma \to 1$.

An *incompressible* fluid is one where $\frac{D\rho}{Dt} = 0$ and so $\nabla \cdot \mathbf{u} = 0$. This approximation can also be considered the limit $\gamma \to \infty$. A completely incompressible fluid will not transmit sound waves.

3.6. **The Lame-Enden equation.** Taking a polytropic equation of state and hydrostatic equilibrium, it is possible to solve for the density, potential and gravitational potential as a function of radius. The polytropic equation of state with index n

$$p = K\rho^{1+\frac{1}{n}}$$

Manipulating derivatives and using a derivative operator D

$$\frac{1}{\rho}Dp = \frac{1}{\rho}D\left(K\rho^{1+\frac{1}{n}}\right) = K(n+1)D\left(\rho^{\frac{1}{n}}\right)$$

Hydrostatic equilibrium is

$$-\frac{\partial \Phi}{\partial r} = \frac{1}{\rho}\frac{\partial p}{\partial r}$$

Combing the previous two equations

$$\frac{d\rho^{1/n}}{d\Phi} = -\frac{1}{K(n+1)}$$

with solution

$$\rho^{\frac{1}{n}} = -\frac{\Phi}{K(n+1)} + \text{constant}$$

or

$$\rho = \rho_c \left(\frac{\Phi_T - \Phi}{\Phi_T - \Phi_c} \right)^n \tag{107}$$

with constants ρ_c, Φ_c the density and potential at the core and $\rho = 0, \Phi_T$ the density and potential at the surface. We define new variables

$$\begin{aligned} \theta &\equiv \frac{\Phi_T - \Phi}{\Phi_T - \Phi_c} \\ \xi &= \left(\frac{4\pi G \rho_c}{\Phi_T - \Phi_c} \right)^{1/2} r \end{aligned} \tag{108}$$

where θ ranges from 0 at the surface to 1 at the core and $\xi = 0$ at the core. Poisson's equation in spherical coordinates gives

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\Phi}{dr} \right) = 4\pi G \rho \tag{109}$$

Combining equations 107, 108, 109 gives

$$\frac{1}{\xi^2} \frac{d}{d\xi} \left(x^2 \frac{d\theta}{d\xi} \right) = -\theta^n \tag{110}$$

and this is known as the *Lane-Emden* equation. A boundary condition is $\frac{d\theta}{d\xi} = 0$ at $\xi = 0$ corresponding to no gradient in potential energy at the core (no force at the core). It is possible to solve the Lane-Emden equation analytically only for $n = 0, 1, 5$.

4. Microphysical Basis for Continuum Equations

4.1. **The particle distribution function.** To describe a distribution of particles we can consider a particle distribution function that depends on position, velocity and time, $f(\mathbf{x}, \mathbf{v}, t)$. Here $f(\mathbf{x}, \mathbf{v}, t) dx^3 du^3$ represents the number of particles found in a volume element of volume $\mathbf{dx^3}$ and in a velocity bin of size dv^3 at time t. The number density (number of particles per unit volume) at position $\mathbf{x}$ and at time t would be

$$n(x,t) = \int_{-\infty}^{\infty} f(\mathbf{x}, \mathbf{v}, t) d^3\mathbf{v}$$

where we perform the integral in 3 dimensions. If each particle has mass m then the density $\rho(\mathbf{x}, t) = mn(\mathbf{x}, \mathbf{t})$. We can consider the average of any quantity Q as

$$\langle Q \rangle = n^{-1} \int Q f(\mathbf{x}, \mathbf{v}, t) d^3\mathbf{v}$$

For example the bulk or average velocity would be

$$\mathbf{u} = n^{-1} \int \mathbf{v} f(\mathbf{x}, \mathbf{v}, t) d^3\mathbf{v}$$

and

$$\int v_i v_j f(\mathbf{x}, \mathbf{v}, t) d^3\mathbf{v} = n\langle v_i v_j \rangle \qquad \text{for} \qquad i \neq j$$

For a single component we can write

$$\int v_i^2 f(\mathbf{x}, \mathbf{v}, t) d^3\mathbf{v} = n\langle v_i^2 \rangle$$

But this is not necessarily the same as nu_i^2 which depends on the square of the average velocity. Usually

$$\langle v_i^2 \rangle \neq u_i^2 \qquad \langle v_i v_j \rangle \neq u_i u_j$$

We can define a total *velocity dispersion*, σ_a, averaged over all directions, as

$$\begin{aligned} \sigma_a^2 &\equiv \frac{1}{3}\left(\langle (v_x - u_x)^2 \rangle + \langle (v_y - u_y)^2 \rangle + \langle (v_z - u_z)^2 \rangle\right) \\ &= \frac{1}{3n} \int |\mathbf{v} - \mathbf{u}|^2 f d^3\mathbf{v} \end{aligned}$$

Evaluating σ_a^2

$$\begin{aligned} \sigma_a^2 &= \frac{1}{3n} \int (v^2 + u^2 - 2\mathbf{u} \cdot \mathbf{v}) d^3\mathbf{v} \\ &= \frac{1}{3}(\langle v^2 \rangle + u^2) - \frac{2}{3n}\mathbf{u} \cdot \int \mathbf{v} d^3\mathbf{v} \\ &= \frac{1}{3}(\langle v^2 \rangle + u^2) - \frac{2}{3}u^2 \\ &= \frac{1}{3}(\langle v^2 \rangle - u^2) \end{aligned}$$

so we can write

$$n\langle v^2 \rangle = \int v^2 f d^3\mathbf{f} = n(u^2 + 3\sigma_a^2)$$

We can think about the velocity v_i as a sum of the mean velocity u_i plus a random component. Let

$$w_{ij} \equiv \langle (v_i - u_i)(v_j - u_j) \rangle = \langle v_i v_j \rangle - u_i u_j$$

Here w_{ij} is a symmetric dispersion tensor with two indexes where each index can assume one of three values (x, y, z). When w_{ij} contains off diagonal components or its diagonal components are not equal we say the dispersion tensor is "anisotropic."

If the system is "isotropic" then the diagonal components would all be the same and the off diagonal components would be zero.

We can write the trace of w as w_{ii} in summation notation and

$$\sigma_a^2 = \frac{1}{3}(\langle v^2 \rangle - u^2) = \frac{w_{ii}}{3} = \frac{1}{3}\text{trace } w$$

If $w_{xx} = w_{yy} = w_{zz}$ then $\sigma_a^2 = w_{xx}$. The dispersion tensor is symmetric. We can decompose the dispersion tensor, w_{ij}, into the sum of a trace component that has zeros off the diagonal and a *symmetric traceless component*, y_{ij};

$$y_{ij} = \frac{w_{ij} + w_{ji}}{2} - \text{trace } w\frac{\delta_{ij}}{3} = \frac{w_{ij} + w_{ji}}{2} - \sigma_a^2 \delta_{ij}$$

Note that y_{ij} can contain components on the diagonal but their sum would be zero. If the system is isotropic then all components of y_{ij} would be zero.

We will associate pressure with the trace of w_{ij} or σ_a^2.

4.2. **The Boltzmann equation.** In the absence of collisions the collisionless Boltzmann equation describes the evolution of the density distribution.

$$\frac{Df}{Dt} = \frac{\partial f(\mathbf{x}, \mathbf{v}, t)}{\partial t} + \frac{\partial f(\mathbf{x}, \mathbf{v}, t)}{\partial \mathbf{x}} \cdot \frac{d\mathbf{x}}{dt} + \frac{\partial f(\mathbf{x}, \mathbf{v}, t)}{\partial \mathbf{v}} \cdot \frac{d\mathbf{v}}{dt} = 0.$$

The derivative here is done with respect to all degrees of freedom of the distribution function. As $\mathbf{v} = d\mathbf{x}/dt$ and $d\mathbf{v}/dt = -\nabla\Phi$ for a force field with potential Φ we can write

$$\frac{\partial f(\mathbf{x}, \mathbf{v}, t)}{\partial t} + \boldsymbol{\nabla}_x f(\mathbf{x}, \mathbf{u}, t) \cdot \mathbf{v} - \boldsymbol{\nabla}_v f(\mathbf{x}, \mathbf{v}, t) \cdot \boldsymbol{\nabla}\Phi = 0. \tag{111}$$

We have used gradient operators

$$\boldsymbol{\nabla}_x = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \qquad \boldsymbol{\nabla}_v = \left(\frac{\partial}{\partial v_x}, \frac{\partial}{\partial v_y}, \frac{\partial}{\partial v_z}\right)$$

Below we often drop the x subscript on the spatial gradient operator. Equation 111 is known as the collisionless Boltzmann equation. It is used to study the kinetic theory of gases, atomic nuclei and for stellar dynamical systems such as galaxies and globular clusters. The collisionless Boltzmann equation is sufficiently complex that it is usually difficult to solve. Equation 111 is sometimes written

$$\frac{Df}{Dt} = 0$$

where the Lagrangian derivative is

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \boldsymbol{\nabla}_x - \boldsymbol{\nabla}\Phi \cdot \boldsymbol{\nabla}_v$$

Here the Lagrangian derivative describes a small element moving in *phase space* or $(\mathbf{x}, \mathbf{v})$. Previously we used a Lagrangian derivative for a small element moving only in Cartesian space.

When collisions are important we can use the full Boltzmann equation by adding a source term that is due to collisions

$$\frac{Df}{Dt} = \left(\frac{\partial f}{\partial t}\right)_C$$

where the term on the right hand side depends on the cross sections of particles and their velocity differences. In many situations collisions conserve mass, momentum and kinetic energy. When these are conserved

$$\int m \left(\frac{\partial f}{\partial t}\right)_C d^3\mathbf{v} = 0$$
$$\int m\mathbf{v} \left(\frac{\partial f}{\partial t}\right)_C d^3\mathbf{v} = 0$$
$$\int mv^2 \left(\frac{\partial f}{\partial t}\right)_C d^3\mathbf{v} = 0$$

4.3. **Conservation of mass.** The simplest continuum equation can be made by integrating the Boltzmann equation over all possible velocities. The first term gives us the time derivative of the particle density as shown in the previous equation. Integrating the first term in the collisionless Boltzmann equation

$$\int_{-\infty}^{\infty} \frac{\partial f(\mathbf{x}, \mathbf{v}, t)}{\partial t} d^3\mathbf{v} \approx \frac{\partial}{\partial t} n(\mathbf{x}, t)$$

As derivatives with $\mathbf{x}$ and $\mathbf{v}$ commute we can integrate the second term in the following way

$$\int_{-\infty}^{\infty} \boldsymbol{\nabla} f(\mathbf{x}, \mathbf{v}, t) \cdot \mathbf{v}\, d^3\mathbf{v} = \boldsymbol{\nabla} \cdot \int f\mathbf{v} d^3\mathbf{v} = \boldsymbol{\nabla} \cdot (n\mathbf{u})$$

where we have rewritten the last term in terms of the average velocity $\mathbf{u}$. The last term in the collisionless Boltzmann equation

$$-\int \boldsymbol{\nabla}_v f(\mathbf{x}, \mathbf{v}, t) \cdot \boldsymbol{\nabla}\Phi d^3\mathbf{v}$$

we can integrate and write in terms of f at infinity and so will be zero. Putting these together with the integral of the collision term (also zero) we find

$$\frac{\partial n}{\partial t} + \boldsymbol{\nabla} \cdot (n\mathbf{u}) = 0$$

To summarize: the integral over velocity space of the Boltzmann equation gives an equation that looks just like the equation for conservation of mass for a fluid.

4.4. **Conservation of momentum.** To derive an equation similar to Euler's equation (conservation of momentum) we multiply the Boltzmann equation by $\mathbf{v}$ and then again integrate over velocity space. Taking the i-the component of the velocity and using summation notation for the other indices

$$\int \left(\frac{\partial f}{\partial t} v_i + \frac{\partial f}{\partial x_j} v_j v_i - \frac{\partial f}{\partial v_j} \frac{\partial \Phi}{\partial x_j} v_i \right) d^3\mathbf{v} = \int \left(\frac{\partial f}{\partial t} \right)_C d^3\mathbf{v} = 0 \tag{112}$$

Consider the first term

$$\int \frac{\partial f}{\partial t} v_i d^3\mathbf{v} = \frac{\partial}{\partial t} \int f v_i d^3\mathbf{v} = \frac{\partial (n v_i)}{\partial t}$$

Consider the second term of equation 112. This can be written

$$\int \frac{\partial f}{\partial x_j} v_j v_i d^3\mathbf{v} = \frac{\partial}{\partial x_j} [n \langle v_j v_i \rangle]$$

We can decompose this in terms of the dispersion tensor ($\mathbf{w}$) and then the traceless component of the dispersion tensor ($\mathbf{y}$) and the average dispersion (σ_a^2)

$$\begin{aligned} \frac{\partial}{\partial x_j} [n \langle v_j v_i \rangle] &= \frac{\partial}{\partial x_j} [n(u_i u_j + w_{ij})] \\ &= \frac{\partial}{\partial x_j} [n(u_i u_j + \sigma_a^2 \delta_{ij} + y_{ij})] \\ &= \frac{\partial}{\partial x_j} [n(u_i u_j + y_{ij}) + P\delta_{ij}] \end{aligned}$$

where we define a pressure in terms of the trace of the dispersion tensor

$$P \equiv n\sigma_a^2 = \frac{n w_{ii}}{3}.$$

Altogether the second term in the momentum equation (112) becomes

$$\frac{\partial}{\partial x_j} (n u_i u_j + P\delta_{ij} + n y_{ij})$$

We recognize the first two terms inside the derivative, $n u_i u_j + P\delta_{ij}$ as resembling the stress tensor. The last term $n y_{ij}$ depends in the traceless component of the dispersion tensor and is only non-zero when the velocity distribution is *anisotropic.*

The third term in the momentum equation (112) requires us to evaluate

$$\begin{aligned}\frac{\partial \Phi}{\partial x_j}\int \frac{\partial f}{\partial v_j} v_i \, d^3\mathbf{v} &= \frac{\partial \Phi}{\partial x_j}\delta_{ij}\int\int (f v_i|_{-\infty}^{\infty} - \int f dv_i) dv_k dv_l \\ &= -n\frac{\partial \Phi}{\partial x_j}\delta_{ij} = -n\frac{\partial \Phi}{\partial x_i}\end{aligned}$$

We integrate the integral by parts finding a non-zero term unless $i = j$. So the third term in equation 112 becomes

$$+n\frac{\partial \Phi}{\partial x_i}$$

Altogether our equation for conservation of momentum becomes

$$\frac{\partial}{\partial t}(n u_i) + \frac{\partial}{\partial x_j}(n u_i u_j + P\delta_{ij} + n y_{ij}) + n\frac{\partial \Phi}{\partial x_i} = 0$$

This is familiar! Except for the term associated with anisotropy this looks just like our relation for conservation of momentum in conservation law form.

By making use of the equation of continuity we can manipulate this equation so that it becomes a force equation that resembles Euler's equation

$$\frac{D\mathbf{u}}{Dt} = -\frac{1}{n}\boldsymbol{\nabla} P - \boldsymbol{\nabla}\Phi - \frac{1}{n}\boldsymbol{\nabla}\cdot(n\mathbf{y})$$

where the last term is a divergence of the traceless component of the dispersion tensor. If the velocity dispersion is isotropic then $\mathbf{y} = 0$ and we recover Euler's equation. To summarize: by multiplying the Boltzmann equation by velocity and integrating over all velocities we recover an equation that looks remarkably like Euler's equation.

Here we have integrated over velocity. If one integrates over all space instead one can derive tensor "virial" equations. Integrating only over velocity and working in cylindrical or spherical coordinates the equations, and in the setting of stellar dynamics, the equations are called the *Jeans equations*.

4.5. **Validity of a continuum fluid approximation.** When collisions are frequent we expect the distribution function to become Maxwellian. A crude approximation known as the BGK approximation for the collision term that approximates this condition is

$$\left(\frac{\partial f}{\partial t}\right)_c \approx -\frac{1}{\tau}(f - f_M) \tag{113}$$

where f_M is a Maxwellian distribution that has the same n, u and velocity dispersion as f. The timescale τ is known as the *relaxation time*. We expect τ to be approximately the mean free flight time. The continuum or fluid approximation is good if the collision time dominates and so characteristic timescales are longer than τ. An equivalent way to describe this is to require that the mean free path, λ be smaller than the characteristic lengthscale L or $\lambda \ll L$. The mean free flight time and mean free path are related by $\lambda \sim \sigma\tau$ where σ is the velocity dispersion.

Above we have used a distribution function to show that moments of the distribution function look similar to equations describing conservation of mass and momentum for a fluid. When collisions are taken into account in a more detailed way dissipative or non-conservative processes can be modeled such as viscosity and transport effects such as heat conduction.

5. Extremely short Introduction to relativistic hydrodynamics

To make sure expressions are relativistically invariant (under Lorenz transformations or boosted frame of reference) it is desirable to work with 4 vectors. In a particular reference frame the 4 momentum of a massive particle or photon

$$\mathbf{P} = (E, \mathbf{p}) \tag{114}$$

where E is the energy and $\mathbf{p}$ is the momentum (a three dimensional vector).

From here we set the speed of light $c = 1$. A pseudo metric is

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{115}$$

with

$$g_{00} = -1 \qquad g_{11} = g_{22} = g_{33} = 1 \qquad g_{ij,i\neq j} = 0 \tag{116}$$

The metric gives a Minkowski norm

$$|\mathbf{P}|^2 = g_{\mu\nu} P^\mu P^\nu \tag{117}$$

The norm $|\mathbf{P}|^2 = -m^2$ for a massive particle and 0 for a photon.

A particle with mass m, that is moving with velocity $\mathbf{v}$ with respect to an observer has four-momentum (as seen by the observer)

$$\mathbf{P} = (\gamma m, \gamma \boldsymbol{\beta} m) \tag{118}$$

where $\gamma = (1 - \beta^2)^{-1/2}$ and $\boldsymbol{\beta} = \mathbf{v}/c$ with $\mathbf{v}$ the velocity in three dimensions. To restore units multiply the energy by c^2 and the momentum by c. Either the space or time parts of the metric then gain a factor of c^2.

A Lorenz boost transforms four-vectors

$$\boldsymbol{\Lambda}(\boldsymbol{\beta}) = \begin{pmatrix} \gamma & -\gamma\beta_1 & -\gamma\beta_2 & -\gamma\beta_3 \\ -\gamma\beta_1 & (\gamma-1)\frac{\beta_1^2}{\beta^2} + 1 & (\gamma-1)\frac{\beta_1\beta_2}{\beta^2} & (\gamma-1)\frac{\beta_1\beta_3}{\beta^2} \\ -\gamma\beta_2 & (\gamma-1)\frac{\beta_2\beta_1}{\beta^2} & (\gamma-1)\frac{\beta_2^2}{\beta^2} + 1 & (\gamma-1)\frac{\beta_2\beta_3}{\beta^2} \\ -\gamma\beta_3 & (\gamma-1)\frac{\beta_3\beta_1}{\beta^2} & (\gamma-1)\frac{\beta_3\beta_2}{\beta^2} & (\gamma-1)\frac{\beta_3^2}{\beta^2} + 1 \end{pmatrix} \tag{119}$$

from one relativistic coordinate frame to another. Here $\boldsymbol{\beta} = (\beta_1, \beta_2, \beta_3)$. Lorenz boosts conserve the Minkowski norm.

A particle or an observer can be described by a 4-velocity $\mathbf{u}$. In the observer's frame, he or she is not moving so the 4-velocity $\mathbf{u} = (1, 0, 0, 0)$. The norm $|\mathbf{u}|^2 = -1$. In a different frame $\mathbf{u} = (\gamma, \gamma\boldsymbol{\beta})$. The first component of $\mathbf{u}$ describes time dilation, or how time advances in the observer's frame compared to the reference frame in which the 4-vector is written. $u^0 = \frac{dt}{d\tau}$ relating world time to proper time. For hydrodynamics a relevant quantity is the velocity of a fluid element. We describe this with a 4-velocity $\mathbf{u}$.

Baryon conservation can be written with 4-vectors

$$\boldsymbol{\nabla} \cdot (n\mathbf{u}) = 0 \qquad \text{or} \qquad (nu^i)_{,i} = 0 \tag{120}$$

where n is the number of baryons per unit volume. In an observer frame $\mathbf{u} = (\gamma, \gamma\boldsymbol{\beta})$ which in the non-relativistic limit is $\mathbf{u} \to (1, \mathbf{v})$ with $\mathbf{v}$ the three-vector velocity. Inserting this into above relation we find

$$\frac{\partial n}{\partial t} + \boldsymbol{\nabla} \cdot (n\mathbf{v}) = 0 \tag{121}$$

where the divergence is in three-dimensions. This looks like a normal conservation law, giving us some confidence that the relativistic version is correct.

The condition for adiabatic motions in terms of the 4-velocity is similar

$$\boldsymbol{\nabla} \cdot (s\mathbf{u}) = 0 \qquad \text{or} \qquad (su^i)_{,i} = 0 \tag{122}$$

where s is the entropy per unit volume.

In the frame moving with the fluid the **energy momentum tensor** or **stress-energy tensor** is

$$\mathbf{T} = \begin{pmatrix} e & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix} \tag{123}$$

with p the pressure and e the energy density. Ignoring internal degrees of freedom and photons $e = \rho c^2$. The T^{00} component gives the energy density, and the other diagonal components, T^{xx}, T^{yy}, T^{zz}, the momentum flux or the pressure.

In another coordinate system, the stress energy tensor (for an ideal fluid) is more generally

$$\mathbf{T} = (e + p)\mathbf{u} \otimes \mathbf{u} + p\mathbf{g} \tag{124}$$

with $\mathbf{g}$ the metric tensor with $g_{00} = -1, g_{11} = g_{22} = g_{33} = 1$ and $g_{ij,i\neq j} = 0$. Recall that $\mathbf{u}$ is the flow four vector of the fluid. Using indices of the four vector $\mathbf{u}$

$$T^{ij} = (e + p)u^i u^j + pg^{ij}, \tag{125}$$

or if we lower indices with the metric tensor

$$T_{ij} = (e + p)u_i u_j + p g_{ij}. \tag{126}$$

Here components T^{0i} or T^{i0} with $i > 0$ describe energy flux or momentum density. The components T^{ij} with $i, j > 0$ describe components of momentum flux, as was true when we worked with the stress tensor. We can compare the stress-energy tensor to the stress tensor we previously used in the non-relativistic limit

$$\boldsymbol{\pi} = p\mathbf{g} + \rho \mathbf{u} \otimes \mathbf{u} \tag{127}$$

where $\mathbf{u}$ is only a 3-vector. The two tensors are similar in form, and that's comforting.

The stress energy tensor is a tensor with two indices. Each index of a tensor transfers separately during a coordinate transformation. Here the coordinate transformation is the Lorenz boost.

$$\tilde{T}^{ij} = T^{kl} \Lambda^i_k \Lambda^j_l \tag{128}$$

Using a Lorenz boost we can transform equation 123 into equation 124.

The 4-momentum is conserved, giving rise to four conservation laws. Conservation of energy and momentum is equivalent to

$$T^{ij}_{\ \ ,j} = 0 \tag{129}$$

Writing this out

$$\frac{\partial T^{i0}}{\partial x_0} + \frac{\partial T^{i1}}{\partial x_1} + \frac{\partial T^{i2}}{\partial x_2} + \frac{\partial T^{i3}}{\partial x_3} = 0 \tag{130}$$

or

$$\frac{\partial T^{it}}{\partial t} + \frac{\partial T^{ix}}{\partial x} + \frac{\partial T^{iy}}{\partial y} + \frac{\partial T^{iz}}{\partial z} = 0 \tag{131}$$

for each index i.

6. Acknowledgements

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