Homework set #2. AST 233, Fall 2024

On hyperbolic orbits and applications of hyperbolic orbits

Due date: Monday Sept 23, 2024, midnight. Please upload your solutions to blackboard.

1. **Maximum deflection of a space craft near a planet**

   (M+D problem 2.3).

   A very low mass test particle approaches a planet of mass M and radius R from infinity with initial speed v_∞ and impact parameter b. The test particle's mass m
   M.

   a) Use the particle's energy, angular momentum and orbit to show that its orbital eccentricity

   $$e = 1 + 2\frac{v_\infty^2}{v_{esc}^2}$$

   where v_{esc} is the escape velocity from M at pericenter q.

   Hints: you need the following three equations Energy per unit mass

   $$E = \frac{1}{2}v_\infty^2 = -\frac{GM}{2a}$$

   Pericenter in terms of semi-major axis and eccentricity

   $$q = a(1-e)$$

   and an equation for the escape velocity at pericenter

   $$v_{esc} = \sqrt{\frac{2GM}{q}}$$

   b) Show that the deflection angle of deflection ψ of the test particle satisfies

   $$\sin(\psi/2) = \frac{1}{e}$$

   c) Given that pericenter distance $q > R$ to avoid collision, calculate the maximum orbital deflection angle for a space craft skimming Jupiter with $v_\infty = 10$ km/s.

2. **On gravitational focusing and the accretion rate of embryos in a planetesimal disk**

   Consider a planetary embryo of mass M and radius R moving through a sea of planetesimals of mass m and number density n. Assume that M is moving with velocity V_{drift} with respect to the mean velocity of the planetesimals and $M > m$ (the embryo mass is big compared to the planetesimals).

   The velocity dispersion of the planetesimals is σ. Assume that anything that hits embryo M is added to M so that M increases in mass. We would like to estimate the mass accretion rate $\dot{M}$ and how does it depend on M?

   When gravitational focusing is important the accretion rate is amplified and given by $\dot{M} \approx mn\pi\frac{GMR}{V_{rel}}$ where V_{rel} is the relative velocity. Otherwise $\dot{M} \sim mn\pi R^2 V_{rel}$. These depend on the relative velocity between projectile and embryo.

   When the planetesimal dispersion is high the relative velocity setting the velocity of a collision is set by the velocity dispersion, whereas if the velocity dispersion is low, the relative velocity is set by V_{drift}.

   There are three velocities in this problem, $\sqrt{GM/R}$, σ, V_{drift}. Discuss the possible regimes. Estimate $\dot{M}$ and how $M/\dot{M}$ (setting the time for doubling the embryo mass) depends on M in all 6 regimes.

   Here is the beginning of a list of the different regimes ....

   - $V_{drift} < \sqrt{GM/R} < \sigma$
   - $V_{drift} < \sigma < \sqrt{GM/R}$

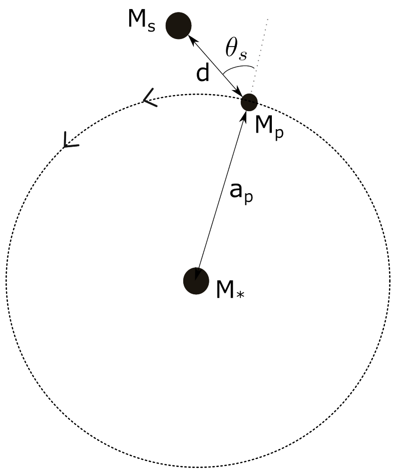

Figure 1: A fast encounter by a star, M_s passing through the ecliptic causes a change in semi-major axis and eccentricity of the planet M_p in orbit about star M_*. Assume that the orbit is initially circular and in the counter clockwise direction.

3. **Excitation of eccentricity from a gravitational encounter**

   Consider a planet of mass M_p at zero orbital inclination and in a circular orbit with radius a_p in orbit about its host star M_*. The initial velocity of the planet is $V_p = \sqrt{GM_p/a_p}$.

   A star of mass M_s moving at a very high speed, V_0, passes through the ecliptic plane that contains the planet's orbit. The star's trajectory is perpendicular to the ecliptic orbital plane. It passes through the ecliptic at a radius r_s from the host star. At the moment that the star M_* passes through the ecliptic, it is a distance d from the planet. The point at which it passes through the ecliptic, is described by angle θ_s as shown in Figure 1.

   The angular momentum of an elliptical orbit per unit mass $L = rv_\theta = \sqrt{GM_* a(1-e^2)}$ (assuming that $M_p \ll M_*$) where a is the orbital semi-major axis, e is the orbital eccentricity, r is the current radius, and v_θ is the tangential velocity component.

   The orbital energy per unit mass $E = \frac{1}{2}(v_r^2 + v_\theta^2) - \frac{GM_*}{r} = -\frac{GM}{2a}$ where v_r is the radial velocity component.

   a. Using the impulse approximation, estimate the size and direction of the velocity kick given to the planet. This is the change in the planet's velocity vector caused by the gravitational encounter.

   b. Suppose the planet's velocity vector has the same *magnitude* after the gravitational encounter as beforehand. Are the planet's energy and angular momentum both changed by the encounter? Are the planet's semi-major axis and eccentricity both changed by the encounter?

   c. Suppose the planet's velocity vector has a different magnitude after the encounter as beforehand. Can the planet's eccentricity remain at its initial zero value?

   d. Assume that the change in the planet's velocity $|\Delta v| \ll V_p$. Show that the change in semi-major axis $\delta a \sim a_p \frac{2\Delta v_\theta}{V_p}$ where Δv_θ is the change in the planet's tangential velocity component caused by the gravitational encounter.

4. **On the relaxation and crossing times in a cluster**

   A Molecular cloud has mass $M \sim 10^5 M_\odot$ and is about $r \sim 1$ pc big. The molecular cloud forms stars.

   a) Estimate the stellar velocity dispersion σ via dimensional analysis.

   b) Estimate the crossing time of the cluster, $t_{cross} \sim r/\sigma$.

   c) Estimate the star cluster's *relaxation time*.

   d) At what age would you expect to see some mass segregation (higher mass stars in the center?)

5. **Disk Heating**

We consider a disk composed of particles with mass m that is orbiting a heavy central object with mass M_*. At a radius r from the central object, the surface density of particles (a mass per unit area) is σ_m.

The inclination of the disk particles at a radius r increases via gravitational scattering at a rate

$$\frac{d}{dt}(\langle i^2 \rangle)^2 \sim \frac{r^2\Omega}{M_*^2}\sigma_m m \tag{1}$$

where $\Omega = \sqrt{\frac{GM_*}{r^3}}$ is the angular rotation rate of a particle at orbital radius r.

The quantity $\langle i^2 \rangle$ is the inclination variance (averaged over the distribution of particles). A typical or mean value for the inclination would be about $\bar{i} \sim \sqrt{\langle i^2 \rangle}$. The disk thickness $h \sim \bar{i}r$ at radius r.

An increase in inclination $\bar{i}$ is equivalent to disk thickening. Gravitational scattering is a mechanism for heating a disk, in the sense of increasing the disk's velocity dispersion and thickness.

a. Show that the mean inclination $\bar{i}(t) \propto t^\alpha$ with exponent α and find α.

b. Assuming that the disk surface density is approximately constant, how much faster is the heating rate at a radius of 1 AU compared to that at 10 AU?

c. Suppose particles in the disk coagulate or stick together. The mass surface density remains fixed but the particle size increases by a factor of 2. How much does the heating rate change?

d. Suppose the total mass in disk particles $M_d \sim r^2\sigma_m$ is only 1% of M_* and the ratio m/M_* is 10^{-9} (somewhat smaller than the ratio of Pluto's mass to that of the Sun). About how many orbital periods is required for the disk to become significantly thick, with $\langle i^2 \rangle \sim 1$?