AST242 LECTURE NOTES PART 6

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1. Ideal Magnetohydrodynamics (MHD)

We start with Maxwell's equations in rationalized MKS units,

$$\nabla \cdot \mathbf{B} = 0 \tag{1}$$

$$\nabla \cdot \mathbf{E} = \frac{q}{\epsilon_0} \tag{2}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} \tag{3}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{4}$$

Here $\mathbf{j}$ is the current density and q is the charge density. In these units the parameter μ_0 is the magnetic permeability of vacuum and ϵ_0 is the dielectric constant of vaccum. The speed of light $c^2 = 1/(\mu_0\epsilon_0)$.

Ohm's law

$$\mathbf{j} = \sigma(\mathbf{E} + \mathbf{u} \times \mathbf{B}) \tag{5}$$

where σ is the electric conductivity. This follows as material moving with velocity $\mathbf{u}$ is subject to a total electric field $\mathbf{E} + \mathbf{u} \times \mathbf{B}$.

For ideal MHD we are going to take the non-relativistic limit and that of high conductivity. For the non-relativistic limit we ignore the term $\frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}$ in the third of Maxwell's equations above. Thus we approximate

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} = \mu_0 \sigma(\mathbf{E} + \mathbf{u} \times \mathbf{B}) \tag{6}$$

where I have used Ohm's law for $\mathbf{j}$. Taking the curl of the equation

$$\nabla \times (\nabla \times \mathbf{B}) = \mu_0 \sigma(\nabla \times \mathbf{E} + \nabla \times (\mathbf{u} \times \mathbf{B})) \tag{7}$$

The vector identity

$$\nabla \times (\nabla \times \mathbf{B}) = -\nabla^2 \mathbf{B} - \nabla(\nabla \cdot \mathbf{B}) = -\nabla^2 \mathbf{B} \tag{8}$$

since $\nabla \cdot \mathbf{B} = 0$. Using the vector identity and the last of Maxwell's equations, equation (7) becomes

$$-\nabla^2 \mathbf{B} = \mu_0 \sigma \left(-\frac{\partial \mathbf{B}}{\partial t} + \nabla \times (\mathbf{u} \times \mathbf{B}) \right) \tag{9}$$

We rewrite this

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) + \frac{1}{\sigma \mu_0} \nabla^2 \mathbf{B} \tag{10}$$

1.1. **Magnetic diffusivity and Magnetic Reynolds Number.** If the fluid is not moving, $\mathbf{u} = 0$, then the above equation looks like

$$\frac{\partial \mathbf{B}}{\partial t} = \frac{1}{\sigma \mu_0} \nabla^2 \mathbf{B} \tag{11}$$

which is a diffusion equation and we can define the diffusion coefficient

$$\eta \equiv \frac{1}{\sigma \mu_0} \tag{12}$$

This coefficient, η, is called the **magnetic diffusivity**. The magnetic field diffuses or leaks through the material. For laboratory conductors the decay timescale for oppositely directed fields to decay is fairly short. However in astrophysical plasmas the decay timescale can be long.

We can define what is called a magnetic Reynolds number, $\mathcal{R}_M$, using our diffusion coefficient, the magnetic diffusivity,

$$\mathcal{R}_M \equiv \frac{LV}{\eta} \tag{13}$$

for a system with size-scale L and typical velocity V. If $\mathcal{R}_M$ is large then largescale motions dominate leakage. The MHD approximation is a good when when the magnetic Reynolds number $\mathcal{R}_M \gg 1$ is large.

1.2. **Magnetic Field Lines Frozen In.** In the limit of high conductivity equation (10) becomes

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) \tag{14}$$

This equation is sometimes called the induction equation. This equation is in exactly the same form for as the evolution of vorticity in an inviscid barotropic flow. Thus we can make the same conclusions: the magnetic field is frozen into the fluid as it moves. The magnetic flux through a surface remains constant as this surface moves in the fluid. Field lines move with the fluid.

In the limit of high conductivity Ohm's law

$$\frac{\mathbf{j}}{\sigma} = \mathbf{E} + \mathbf{u} \times \mathbf{B} \to 0 \qquad \text{or} \qquad \mathbf{E} = -\mathbf{u} \times \mathbf{B} \tag{15}$$

If we take $\mathbf{B}\cdot$ both sides of the equation we find

$$\mathbf{E} \cdot \mathbf{B} = 0 \tag{16}$$

implying that the electric and magnetic fields are perpendicular.

1.3. **The Lorentz force.** We consider that the plasma is composed of positive and negative particles. Each one obeys a continuity equation

$$\frac{\partial n^+}{\partial t} + \nabla \cdot (n^+ \mathbf{u}^+) = 0 \qquad \frac{\partial n^-}{\partial t} + \nabla \cdot (n^- \mathbf{u}^-) = 0 \tag{17}$$

where n^+, n^- are the number densities. We can define a bulk density and bulk velocity by

$$\rho = m^+ n^+ + m^- n^- \tag{18}$$

$$\mathbf{u} = \frac{m^+ n^+ \mathbf{u}^+ + m^- n^- \mathbf{u}^-}{m^+ n^+ + m^- n^-} \tag{19}$$

where m^+, m^- are the masses of the particles. Using these definitions conservation of mass is as usual

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \tag{20}$$

We define a charge and current density

$$q = n^+ e^+ + n^- e^- \tag{21}$$

$$\mathbf{j} = n^+ e^+ \mathbf{u}^+ + n^- e^- \mathbf{u}^- \tag{22}$$

We now consider the momentum equations for each component of the fluid taking into account the Lorentz force

$$\mathbf{F} = e(\mathbf{E} + \mathbf{u} \times \mathbf{B}) \tag{23}$$

$$\begin{aligned} m^+ n^+ \left(\frac{\partial \mathbf{u}^+}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u}^+ \right) &= e^+ n^+ (\mathbf{E} + \mathbf{u}^+ \times \mathbf{B}) - f^+ \nabla p \\ m^- n^- \left(\frac{\partial \mathbf{u}^-}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u}^- \right) &= e^- n^- (\mathbf{E} + \mathbf{u}^- \times \mathbf{B}) - f^- \nabla p \end{aligned} \tag{24}$$

If we add the pressure terms we will get the total pressure force on the fluid element. We sum the previous two equations to find

$$\rho \left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} \right] = q\mathbf{E} + \mathbf{j} \times \mathbf{B} - \nabla p \tag{25}$$

Let's reconsider equation (6) (Maxwell's equation but neglecting the relativistic term)

$$\mu_0 \mathbf{j} = \nabla \times \mathbf{B} \tag{26}$$

Using this equation the Lorentz force becomes

$$q\mathbf{E} + \frac{1}{\mu_0}(\nabla \times \mathbf{B}) \times \mathbf{B} \tag{27}$$

The term proportional to $\mathbf{E}$ we assume is negligible because we do not expect charge separation. With this in mind our equation for momentum conservation (Euler's equation) becomes

$$\rho \left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} \right] = \frac{1}{\mu_0}(\nabla \times \mathbf{B}) \times \mathbf{B} - \nabla p \tag{28}$$

The right hand side shows that our fluid element has an additional force on it due to the magnetic field.

1.4. **Magnetic stress tensor.** It may be useful later on to write our equation for conservation of momentum in conservation law form. For our Lorentz force density

$$\mathbf{f} = \mu_0^{-1}(\nabla \times \mathbf{B}) \times \mathbf{B} \tag{29}$$

Since the force is a cross product with $\mathbf{B}$ the force is always perpendicular to the field lines.

We use summation notation to find a different form for the force density,

$$\begin{aligned} f_i &= \epsilon_{ijk}\epsilon_{jlm}B_{m,l}B_k \\ &= \epsilon_{jki}\epsilon_{jlm}B_{m,l}B_k \\ &= (\delta_{kl}\delta_{im} - \delta_{km}\delta_{il})B_{m,l}B_k \\ &= B_{i,k}B_k - B_{k,i}B_k \end{aligned} \tag{30}$$

so that

$$\mathbf{f} = \mu_0^{-1}\left[(\mathbf{B}\cdot\nabla)\mathbf{B} - \nabla\left(\frac{B^2}{2}\right)\right] \tag{31}$$

With this force Euler's equation becomes

$$\rho\left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right] = \mu_0^{-1}\left[(\mathbf{B}\cdot\nabla)\mathbf{B} - \nabla\left(\frac{B^2}{2}\right)\right] - \nabla p \tag{32}$$

The $\mathbf{B}\cdot\nabla)\mathbf{B}$ term is referred to as the force due to magnetic tension. The term proportional to ∇B^2 is described as a force due to magnetic pressure.

We can also describe $\mathbf{f}$ in terms of a divergence

$$\mathbf{f} = \nabla\cdot\mathbf{m} \qquad \text{with} \qquad f_i = \frac{\partial m_{ij}}{\partial x_j} \tag{33}$$

where $\mathbf{m}$ is a magnetic stress tensor and

$$\mathbf{m} = \mu_0^{-1}\left(\mathbf{B}\otimes\mathbf{B} - \frac{1}{2}B^2\mathbf{g}\right) \tag{34}$$

or with indexes

$$m_{ij} = \mu_0^{-1}\left(B_iB_j - \frac{1}{2}B^2\delta_{ij}\right) \tag{35}$$

Here $\mathbf{g}$ is the coordinate tensor we used previous for the part of the stress tensor dependent on pressure. Note that the second term is diagonal. In our equation for conservation of momentum this term would become $-\nabla\left(\frac{B^2}{2\mu_0}\right)$. Its sign is the same as the pressure term. This means it acts just like a pressure. The magnetic pressure

$$P_{mag} = \frac{B^2}{2\mu_0} \tag{36}$$

in rationalized MKS units with B in Tesla. In cgs

$$P_{mag} = \frac{B^2}{8\pi} \tag{37}$$

with B in units of Gauss. The other term in our force density, $\mu_0^{-1}(\nabla\cdot\mathbf{B})\mathbf{B}$, is described as a magnetic tension along the field lines. It arises from a term in the magnetic stress tensor that looks similar to the ram pressure term we saw in the stress tensor for our equation for conservation of momentum. In conservation law form we can write our conservation of momentum as

$$\frac{\partial(\rho \mathbf{u})}{\partial t} + \nabla \cdot (\boldsymbol{\pi} - \mathbf{m}) = -\rho \nabla \Phi \tag{38}$$

where $\boldsymbol{\pi} = p\mathbf{g} + \rho \mathbf{u} \otimes \mathbf{u}$ and the magnetic stress tensor $\mathbf{m}$ is given by equation (34).

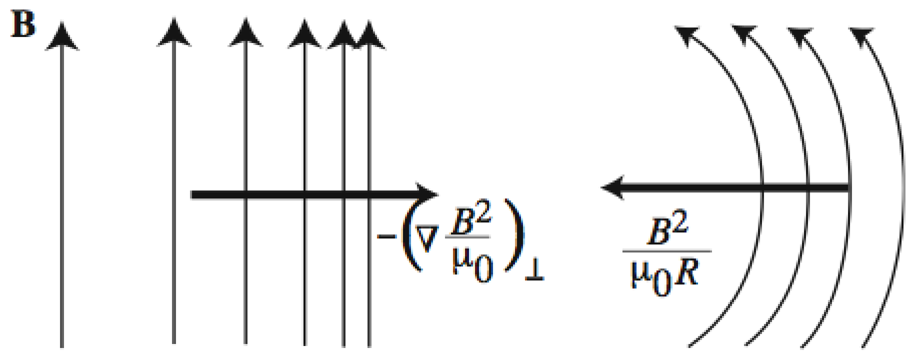

FIGURE 1. The force due to magnetic pressure is shown on the left. The curvature force due to magnetic tension is shown on the right. Both are perpendicular to the field lines. This figure is taken from Kip Thorne's notes.

1.5. **Magnetic energy.** The magnetic energy density is

$$\frac{B^2}{2\mu_0} \tag{39}$$

and is then included as an additional term in our total energy per unit volume E. The total energy flux picks up an an additional term the Poynting flux, $\mu_0^{-1}\mathbf{E} \times \mathbf{B}$, which we can neglect in the MHD approximation because we neglect the electric field. Just as viscosity increases entropy through viscous dissipation, electrical conductivity and associated magnetic dissipation will give an entropy increase through Ohmic dissipation. Our equation for conservation of energy

$$\frac{\partial}{\partial t}\left[\rho\left(\frac{u^2}{2} + e + \Phi\right) + \frac{B^2}{2\mu_0}\right] + \nabla \cdot \left[\rho \mathbf{u}\left(\frac{u^2}{2} + h + \Phi\right) + \frac{\mathbf{E} \times \mathbf{B}}{\mu_0}\right] = 0 \tag{40}$$

where I have neglected $\dot{\Phi}$, viscous dissipation, self-gravity, heating and cooling and thermal electric conduction.

1.6. **The equations of ideal MHD.** We have assumed sub-relativistic flow, and high conductivity. In this limit we find that the magnetic field is frozen into the fluid. We also assume no charge separation. Let us summarize the equations needed to describe the system.

Conservation of mass

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \tag{41}$$

Conservation of momentum

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} \right) = \frac{1}{\mu_0} (\nabla \times \mathbf{B}) \times \mathbf{B} - \nabla p - \rho \nabla \Phi \tag{42}$$

where I have added a gravitational term.

And because we now have an additional vector field we need to evolve it as well. We have the induction equation

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) \tag{43}$$

These three equations (along with an equation of state, and an equation for conservation of energy and $\nabla \cdot \mathbf{B} = 0$) represent the ideal MHD approximation.

1.7. **Notation and Units.** The electric field has dropped out of all the equations. The magnetic field appears in the induction equation and in the Lorentz force. The induction equation contains B on both sides of the equation so it doesn't matter what units are used for B. For the Lorentz force the magnetic field enters in as B^2/μ_0, in rationalized MKS units so the magnetic field is in Tesla. We could replace B^2/μ_0 with $B^2/(4\pi)$ and work in cgs with magnetic field in units of Gauss.

1.8. **MHD Magnetostatics.** We look at our evolution equations setting all time derivatives to zero and setting the velocity to zero. We find only one interesting equation which is a hydrostatic equilibrium equation

$$\frac{1}{\mu_0} (\nabla \times \mathbf{B}) \times \mathbf{B} = \nabla p + \rho \nabla \Phi \tag{44}$$

This equation is used to look at the structures of coronal loops, sunspots and the magnetic field above the magnetic plane.

Let us reformulate the magnetostatic equation above in terms of a vector potential, $\mathbf{A}$

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{45}$$

This automatically satisfies $\nabla \cdot \mathbf{B} = 0$.

$$\nabla \times \mathbf{B} = \nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} \tag{46}$$

Now consider the Lorenz force

$$\mathbf{f} = \mu_0^{-1}(\nabla \times \mathbf{B}) \times \mathbf{B} = \mu_0^{-1}\left[\nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A}\right] \times (\nabla \times \mathbf{A}) \tag{47}$$

In two dimensions this is much simpler. Assume for example that $\mathbf{B}$ lies in the $\mathbf{y}, \mathbf{z}$ plane. Then we may choose $\mathbf{A}$ to have a single component in the $\mathbf{x}$ direction that only depends on y, z. In this case the above is particularly simple since $\nabla \cdot \mathbf{A} = 0$

$$\mathbf{f} = -\mu_0^{-1}(\nabla^2 A)\boldsymbol{\nabla} A \tag{48}$$

Why did I do this exactly? I thought it would be more interesting than it was.

Lots more can be done here! For example we can look at how magnetic pressure can give buoyancy. There is a nice section on in Pringle and King on magnetic instability in a static atmosphere. Clarke and Carswell discuss magnetic stabilization of the Rayleigh-Taylor instability.

2. Alfvén and magnetosonic waves

Alfvén waves are hydromagnetic waves in an incompressible fluid. Here we follow the derivation by Jackson, and don't assume incompressibility just yet. If the gas is not necessarily assumed to be incompressible then we would call the waves magnetosonic waves.

We start with our equations of ideal MHD:

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u}\right) = \frac{1}{\mu_0}(\nabla \times \mathbf{B}) \times \mathbf{B} - \nabla p \tag{49}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) \tag{50}$$

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \mathbf{u}) \tag{51}$$

where the first equation is for momentum, the second is the induction equation and the last equation the continuity equation. We assume unperturbed flow with $\mathbf{B}_0$ independent of position and $\mathbf{u}_0 = 0$.

We linearize the three equations only taking first order terms

$$\frac{\partial \rho_1}{\partial t} = -\rho_0(\nabla \cdot \mathbf{u}_1) \tag{52}$$

$$\rho_0 \frac{\partial \mathbf{u}_1}{\partial t} = -c_s^2 \nabla \rho_1 - \mu_0^{-1}\mathbf{B_0} \times (\nabla \times \mathbf{b}_1) \tag{53}$$

$$\frac{\partial \mathbf{b}_1}{\partial t} = \nabla \times (\mathbf{u}_1 \times \mathbf{B}_0) \tag{54}$$

where c_s is the acoustic sound speed.

We take the time derivative of the second equation

$$\frac{\partial^2 \mathbf{u}_1}{\partial t^2} = -\frac{c_s^2}{\rho_0}\nabla\frac{\partial \rho_1}{\partial t} - \frac{\mathbf{B}_0}{\mu_0\rho_0} \times (\nabla \times \frac{\partial \mathbf{b}_1}{\partial t}) \tag{55}$$

$$= c_s^2\nabla(\nabla\cdot\mathbf{u}_1) - \frac{\mathbf{B}_0}{\mu_0\rho_0} \times (\nabla \times \nabla \times (\mathbf{u}_1 \times \mathbf{B}_0)) \tag{56}$$

where we have inserted expressions from the linearized equation of continuity and induction equation.

We define a velocity vector

$$\mathbf{v}_A = \frac{\mathbf{B}_0}{\sqrt{\mu_0\rho_0}} \tag{57}$$

and insert it into the previous equation

$$\frac{\partial \mathbf{u}_1}{\partial t^2} = c_s^2\nabla(\nabla\cdot\mathbf{u}_1) - \mathbf{v}_A \times (\nabla \times \nabla \times (\mathbf{u}_1 \times \mathbf{v}_A)) \tag{58}$$

Henceforth we will refer to the length of this velocity vector, $|\mathbf{v}_A|$, as the Alfven speed.

This looks pretty nasty but it is essentially a wave equation for $\mathbf{u}_1$. The velocity perturbation can be related to the magnetic field perturbation, $\mathbf{b}_1$, using the induction equation. Note that the leftmost term on the right hand side is zero if the fluid is incompressible.

We assume a perturbation in the following form

$$\mathbf{u} = \mathbf{u}_1 e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} \tag{59}$$

We derive the following relation

$$-\omega^2\mathbf{u}_1 = -c_s^2(\mathbf{k}\cdot\mathbf{u}_1)\mathbf{k} + \mathbf{v}_A \times (\mathbf{k}\times\mathbf{k}\times(\mathbf{u}_1\times\mathbf{v}_A)) \tag{60}$$

We can use the following vector identity

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B}) \tag{61}$$

so that

$$\mathbf{k}\times(\mathbf{u}_1\times\mathbf{v}_A) = \mathbf{u}_1(\mathbf{k}\cdot\mathbf{v}_A) - \mathbf{v}_A(\mathbf{k}\cdot\mathbf{u}_1) \tag{62}$$

Equation (60) becomes

$$\begin{aligned} -\omega^2\mathbf{u}_1 + c_s^2(\mathbf{k}\cdot\mathbf{u}_1)\mathbf{k} &= \mathbf{v}_A\times[\mathbf{k}\times(\mathbf{u}_1(\mathbf{k}\cdot\mathbf{v}_A) - \mathbf{v}_A(\mathbf{k}\cdot\mathbf{u}_1))] \\ &= \mathbf{v}_A\times(\mathbf{k}\times\mathbf{u}_1)(\mathbf{k}\cdot\mathbf{v}_A) + \mathbf{v}_A\times(\mathbf{v}_A\times\mathbf{k})(\mathbf{k}\cdot\mathbf{u}_1) \end{aligned} \tag{63}$$

We use the vector identity again

$$\begin{aligned} -\omega^2\mathbf{u}_1 + c_s^2(\mathbf{k}\cdot\mathbf{u}_1)\mathbf{k} &= [\mathbf{k}(\mathbf{v}_A\cdot\mathbf{u}_1) - \mathbf{u}_1(\mathbf{v}_A\cdot\mathbf{k})](\mathbf{k}\cdot\mathbf{v}_A) \\ &\quad + \left[\mathbf{v}_A(\mathbf{v}_A\cdot\mathbf{k}) - \mathbf{k}v_A^2\right](\mathbf{k}\cdot\mathbf{u}_1) \end{aligned} \tag{64}$$

or

$$-\omega^2 \mathbf{u}_1 + (c_s^2 + v_A^2)(\mathbf{k}\cdot\mathbf{u}_1)\mathbf{k} = (\mathbf{k}\cdot\mathbf{v}_A)\left[\mathbf{k}(\mathbf{v}_A\cdot\mathbf{u}_1) - \mathbf{u}_1(\mathbf{v}_A\cdot\mathbf{k}) + \mathbf{v}_A(\mathbf{k}\cdot\mathbf{u}_1)\right] \tag{65}$$

This equation should let us discuss all magnetosonic waves. Taking the limit of $c_s = 0$ is equivalent to assuming that the magnetic pressure is much larger than the gas pressure. In this case all waves propagate at the Alfven velocity.

2.1. **Wavevector is perpendicular to the magnetic field.** If $\mathbf{k}$ is perpendicular to $\mathbf{B}_0$ (and so $\mathbf{v}_A$) then the right hand side of Equation (65) is zero. In this case the above equation becomes

$$-\omega^2 \mathbf{u}_1 + (c_s^2 + v_A^2)(\mathbf{k}\cdot\mathbf{u}_1)\mathbf{k} = 0 \tag{66}$$

and $\mathbf{u}_1$ must be parallel to $\mathbf{k}$. This means the waves are *longitudinal*. The dispersion relation

$$\omega^2 = (c_s^2 + v_A^2)k^2 \tag{67}$$

so the wave speed is

$$s = \sqrt{c_s^2 + v_A^2} \tag{68}$$

It's a fast wave with velocity larger than both sound and Alfven velocities. Both magnetic and gas pressure resist the compression and so contribute to the restoring force.

2.2. **Wave vector parallel to the magnetic field, sonic and transverse waves.** If $\mathbf{k}$ is parallel to $\mathbf{B}_0$ then $\mathbf{k}(\mathbf{v}_A\cdot\mathbf{u}_1) = \mathbf{v}_A(\mathbf{k}\cdot\mathbf{u}_1)$. Equation (65) becomes

$$(v_A^2 k^2 - \omega^2)\mathbf{u}_1 = -(c_s^2 + v_A^2)(\mathbf{k}\cdot\mathbf{u}_1)\mathbf{k} + 2\mathbf{k}(\mathbf{v}_A\cdot\mathbf{u}_1)(\mathbf{k}\cdot\mathbf{v}_A) \tag{69}$$

This can be simplified by assuming $\mathbf{k} = a\mathbf{v}_A$ (with $a^2 = k^2/v_A^2$) to manipulate the equation

$$(v_A^2 k^2 - \omega^2)\mathbf{u}_1 = \left(1 - \frac{c_s^2}{v_A^2}\right) k^2 (\mathbf{v}_A\cdot\mathbf{u}_1)\mathbf{v}_A \tag{70}$$

We consider two cases:
a) $\mathbf{u}_1$ is parallel to k and $\mathbf{v}_A$. The waves are longitudinal. The dispersion relation is

$$(v_A^2 k^2 - \omega^2) = \left(1 - \frac{c_s^2}{v_A^2}\right) k^2 v_A^2 \tag{71}$$

which simplifies to

$$\omega^2 = c_s^2 k^2 \tag{72}$$

These are ordinary acoustic waves.

b) $\mathbf{u}_1$ is perpendicular to $\mathbf{v}_A$ so that $\mathbf{u}_1 \cdot \mathbf{v}_A = 0$ and

$$\omega^2 = k^2 v_A^2 \tag{73}$$

These are transverse waves (like seismic s waves) and are due to the magnetic tension.

Here we have considered simple cases which allow us to find simple solutions to equation 65. In general the wavevector have components both perpendicular and parallel to the field lines and the dispersion relation will depend on the angle between the wavevector and the field lines.

2.3. **The Alfvén speed.** The Alfvén speed

$$\begin{aligned} v_A &\equiv \sqrt{\frac{B_0^2}{\mu_0 \rho_0}} \qquad \text{MKS} \\ &\equiv \sqrt{\frac{B_0^2}{4\pi \rho_0}} \qquad \text{cgs} \end{aligned} \tag{74}$$

where each expressions depends on the units adopted. For MKS the magnetic field should be in units of Tesla. For cgs the magnetic field should be in units of Gauss. Remember a magnetic pressure could be defined with

$$\begin{aligned} P_{mag} &\equiv \frac{B^2}{2\mu_0} \qquad \text{MKS} \\ &\equiv \frac{B^2}{8\pi} \qquad \text{cgs} \end{aligned} \tag{75}$$

We can see that the square of the Alfvén speed is approximately the magnetic pressure divided by the density. Recall that the square of the acoustic sound speed is approximately equal to the gas pressure divided by the density. For longitudinal waves ($\mathbf{k}$ is parallel to $\mathbf{u}_1$) the magnetic pressure resists the compression of the fluid. Transverse waves don't exist in an ideal gas but do in a magnetic fluid where the magnetic field provides tension. The magnetic tension also scales with the square of the magnetic field and so transverse waves propagate with the Alfvén velocity.

2.4. **Classifying hydromagnetic waves from the Velocity perturbation direction.** Let us classify waves in another way. Consider the plane containing $\mathbf{k}$ and $\mathbf{B}$, and the velocity perturbation vector $\mathbf{u}_1$ either in this plane or perpendicular to it.

We first consider the case where $\mathbf{u}_1 \propto \mathbf{v}_A \times \mathbf{k}$, so $\mathbf{u}_1$ is perpendicular to the plane containing $\mathbf{B}$ and $\mathbf{k}$. In this case the dispersion relation (equation 65) becomes

$$-\omega^2 \mathbf{u}_1 = (\mathbf{k} \cdot \mathbf{v}_A)^2 \mathbf{u}_1 \tag{76}$$

We find $\omega^2 = v_A^2 k^2$ and have Alfven waves. Because $\mathbf{u}_1$ is perpendicular to $\mathbf{k}$ the waves are transverse only.

We now consider the case where $\mathbf{u}_1$ lies in the plane containing $\mathbf{B}$ and $\mathbf{k}$. Let $\mathbf{k}$ be in the positive $\mathbf{x}$ direction. Furthermore let ψ be the angle between the two vectors $\mathbf{v}_A$ and $\mathbf{k}$ so that $\mathbf{v_A} = v_A(\cos\psi, \sin\psi, 0)$ and $\mathbf{k}\cdot\mathbf{v}_A = kv_A\cos\psi$. The dispersion relation (equation 65) becomes

$$\begin{aligned} (-\omega^2 + (c_s^2 + v_A^2)k^2)u_{1,x} &= k^2 v_A^2 \cos\psi\left[(\cos\psi u_{1,x} + \sin\psi u_{1,y}) - \cos\psi u_{1,x} + \cos\psi u_{1,x}\right] \\ -\omega^2 u_{1,y} &= k^2 v_A^2 \cos\psi\left[-u_{1,y}\cos\psi + \sin\psi u_{1,x}\right] \end{aligned}$$

Simplify the first equation and use the second to solve for $u_{1,y}$

$$\left(-\omega^2 + (c_s^2 + v_A^2)k^2\right) u_{1,x} = k^2 v_A^2 \cos\psi(\sin\psi u_{1,y} + \cos\psi u_{1,x}) \tag{77}$$

$$u_{1,y} = (k^2 v_A^2 \cos^2\psi - \omega^2)^{-1} k^2 v_A^2 \sin\psi\cos\psi u_{1,x} \tag{78}$$

Inserting the expression for $u_{1,y}$ into the previous equation

$$\left[-\omega^2 + c_s^2 k^2 + (1-\cos^2\psi)v_A^2 k^2\right](k^2 v_A^2\cos^2\psi - \omega^2) = k^4 v_A^4 \cos^2\psi\sin^2\psi$$

$$\omega^4 - \omega^2 k^2(c_s^2 + v_A^2) + k^4 v_A^2 \cos^2\psi = 0 \tag{79}$$

which is a quartic equation for ω but only involving ω^4 and ω^2 and constant terms. The solution is

$$\frac{\omega^2}{k^2} = \frac{1}{2}\left[(v_a^2 + c_s^2) \pm \sqrt{(v_A^2 + c_s^2)^2 - 4v_A^2 c_s^2 \cos^2\psi}\right] \tag{80}$$

The positive roots are called “fast” waves and the negative ones are “slow” waves.

The previous two cases are independent. It would be nice to show this here.

Notes: For high enough frequencies the frequency can be comparable to the oscillation frequency of individual particles in the plasma. In this case the single fluid model breaks down.

3. Acknowledgements

Drawing heavily on Pringle and King, Clarke and Carswell and and old favorite of mine called Magnetohydrodynamics by T. G. Cowling, 1976. The figure is the same as Figure 18.3 from Kip Thorne’s notes. Following Jackson’s Classical Electrodynamics for Alfvén waves.