Homework set #4. AST 233, Fall 2024

On epicycles and Lindblad resonances

Due date: Monday Oct 28, 2024, midnight. Please upload your solutions to blackboard.

1. **The density distribution for a flat rotation curve**

   In a galaxy, the velocity of a star in a circular orbit v_c is independent of radius.

   Assume that the mass distribution is spherically symmetric.

   a) Find an expression for the density distribution of matter $\rho(r)$ where r is radius from the center of the galaxy. Your answer should depend upon r, v_c, G.

   Hints: The mass within radius r is

   $$M(r) = \int_0^r dr' \; 4\pi r'^2 \rho(r').$$

   Balancing centripetal acceleration with gravitational acceleration the tangential velocity component $v_\theta = v_c$ obeys

   $$\frac{v_\theta^2}{r} = \frac{GM(r)}{r^2}.$$

   b) Show that the gravitational potential

   $$\Phi(r) = v_c^2 \ln r + \text{constant}.$$

   Hint: The radial component of the force is minus the gradient of the potential.

2. **On the location of Lindblad resonances**

   Consider a barred spiral galaxy with a flat rotation curve with circular velocity $v_c = 200$ km/s. A bar pattern has pattern speed

   $$\Omega_b = 40 \text{ km s}^{-1}\text{kpc}^{-1}.$$

   a) At what radius are stars or gas in the disk corotating with the bar pattern?

   By corotating we mean the radius at which the angular rotation rate $\Omega(r) = \Omega_b$.

   b) Show that the epicyclic frequency in a galaxy with a flat rotation curve

   $$\kappa(r) = \sqrt{2}\Omega(r).$$

   c) Find the radii of the 2:1 inner and outer Lindblad resonances. These satisfy

   $$\kappa^2 = m^2(\Omega - \Omega_b)^2$$

   with integer $m = 2$.

   Sometimes there are star forming rings at inner or outer Lindblad resonances.

3. **On the relationship between mean motion and Lindblad resonances**

   A **mean motion resonance** between a satellite and planet is a commensurability where the mean motion n of the satellite and that of the planet n_p satisfy $j_s n \approx j_p n_p$ for non-zero integers j_s, j_p. A **first order mean motion resonance** is one where j_s is 1 larger or smaller than j_p.

   A **Lindblad resonance** for a particle in orbit with mean motion n and perturbed by a pattern with frequency Ω_p is a commensurability where $\kappa^2 = m^2(n - \Omega_p)^2$. Here Ω_p represents a pattern speed but here we set $\Omega_p = n_p$ equal to a planet's mean motion. The frequency κ is the epicyclic frequency of the particle.

   a) Consider a first order mean motion resonance with

   $$(j+1)n \sim j n_p.$$

   Show that this mean motion resonance is also a Lindblad resonance with pattern speed equal to n_p, the mean motion of the planet. Find integer m for the Lindblad resonance condition: $\kappa^2 = m^2(n - n_p)^2$.

   Hint: in a Keplerian setting, the mean motion is approximately equal to the epicyclic frequency.

b) The j-th first order mean motion resonance is at orbital semi-major axis a_j. If the planet's semi-major axis is a_p, show that

$$a_j = \left(\frac{j+1}{j} \right)^{\frac{2}{3}} a_p \tag{1}$$

Hint: $n \propto a^{-3/2}$.

c) Show that the distance between the j-th first order mean motion resonance and the $j+1$-th mean motion resonance is approximately

$$\frac{a_{j+1} - a_j}{a_p} \sim \frac{2}{3} j^{-2} \tag{2}$$

in the limit of large j. This implies that the resonances get closer and closer together.

Hints: The Taylor series of

$$(1+x)^{\frac{2}{3}} \sim 1 + \frac{2}{3}x$$

for small x.

4. **On Type 2 planet migration**

A large planet opens up a gap in a gaseous circumstellar disk. The gas disk's kinematic viscosity ν is described with a dimensionless α parameter, $\nu \sim \alpha h^2 \Omega$ where h is the disk thickness, $\Omega = \sqrt{GM_*/r^3}$ is the orbital angular rotation rate and M_* is the mass of the star. Assume that the disk aspect ratio $h/r = 0.1$, the dimensionless $\alpha = 10^{-3}$ and $M_* = 1 M_\odot$. While α is independent of radius h, ν, Ω are functions of radius.

Because the planet pushes the disk away from its orbit (by driving spiral density waves into the disk) the planet migrates radially inward at a speed $\dot{r}$ that is set by the viscous inflow rate of the disk $\dot{r} \sim -\frac{\nu}{r}$.

a) Show that the planet's migration rate $\dot{r} \propto r^{-\frac{1}{2}}$.

b) Show that a solution is

$$r^{\frac{3}{2}} = r_0^{\frac{3}{2}} - A(t - t_0)$$

with constant A and find the constant A as a function of $G, M_*, h/r, \alpha$.

c) If the planet starts at $r_p = 1$ AU, how long (in years) does it take for the planet to reach a distance of 0.1 AU from its host star?

Hint: $\sqrt{\frac{GM_\odot}{(1 \text{ AU})^3}} = \frac{2\pi}{1 \text{ year}}$.

5. **Another short program in *rebound*.**

This problem is available in html form as https://astro.pas.rochester.edu/~aquillen/ast233/lectures/ps04.html and as a python notebook as https://astro.pas.rochester.edu/~aquillen/ast233/lectures/ps04.ipynb