Homework set #5. AST 233, Fall 2024

On distributions of stars

Due date: Monday Nov 11, 2024, midnight. Please upload your solutions to blackboard.

1. **Finding a pattern speed from a phase space distribution**

   Consider a phase space density distribution function $f(\mathbf{x}, \mathbf{v}, t)$ that depends on time with numbers of stars per unit volume in phase space that looks like

   $$f(\mathbf{x}, \mathbf{v}, t) = g(x - ut, y, z, v_x, v_y, v_z).$$

   Here u is the velocity of a wave that passes through the distribution of stars in the galaxy. In other words the distribution function is fixed in a frame that moves with velocity u in the x direction.

   a) Show that the number density distribution $n(\mathbf{x}, t) = \int f() d^3\mathbf{v}$ can be written as a function that depends on $x - ut$ and so the system exhibits a traveling density wave.

   The collisionless Boltzmann equation

   $$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \boldsymbol{\nabla} f - \boldsymbol{\nabla}_v f \cdot \boldsymbol{\nabla} \Phi = 0, \tag{1}$$

   where spatial gradient $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$ and velocity $\nabla_{\mathbf{v}} = \left(\frac{\partial}{\partial v_x}, \frac{\partial}{\partial v_y}, \frac{\partial}{\partial v_z}\right)$.

   b) Show that the collisionless Boltzman equation is equivalent to

   $$-u\frac{\partial g}{\partial x} + \mathbf{v} \cdot \boldsymbol{\nabla} g - \boldsymbol{\nabla}_v g \cdot \boldsymbol{\nabla} \Phi = 0. \tag{2}$$

   c) Consider a specific location $\mathbf{x}$ in the galaxy. At this location we identify a peak in the velocity distribution. In other works at $\mathbf{x}$ there is a velocity $\mathbf{v}^*$ such that $\boldsymbol{\nabla}_v f(\mathbf{x}, \mathbf{v}^*) = 0$.

   Show that at $\mathbf{x}, \mathbf{v}^*$ the pattern wave speed

   $$u = \frac{(\mathbf{v} \cdot \boldsymbol{\nabla}) f(\mathbf{x}, \mathbf{v}^*)}{\frac{\partial f(\mathbf{x}, \mathbf{v}^*)}{\partial x}}$$

   If there is a peak in the velocity distribution function, it is possible to estimate the wave speed of the pattern u from the spatial gradients of the distribution function.

   This is related to the Weinberg-Tremaine method for measuring pattern speeds of bar or spiral wave like patterns in disk galaxies.

   Note: $\frac{\partial f}{\partial x} = \frac{\partial g}{\partial x}$.

2. **On the effect of a black hole on the stellar velocity dispersion**

   Jeans equation for a static isotropic spherically symmetric stellar distribution gives

   $$\rho \frac{\partial \Phi}{\partial r} + \frac{\partial (\rho \sigma^2)}{\partial r} = 0 \tag{3}$$

   where $\rho(r)$ is the mass density of stars, $\Phi(r)$ is the gravitational potential and σ^2 is the velocity dispersion.

   A commonly used density distribution for a stellar cluster is the Plummer model. A Plummer model with mass M_0, core radius a and isotropic velocity dispersion has stellar density distribution $\rho_P(r)$, velocity dispersion σ_P^2 and gravitational potential $\Phi_P(r)$,

   $$\rho_P(r) = \frac{3M_0}{4\pi a^3}\left(1 + \frac{r^2}{a^2}\right)^{-\frac{5}{2}} \tag{4}$$

   $$\Phi_P(r) = -\frac{GM_0}{a}\left(1 + \frac{r^2}{a^2}\right)^{-\frac{1}{2}} \tag{5}$$

   $$\sigma_P^2(r) = \frac{1}{6}\frac{GM_0}{a}\left(1 + \frac{r^2}{a^2}\right)^{-\frac{1}{2}} \tag{6}$$

   $$\frac{d\Phi_P(r)}{dr} = \frac{GM_0 r}{a^2}\left(1 + \frac{r^2}{a^2}\right)^{-\frac{3}{2}}. \tag{7}$$

   At the center of the cluster there is a black hole of mass M_{bh}. Our goal is to find the velocity dispersion as a function of radius r.

   The Jeans equation becomes

   $$\rho_P\left(\frac{d\Phi_P}{dr} + \frac{GM_{bh}}{r^2}\right) + \frac{d(\rho_P \sigma^2)}{dr} = 0 \tag{8}$$

Define the difference in velocity dispersion as

$$\Delta\sigma^2 = \sigma^2 - \sigma_P^2 \tag{9}$$

a) Show that

$$\Delta\sigma^2(r) = -\rho_P^{-1}\int^r \rho_P(r')\frac{GM_{bh}}{r'^2}dr' + \frac{C}{\rho_P(r)} \tag{10}$$

with constant C.

b) Find an expression (in r) for $\Delta\sigma^2(r)$ the difference in the velocity dispersion.

Hints: The indefinite integral

$$-\int^r (1+r'^2)^{-\frac{5}{2}}\frac{dr'}{r'^2} = \frac{\left(\frac{8}{3}r^4+4r^2+1\right)}{(1+r^2)^{\frac{3}{2}}r} + \text{constant} \tag{11}$$

can be found using Wolfram alpha or other sites.

The requirement that σ^2 is finite at large radius sets the constant C.

The result should be a sum of terms where each one is an integer power of r or a simple function of r. I found a sum of polynomials, a term proportional to $1/r$, and a term proportional to $(1 + r^2/a^2)^{\frac{5}{2}}$.

3. **The vertical oscillation frequency**

Consider a gravitational potential that is nearly round

$$\Phi(x, y, z) = \Phi(R, z) = f(R^2 + q^2z^2) \tag{12}$$

with $R^2 = x^2 + y^2$ and with axis ratio q that is close to 1. The function f determines the shape of the potential.

a) For circular orbits in the midplane with $z = 0$, compute the angular rotation rate $\Omega(R)$.

b) With a Taylor series in z show that the gravitational potential near the midplane

$$\Phi(R, z) \approx \Phi(R, 0) + q^2\Omega(R)^2\frac{z^2}{2} + ... \tag{13}$$

c) Consider an orbit that is nearly circular and nearly in the midplane, but oscillates vertically so that it travels a small distance above and below the midplane. Show that the vertical oscillation frequency

$$\nu = q\Omega.$$

d) With $q > 1$ is the galaxy oblate (like a pancake) or prolate (like a cigar)?

Hint: Consider the shape of an equipotential surface.

4. **On Hydrostatic equilibrium**

We are interested in the relationship between the scale height of a thin gaseous disk in circular motion and its temperature, setting its thermal or sound speed.

Hydrostatic equilibrium gives a relation between the vertical pressure gradient and the vertical force due to gravity

$$\frac{1}{\rho}\frac{\partial p}{\partial z} = -\frac{\partial \Phi}{\partial z} \tag{14}$$

where Φ is the gravitational potential, p is the pressure, and ρ is the mass density.

In cylindrical coordinates, R, z, we assume that the gravitational potential $\Phi(R, z) = f(\sqrt{R^2 + z^2})$ (for some function f) so that it is spherically symmetric.

a) Expand the gravitational potential Φ near the midplane as a function of z in a Taylor series. Show that

$$\Phi(R, z) = \Phi(R, z)|_{z=0} + \Omega(R)^2\frac{z^2}{2} + ...$$

for small z where $\Omega(R)^2 = \frac{1}{R}\frac{d\Phi}{dR}|_{z=0}$ and $\Omega(R)$ is the angular rotation rate for a circular orbit of radius R in the midplane.

We assume that locally at a particularly radius R we can ignore the dependence on R and only look at the vertical degree of freedom. The vertical pressure gradient

$$\frac{\partial p}{\partial z} = \frac{\partial p}{\partial \rho}\frac{\partial \rho}{\partial z}.$$

If the disk is an ideal gas then pressure

$$p = nk_BT = \frac{\rho k_B T}{\bar{m}}$$

where k_B is the Boltzmann constant, T is the temperature, $\bar{m}$ is the mean molecular mass, and n is the number density of particles. The quantity $v_{th} = \sqrt{\frac{k_BT}{\bar{m}}}$ is called a thermal velocity. The derivative

$$\frac{dp}{d\rho} = \frac{k_BT}{\bar{m}} = v_{th}^2.$$

The quantities $\rho_0, T, h, v_{th}, \Omega$ can vary with radius R but we ignore this dependence when estimate conditions for vertical hydrostatic equilibrium.

Often people use the approximation $v_{th} \sim c_s$ where c_s is the sound speed. For a particulate disk we would approximate $v_{th} \sim \sigma_*$ where σ_* is the velocity dispersion.

b) Show that the condition of hydrostatic equilibrium (vertical pressure balance, equation 14) is equivalent to

$$\frac{1}{\rho}\frac{d\rho}{dz}v_{th}^2 = -\Omega^2 z. \tag{15}$$

We assume that the gas obeys the ideal gas law and that temperature T and mean molecular mass $\bar{m}$ do not vary with z.

c) Show that a solution to equation 15 is

$$\rho(z) = \rho_0 e^{-\frac{z^2}{2h^2}}$$

and find a relation for the scale height h in terms of Ω and v_{th}. Here ρ_0 is the density in the midplane.

d) A Keplerian disk has disk temperature set by radiation balance between emission and absorption of light from a central star. The balance between emission and absorption $\sigma_{SB}T^4 \sim L_*/(4\pi R^2)$.

Use hydrostatic equilibrium to show that the disk scale height $h \propto r^{5/4}$.

Hint: A Keplerian disk has $\Omega \propto r^{-3/2}$.

5. **Another short program in *rebound*.**

   This problem is available in html form as https://astro.pas.rochester.edu/~aquillen/ast233/lectures/ps05.html and as a python notebook as https://astro.pas.rochester.edu/~aquillen/ast233/lectures/ps05.ipynb