Homework set #3. AST 233, Fall 2024

On Keplerian orbits

Due date: Monday Oct 7, 2024, midnight. Please upload your solutions to blackboard.

1. **On the ellipse shape of a Keplerian orbit**

   The equation for a Keplerian orbit is

   $$r(f) = \frac{a(1-e^2)}{1 + e\cos f} \tag{1}$$

   where r is radius from a central mass, a is the semi-major axis, e is the eccentricity, and f is the true anomaly which is a polar coordinate angle taken from pericenter. With coordinate system with origin at the central mass and with the xy plane containing the orbit, $x = r\cos f, y = r\sin f$. For $e < 1$, the orbit is an ellipse.

   Show that the orbital equation can usually be transformed into an equation in the form:

   $$a_2x^2 + a_1x + a_0 + b_2y^2 = 0$$

   with constant coefficients a_2, a_1, a_0, b_2. By completing the square, the orbital equation can be written in the form

   $$\frac{(x-x_0)^2}{a^2} + \frac{y^2}{b^2} = 1$$

   describing an ellipse, with coefficients a, b equal to its semi-major and semi-minor axes.

2. **The orbit velocity**

   In polar coordinates the radial component of velocity is $v_r = \dot{r}$. An object moves in an elliptic orbit with semi-major axis a and eccentricity e.

   a) At what radius is the maximum radial velocity amplitude attained? In other words at what radius is $|v_r|$ a maximum?

   b) Find an expression for the true anomaly of the points in the orbit where $|v_r|$ is maximum.

   c) Find an expression for the value of $|v_r|$ when it is maximum.

   Hint: Consider an extremum of the effective potential energy per unit mass which is $V_{eff}(r) = \frac{L^2}{2r^2} - \frac{GM}{r}$. Here $L = \sqrt{GMa(1-e^2)}$ is the angular momentum per unit mass which depends on the semi-major axis and eccentricity. You will need an expression for energy in terms of a, e and V_{eff} and v_r.

3. **On planetary orbit databases**

   What are the orbital elements of your favorite solar system object?

   https://ssd.jpl.nasa.gov/about/

   https://ssd.jpl.nasa.gov/tools/sbdb_lookup.html#/

   Specify the units of your answers.

4. **On time averaging, the eccentric anomaly and Kepler's equation**

   A low mass particle is in an elliptical Keplerian orbit with semi-major axis a, eccentricity e and orbital period $P = 2\pi/n$ with mean motion $n = \sqrt{GM/a^3}$. The central object has mass M.

   a) What is the average radius of the orbit

   $$\bar{r} = \frac{1}{P}\int_0^P r(t)dt \tag{2}$$

   Hints: The orbit obeys $r = a(1 - e\cos E)$ where E is the eccentric anomaly. Kepler's equation is $M = E - e\sin E$ where M is the mean anomaly. The time derivative of the mean anomaly $\dot{M} = n$. Write $\bar{r}$ in terms of an integral of dE with $dt = \frac{dt}{dE}dE = \frac{dt}{\dot{E}}$. Take the time derivative of Kepler's equation to find an expression for $\dot{E}$ that will allow you to integrate the expression for $\bar{r}$.

   b) Does the particle spend more time with radius $r > a$ or with $r < a$?

5. **A short program in *rebound*.**

   Your goal is to numerically measure the precession rates of a massless particle in a system containing a single planet.

   This problem is available in html form as https://astro.pas.rochester.edu/~aquillen/ast233/lectures/ps03_p5.html and as a python notebook as https://astro.pas.rochester.edu/~aquillen/ast233/lectures/ps03_p5.ipynb