Homework set #6. AST 233, Fall 2024

On resonances and tidal evolution

Due date: Thursday Dec 12 2024, midnight. Please upload your solutions to blackboard.

1. **Tidal outward drift of the Moon**

   Due to tides excited in the Earth by the Moon, the semi-major axis a of the Earth/Moon's orbit approximately obeys a draft rate of

   $$|\dot{a}| = \frac{3k_{2,\oplus}}{Q_\oplus}\frac{M_m}{M_\oplus}\left(\frac{R_\oplus}{a}\right)^5 na \qquad (1)$$

   where $n = \sqrt{G(M_\oplus + M_m)/a^3}$ is the mean motion, M_m is the mass of the Moon, $M_\oplus$ is the mass of the Earth, and $R_\oplus$ is the mean equatorial radius of the Earth. The coefficients $k_{2\oplus}, Q_\oplus$ are the Love number and energy dissipation factor of the Earth.

   Lunar laser ranging experiments using the laser reflectors left by the Apollo and Lunakhod missions find that the Moon is receding from the Earth at a current rate of $\dot{a} \approx +10^{-9}\text{m s}^{-1}$. (Following Murray and Dermott's book!) We can rewrite the semi-major axis drift rate as

   $$\dot{a} = 10^{-9}\text{m s}^{-1}\left(\frac{a}{60.3R_\oplus}\right)^{-\frac{11}{2}} \qquad (2)$$

   $$= 5\times 10^{-9}R_\oplus\text{yr}^{-1}\left(\frac{a}{60.3R_\oplus}\right)^{-\frac{11}{2}} \qquad (3)$$

   where $a = 60.3R_\oplus$ is the Moon's current orbital semi-major axis.

   a. Find an expression for $a(t)$.

   b. Estimate how long ago the Moon's orbit had a semi-major axis of 1 Earth Radius. In other words, find t such that $a(t) = 1R_\oplus$.

   This time should be shorter than the age of the Earth, implying that the tidal drift rate must have been slower in the past!

   c. What is the ratio of the orbital angular momentum associated with the Moon's orbit and the spin angular momentum of the Earth?

   Hints: The moon's mass is about 1/100 of the Earth. The Earth's spin angular momentum is about

   $$L_s \approx \frac{2}{5}M_\oplus R_\oplus^2\Omega$$

   where $2\pi/\Omega = 1$ day. (Ignoring it's internal density distribution).n The orbital angular momentum of the moon is about

   $$L_o \approx M_m\sqrt{GM_\oplus a} = M_m n a^2$$

   where $2\pi/n = 1$ month. This problem is quick to do if you use the ratio of 1 month to 1 day, the ratio of the moon to Earth mass and the ratio of orbital semi-major axis to Earth's radius.

   Using the ratio of spin to orbital angular momentum infer how much faster the Earth could have been spinning just after the Earth/Moon system was born.

2. **Numerical problem on resonance capture.** This problem illustrates how you can produce stargrazing comets by migrating a planet inward. The problem is posted here and uses rebound: https://astro.pas.rochester.edu/~aquillen/ast233/lectures/ps06h.pdf or https://astro.pas.rochester.edu/~aquillen/ast233/lectures/ps06.html

3. **The Kepler map**

   Area preserving maps can illustrate remarkable complexity and can serve as approximate models for chaos in celestial mechanics.

   A generalization of the standard map (https://en.wikipedia.org/wiki/Standard_map) is the Kepler map which is the following map with $\gamma = 1.5$.

$$\begin{aligned} p_{n+1} &= p_n + K \sin\theta_n \\ \theta_{n+1} &= \theta_n + \text{sign}(p_{n+1})|p_{n+1}|^\gamma \qquad (4) \\ &= \theta_n + \text{sign}(p_n + K\sin\theta_n)\,|p_n + K\sin\theta_n|^\gamma \end{aligned}$$

in phase space with $\theta \in [0, 2\pi]$ modulo 2π and $p \in [-\infty, \infty]$. With $\gamma = 1$ the map is the same as the standard map.

Starting with an initial position θ_0, p_0 the map is iterated. Each point, labelled with θ_n, p_n is used to generate the next point in the map, θ_{n+1}, p_{n+1}, and the subscript denotes the number of iterations. A set of iterations generated from a single initial condition is called an *orbit*.

The Kepler map is relevant for the dynamics of comets. In this setting the variable p represents orbital energy and the angle represents the angle of the comets orbit with respect to Jupiter. For a derivation see https://www.researchgate.net/publication/221957457_The_Kepler_map_in_the_three-body_problem

a. Show that the Kepler map is an area preserving map.

Hint: compute the Jacobian J and show that its determinant is 1.

$$J = \begin{pmatrix} \frac{\partial p_{n+1}}{\partial p_n} & \frac{\partial p_{n+1}}{\partial \theta_n} \\ \frac{\partial \theta_{n+1}}{\partial p_n} & \frac{\partial \theta_{n+1}}{\partial \theta_n} \end{pmatrix}$$

b. Find all the fixed points of the map. Which fixed points are stable and which ones are unstable?

Hint: fixed points satisfy $p_{n+1} = p_n$ and $\theta_{n+1} = \theta_n$modulo 2π. To figure out whether a fixed point is stable or unstable, you can examine Figure 1.

The resonant islands get closer together at large p. Because the resonant widths are similar, the resonances increasingly overlap at larger p causing larger chaotic regions. Because the chaotic regions are connected,

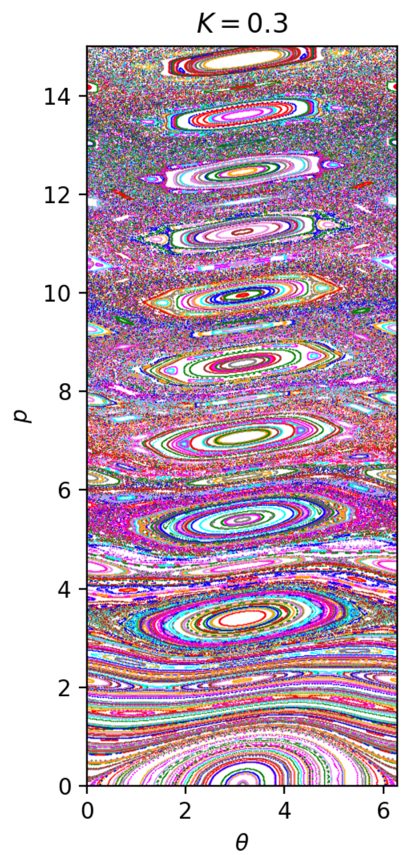

Figure 1: Different orbits of the Kepler map (with $\gamma = 1.5$) are shown with different color points.

a particle can jump across resonances, increasing to very large p.

4. **On conserved quantities relevant for a first order resonance**

A first order mean motion resonance with a planet, restricted to a plane, can be modeled with the Hamiltonian

$$\begin{aligned} H(\Lambda, \lambda, \Gamma, \gamma) = & -\frac{1}{2\Lambda^2} \\ & +\epsilon(a)\mu_p e(\Gamma)\cos(j\lambda - (j+1)\lambda_p + \gamma) \end{aligned} \tag{5}$$

where Poincare momenta

$$\begin{aligned} \Lambda &\equiv \sqrt{a} \\ \Gamma &\equiv \sqrt{a}(1 - \sqrt{1-e^2}). \end{aligned} \tag{6}$$

This describes the dynamics of a particle with semi-major axis a, eccentricity e, and mean longitude λ. The longitude of pericenter $\varpi = -\gamma$. The minus sign ensures that the momenta and angles are canonical. The canonical momenta and their associated angles are (in pairs) Λ, λ, and Γ, γ.

The planet has mass ratio (mass divided by that of the central star) μ_p and mean longitude $\lambda_p = n_p t$ where n_p is the planet's mean motion and t is time. We have adopted units with $GM_* = 1$. The coefficient ϵ is dependent upon semi-major axis (and so Λ). The eccentricity is dependent on the Poincare momentum Γ.

a. Show that if eccentricity is small $e < 1$,

$$\Gamma \approx \sqrt{a}\frac{e^2}{2} \tag{7}$$

and

$$e \approx \sqrt{\frac{2\Gamma}{\Lambda}}. \tag{8}$$

in terms of the Poincare momenta.

Hence the Hamiltonian

$$\begin{aligned} H(\Lambda, \lambda, \Gamma, \gamma) = & -\frac{1}{2\Lambda^2} \\ & +\epsilon'(\Lambda)\mu_p \Gamma^{\frac{1}{2}}\cos(j\lambda - (j+1)n_p t + \gamma) \end{aligned} \tag{9}$$

and there is a new coefficient $\epsilon'(\Lambda) = \epsilon(\Lambda)\sqrt{2/\Lambda}$ that depends on semi-major axis.

b. Show that the quantity E_J

$$\begin{aligned} E_J = & -\frac{1}{2\Lambda^2} + \epsilon'(\Lambda)\mu_p\Gamma^{\frac{1}{2}}\cos(j\lambda - (j+1)n_p t + \gamma) \\ & - (j+1)n_p\Gamma \end{aligned} \tag{10}$$

is conserved.

Hints: This is the Jacobi integral. You can show it is conserved by making a canonical transformation with generating function which is a function of old coordinate angles and new momenta

$$F_2(\lambda, \gamma, I, J) = (j\lambda - (j+1)n_p t + \gamma)I + \lambda J. \tag{11}$$

The new Hamiltonian $K = H + \frac{\partial F_2}{\partial t}$ should be time independent and so conserved.

c. Show that the quantity

$$J = \Lambda - j\Gamma = \sqrt{a} - j\sqrt{a}e^2/2 \tag{12}$$

is conserved.

This implies that there are coupled oscillations in eccentricity and semi-major axis near this resonance.

Hint: After doing the canonical transformation you should find that the new Hamiltonian is not dependent on one of the canonical angles and so Hamilton's equations imply that its associated momentum is conserved.