AST233 Lecture notes – On Applications of a Hyperbolic orbit

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Contents

1 The Hyperbolic orbit

1.1 Two bodies, velocity changes

The center of mass $\mathbf{X}_{com}$

$$\mathbf{X}_{com} = \frac{\sum_i \mathbf{x}_i}{\sum_i m_i}$$

The center of velocity $\mathbf{V}_{com} = \dot{\mathbf{X}}_{com}$ is found by taking the time derivative of $\mathbf{X}_{com}$

$$\mathbf{V}_{com} = \dot{\mathbf{X}}_{com} = \frac{\sum_i \dot{\mathbf{x}}_i}{\sum_i m_i}$$

Two bodies, M, m with velocities $\mathbf{v}_m, \mathbf{v}_M$, we define a **relative velocity**

$$\mathbf{v} = \mathbf{v}_M - \mathbf{v}_m$$

Our goal is to find a relation for the velocities of m and M in terms of the relative velocity

$$\begin{aligned} m\mathbf{v}_m + M\mathbf{v}_M &= \text{constant} = (m+M)\mathbf{V}_{com} \\ m\mathbf{v}_m + M\mathbf{v}_M - M\mathbf{v}_m + M\mathbf{v}_m &= \text{constant} \\ (m+M)\mathbf{v}_m + M\mathbf{v} &= \text{constant} \end{aligned} \tag{1}$$

$$\mathbf{v}_m = -\frac{M}{m+M}\mathbf{v} + \text{constant} \tag{2}$$

$$\mathbf{v}_M = \frac{m}{m+M}\mathbf{v} + \text{constant} \tag{3}$$

Velocity changes

$$\Delta\mathbf{v}_m = -\frac{M}{m+M}\Delta\mathbf{v} \tag{4}$$

$$\Delta\mathbf{v}_M = \frac{m}{m+M}\Delta\mathbf{v} \tag{5}$$

With these we can relate the change in relative velocity to those experience by each of the two mass.

1.2 The two body problem

Two bodies M_1, M_2 with positions $\mathbf{r}_1, \mathbf{r}_2$ and velocities $\mathbf{v}_1, \mathbf{v}_2$. have total energy

$$E = M_1\frac{v_1^2}{2} + M_2\frac{v_2^2}{2} - \frac{GM_1M_2}{|\mathbf{r}_1 - \mathbf{r}_2|} \tag{6}$$

a sum of kinetic and potential terms. We show that this is equal to

$$E = (M_1 + M_2)\frac{\mathbf{V}_{com}^2}{2} + \mu\frac{\mathbf{v}^2}{2} + \frac{G(M_1+M_2)\mu}{|\mathbf{r}|} \tag{7}$$

The potential terms are equivalent because $(M_1 + M_2)\mu = M_1 M_2$. The kinetic terms are equivalent because

$$\begin{aligned}
(M_1 + M_2)V_{com}^2 + \mu v^2 &= \frac{1}{M_1+M_2}\left[(M_1 v_1 + M_2 v_2)^2 + M_1 M_2 (v_1 - v_2)^2\right] \\
&= \frac{1}{M_1+M_2}\left(M_1^2 v_1^2 + M_2^2 v_2^2 + 2M_1 M_2 v_1 v_2 + M_1 M_2 (v_1^2 + v_2^2 - 2v_1 v_2\right) \\
&= \frac{1}{M_1+M_2}\left[(M_1+M_2)^2 (v_1^2 + v_2^2)\right] = M_1 v_1^2 + M_2 v_2^2.
\end{aligned}$$

The first term in equation 7 is a coasting body of total mass $M = M_1 + M_2$ with a constant velocity $\mathbf{V}_{com}$ corresponding to the velocity of the center of mass. The reduced mass

$$\mu = \frac{M_1 M_2}{M_1 + M_2}$$

The second two terms in 7 are a Keplerian system of reduced mass μ in orbit about a large mass $M = M_1 + M_2$. For the Keplerian system, the coordinate is the relative position $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$ with relative velocity $\mathbf{v} = \mathbf{v}_1 - \mathbf{v}_2$.

1.3 Angular momentum in polar coordinates

A single body at position $\mathbf{r}$ with velocity $\mathbf{v}$. Together tho vectors $\mathbf{r}, \mathbf{v}$ give us a plane for the orbit. Coordinate

$$\mathbf{r} = r\hat{\mathbf{r}}$$

Velocity

$$\begin{aligned}
\mathbf{v} &= v_x \hat{\mathbf{x}} + v_y \hat{\mathbf{y}} & (8)\\
&= v_r \hat{\mathbf{r}} + v_\theta \hat{\boldsymbol{\theta}} & (9)
\end{aligned}$$

where we take x, y to be coordinates spanning the plane containing both $\mathbf{r}$ and $\mathbf{v}$.

$$v_\theta = r\dot{\theta}$$

where θ is an angle on the xy plane.

Angular momentum per unit mass

$$\begin{aligned}
\mathbf{L} &= \mathbf{r} \times \mathbf{v} & (10)\\
&= r v_\theta \hat{\mathbf{z}} \\
&= r^2 \dot{\theta} \hat{\mathbf{z}} & (11)
\end{aligned}$$

The angular momentum is only sensitive to the tangential velocity component.

1.4 Conservation of Angular momentum

With a radial force law the force on a particle i associated with a particle j is $\mathbf{F}_{ij} \propto \mathbf{r}_i - \mathbf{r}_j$ is proportional to the vector between the two particles. Let us adopt $\mathbf{F}_{ij}(\mathbf{r}_i, \mathbf{r}_j) = a_{ij}(\mathbf{r}_i - \mathbf{r}_j)$ with $a_{ii} = 0$. The force on particle i is opposite to that on particle j and this implies that a_{ij} is symmetric. The total angular momentum $\mathbf{L} = \sum_i m_i \mathbf{r}_i \times \mathbf{v}_i$ where we are summing over particles.

The change in angular momentum

$$\begin{aligned}
\dot{L} &= \sum_i m_i \left(\dot{\mathbf{r}}_i \times \mathbf{v}_i + \mathbf{r}_i \times \dot{\mathbf{v}}_i\right) \\
&= \sum_i m_i \left(\mathbf{v}_i \times \mathbf{v}_i + \mathbf{r}_i \times \sum_j \mathbf{F}_{ij}/m_i\right) \\
&= \sum_{i,j} \mathbf{r}_i \times \mathbf{F}_{ij} \\
&= \sum_{i,j} \mathbf{r}_i \times a_{ij}(\mathbf{r}_i - \mathbf{r}_j) \\
&= \sum_{i,j} -a_{ij} \mathbf{r}_i \times \mathbf{r}_j \\
&= 0
\end{aligned} \tag{12}$$

Here a_{ij} is symmetric but $\mathbf{r}_i \times \mathbf{r}_j = -\mathbf{r}_j \times \mathbf{r}_i$ and is antisymmetric. For every pair i, j the coefficients a_{ij} and a_{ji} have the same sign, but the cross product factors have opposite signs and so the two terms cancel. As a consequence $\dot{L} = 0$ making the total angular momentum $\mathbf{L}$ a conserved quantity.

When forces are only applied along vectors connecting particles, angular momentum conservation is assured. Potentials that are two-body interactions of functions of interparticle distance fall into this category.

1.5 Keplerian orbit

Radial force with $\mathbf{r}$ the vector between two masses

$$\frac{d^2\mathbf{r}}{dt^2} = -\frac{G(M+m)}{r^2}\hat{\mathbf{r}} \tag{13}$$

$$\ddot{r} - r\dot{\theta}^2 = -\frac{G(M+m)}{r^2} \tag{14}$$

Angular momentum per unit mass

$$h \equiv r^2\dot{\theta} = L$$

It is useful to work with inverse radius

$$\begin{aligned} u &\equiv \frac{1}{r} \\ \dot{u} &= -\frac{\dot{r}}{r^2} \end{aligned} \tag{15}$$

We cannot find $r(t)$ but we can find $r(\theta)$.

$$\dot{u} = \frac{du}{d\theta}\dot{\theta} \tag{16}$$

$$= \frac{du}{d\theta}\frac{h}{r^2} \tag{17}$$

where I have used angular momentum per unit mass h which is conserved to get rid of $\dot{\theta}$. Putting these together

$$\begin{aligned} \frac{du}{d\theta}\frac{h}{r^2} &= -\frac{\dot{r}}{r^2} \\ \frac{du}{d\theta}h &= -\dot{r} \\ \frac{d\dot{u}}{d\theta}h &= -\ddot{r} \end{aligned} \tag{18}$$

where on the last step I took the time derivative and h is a constant. Now insert equation 17 in to equation 18

$$-\ddot{r} = \frac{d}{d\theta}\left(\frac{du}{d\theta}\frac{h}{r^2}\right)h \tag{19}$$

$$\ddot{r} = -\frac{d^2u}{d\theta^2}h^2u^2 \tag{20}$$

Now we go back to equation 14 and start replacing r with u.

$$r\dot{\theta}^2 = \frac{h^2}{r^3} = h^2u^3$$

$$\frac{G(M+m)}{r^2} = G(M+m)u^2$$

Inserting these two relations into equation 14 and using equation 20 we find

$$\frac{d^2u}{d\theta^2} + u = \frac{G(M+m)}{h^2} \tag{21}$$

This has a solution

$$u = \frac{1 + e\cos(\theta - \varpi)}{p} \qquad \text{or} \qquad r = \frac{p}{1 + e\cos(\theta - \varpi)} \tag{22}$$

with

$$p \equiv \frac{h^2}{G(M+m)} \tag{23}$$

Here ϖ is the longitude of pericenter and sets the angle of the minimum radius r, known as pericenter. The parameter p is called the semi-latus rectum and e is the orbital eccentricity.

The orbits are conic sections.

Ellipses: $0 < e < 1$, and $p = a(1-e^2)$. Pericenter radius is $q = a(1-e)$. Semi-major axis $a > 0$. Orbit is bound so $E < 0$.

Hyperbolas $e > 1$, and $p = |a(e^2-1)|$. Pericenter radius is $q = |a(e-1)|$. Sometimes negative a is used so that energy per unit mass is positive with $E = -\frac{GM}{2a}$ (and that makes sense as the orbit is not bound and $E > 0$).

Parabolas have $e = 1$, and $p = 2q$ where q is pericenter.

In summary

$$p = |a(1-e^2)| \qquad \text{for} \qquad e \neq 1 \tag{24}$$

$$p = 2q \qquad \text{for} \qquad e = 1 \tag{25}$$

Our orbits are described by 3 parameters (see equation 23), a unitless eccentricity e, an orientation angle for the angle of pericenter ϖ, and the semi-latus rectum p. The constant p is the only one that has units and it is in units of length. But note that it involves a ratio of the square of the angular momentum and $G(M+m)$. We should not be surprised that p is related to a, e and so can be written in terms of orbital energy and angular momentum.

In terms of a, e, the orbital energy per unit mass

$$E = -\frac{G(M+m)}{2a}. \tag{26}$$

Equations 23, 24 then gives the angular momentum per unit mass

$$h = \sqrt{G(M+m)|a(1-e^2)|} \tag{27}$$

1.6 The true anomaly

The Keplerian orbit has radius r as a function of angle θ

$$r = \frac{p}{1 + e\cos(\theta - \varpi)} \tag{28}$$

with eccentricity e and $p = a^2(1-e^2)$ with semi-major axis a. We define a new angle $f = \theta - \varpi$, known as true anomaly in the orbital plane,

$$r = \frac{p}{1 + e\cos f}. \tag{29}$$

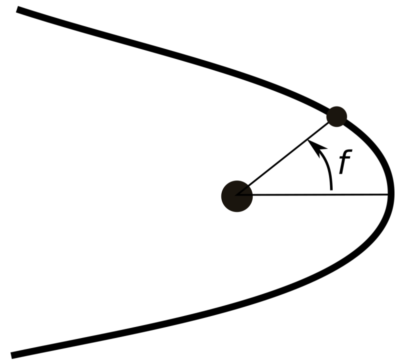

Figure 1: The true anomaly gives the angle of the test mass in the orbital plane with respect to percenter for a test particle in orbit about a larger mass.

Heliocentric coordinates for a small mass

$$x = r\cos f \quad (30)$$

$$y = r\sin f \quad (31)$$

Angles from pericenter are **anomalies** (see Figure 1), whereas angles from a fixed reference direction are **longitudes**.

1.7 Energy and semi-major axis for a hyperbolic encounter

We consider an incoming particle of mass m with velocity V_0 approaching an initially fixed mass M. We define the impact parameter b as shown in Figure 2.

What is the velocity of the center of mass?

$$V_{com} = \frac{m}{m+M} V_0, \quad (32)$$

and it is positive.

Initially the energy is only in the form of kinetic energy. The total energy is $E = \frac{mV_0^2}{2}$. This is equal to the sum of the kinetic energy of the center of mass and the total Keplerian energy of the two body system which we write in terms of the semi-major axis; $-GMm/2a$.

The total energy

$$E = \frac{mV_0^2}{2} = (m+M)\frac{V_{com}^2}{2} - \frac{GMm}{2a}$$

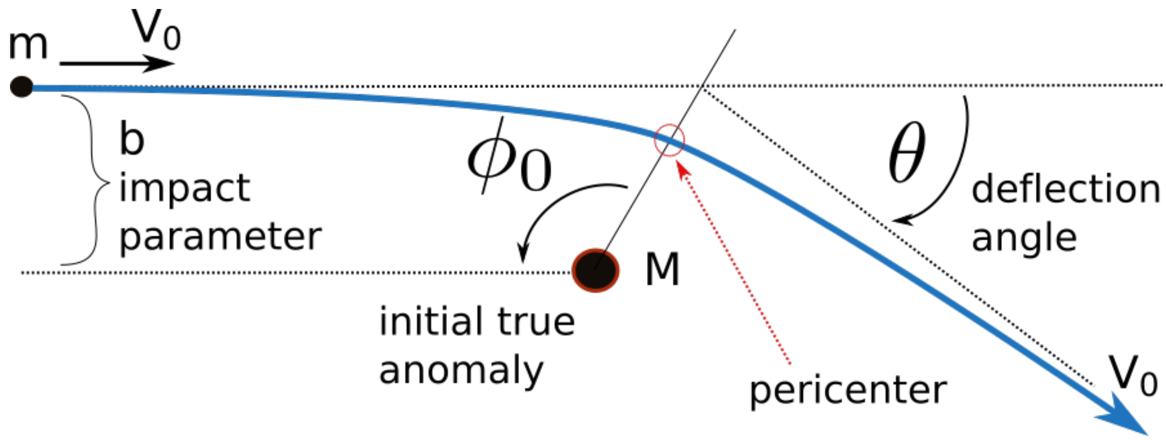

Figure 2: A gravitational encounter with impact parameter b and relative velocity V_0. The orbit is hyperbolic. The angle ϕ is also the true anomaly. Here the angular momentum per unit mass $h = bV_0$.

Note $G(M + m)\mu = GMm$. Insert the center of mass velocity (equation 32) and solve for semi-major axis a

$$a = -\frac{G(M + m)}{V_0^2} \tag{33}$$

Note, no factor of 2 here is correct. Here I am using the convention $E = -\frac{GMm}{2a} > 0$ and $a < 0$ for a hyperbolic (unbound) orbit.

1.8 Angular momentum and eccentricity for a Hyperbolic encounter

We recall that the angular momentum only depends upon the tangential velocity component. With impact parameter b, and velocity V_0, the angular momentum per unit mass

$$h = bV_0. \tag{34}$$

We had two ways to write the semi-latus rectum (equations 23, 24)

$$p = \frac{h^2}{G(m + M)} = |a(e^2 - 1)|$$

Insert the expression for h (equation 34 and solve for e^2

$$e^2 - 1 = \frac{b^2 V_0^4}{G^2(M + m)^2} \tag{35}$$

using the convention $e > 1$ for a hyperbolic orbit.

Notice that we see $G(M+m)/b$ in the expression. Let us define a gravitational velocity scale

$$V_g \equiv \sqrt{\frac{G(M+m)}{b}}. \tag{36}$$

Then

$$e^2 = 1 + \frac{V_0^4}{V_g^4}$$

For $V_0 > V_g$ the eccentricity is large and the orbit strongly hyperbolic. For V_0 small the orbit approaches $e \to 1$ and the orbit is nearly parabolic.

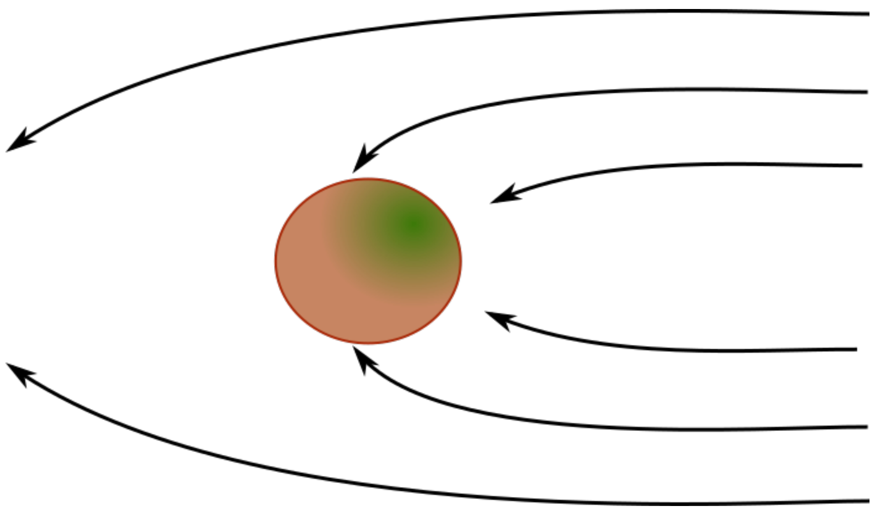

Figure 3: Gravitational focusing

1.9 Gravitational focusing

For a hyperbolic encounter the semi-major axis

$$|a| = \frac{G(M+m)}{V_0^2} \tag{37}$$

and eccentricity

$$e^2 = 1 + \frac{b^2 V_0^4}{G^2(M+m)^2} = 1 + A^{-2} \tag{38}$$

with

$$A = \frac{G(m+M)}{bV_0^2} = \left(\frac{V_g}{V_0}\right)^2, \tag{39}$$

and where the gravitational velocity scale is defined in equation 36. The pericenter radius $q = |a(e-1)|$. Inserting a, e into the equation for pericenter we find that

$$q = b\left(\sqrt{1+A^2} - A\right) \tag{40}$$

The pericenter is a minimum distance between the masses m, M during the encounter. For $V_0 > V_g$ the encounter has $q \sim b$ where as for $V_0 < V_g$ the pericenter distance q is much smaller than b.

When $V_0 > V_g$, the parameter $A < 1$ and pericenter $q \sim b$.

When $V_0 < V_g$, the parameter $A > 1$. In the limit of $V_0/V_g \to 0$, pericenter q approaches 0 (becomes smaller and smaller). Approximating this for small A^{-1},

$$\begin{aligned} \frac{q}{b} &= \sqrt{1+A^2} - A = A\left(\sqrt{A^{-2}+1} - 1\right) && (41) \\ &\sim A\left(1 + A^{-2}/2 - 1\right) \sim A^{-1}/2 && (42) \end{aligned}$$

or

$$q \sim \frac{b}{2}\left(\frac{V_0}{V_g}\right)^2 = \frac{b}{2}\frac{V_0^2 b}{G(M+m)}. \tag{43}$$

What does the pericenter distance have to do with *gravitational focusing*? The pericenter sets the cross section for collisions.

To discuss collisions we consider a mass M passing through a sea of smaller particles of mass m. I am flipping the picture (M vs m) because nothing we did above depends on which of the two particles was more massive. It makes more sense to use notation $M > m$ and have M be the moving particle. The mass M has velocity V_0 with respect to the fixed particles m. We ignore the radius of the m particles assuming that they are small. The number of density of background particles is n.

A collision happens if M passes within a distance R of a smaller particle m where R would be the radius of M. A collision happens if the pericenter distance of the encounter $q < R$. We have introduced a new scale, R, to the problem. Within a sea of particles, the collision rate is set by particles with impact parameter b such that pericenter $q(b) < R$. Let us define b_R to be the critical impact parameter that allows a grazing collision; $q(b_R) = R$. The radius b_R is the largest impact parameter that gives an impact.

If the encounters are slow then $b_R > R$ whereas if the encounters are fast then $b_R = R$. As $b_R > R$ in the slow setting, the collision rate is larger than in the fast setting. This effect is known as gravitational focusing because the encounters themselves pull trajectories toward M, increasing the collision rate. Gravity focuses in the sense that many more trajectories are encounters than estimated using the body's radius alone to estimate the cross section.

Given a velocity V_0, masses M, m and radius R what is the ratio b_R/R?

We solve the equation $q(b_R) = R$ for b_R. Taking equation 40 we can rewrite it as

$$q^2 + 2Abq = b^2$$

Now let $q = R$ and insert $A = \frac{G(m+M)}{bV_0^2}$ (equation 39) giving

$$R^2 + \frac{2G(m+M)R^2}{V_0} = b_R^2.$$

We solve for the **impact parameter that** b_R **that gives pericenter** $q = R$ **for an impact with relative velocity** V_0 **and between two masses** m, M, finding

$$b_R = R\left(1 + \frac{2G(M+m)}{V_0^2 R}\right)^{\frac{1}{2}} \tag{44}$$

This expression is valid in both high and low velocity limits.

It may be useful to define a new quantity

$$V_R \equiv \sqrt{\frac{G(M+m)}{R}}. \tag{45}$$

$$\frac{b_R}{R} = \left(1 + \frac{2V_R^2}{V_0^2}\right)^{\frac{1}{2}}. \tag{46}$$

By introducing a scale R we have also introduced a new velocity scale, V_R. If $V_R > V_0$ then gravitational focusing is a large effect, otherwise $b_R \sim R$.

The accretion rate depends on the number density of planetesimals n, their masses m, and the relative velocity V_0,

$$\dot{M} = nm\pi b_R^2 V_0 = nm\pi V_0 R^2\left(1 + \frac{2V_R^2}{V_0^2}\right) \tag{47}$$

$$= nm\pi V_0 R^2\left(1 + \frac{2G(M+m)}{RV_0^2}\right). \tag{48}$$

This is valid in both high and low velocity limits. Using $R \propto M^{\frac{1}{3}}$, the accretion rate

$$\dot{M} \propto M^{\frac{4}{3}} \qquad \text{low } V_0$$

$$\propto M^{\frac{2}{3}} \qquad \text{high } V_0 \tag{49}$$

A timescale for increasing mass is

$$t_M = \frac{M}{\dot{M}}$$

and is very short, $\propto M^{-1/3}$ for higher mass objects which can gravitationally focus incoming objects. As higher mass objects double their mass faster than lower mass objects, accretion favors growth of a few high mass objects.

1.9.1 The isolation mass

If the largest embryo tends to grow the fastest, the largest embryo would accrete material until it runs out of disk material. Assume a embryo of mass M_{em} is embedded in a disk of mass density Σ at an orbital radius of a_o. The embryo's Hill radius is $R_H = a_o \left(\frac{M_{em}}{3M_*}\right)^{\frac{1}{3}}$. The total mass of an annulus of the disk within the Hill radius of the embryo would be

$$M_d = 2\pi a_o \Sigma 2 R_H = 4\pi a_o^2 \Sigma \left(\frac{M_{em}}{3M_*}\right)^{\frac{1}{3}}. \tag{50}$$

If we set the mass of this annulus $M_d = M_{em}$ equal to embryo mass we find an embryo mass known as the isolation mass

$$M_{em} \sim \frac{(4\pi)^{\frac{3}{2}}}{3^{\frac{1}{2}}} a_o^3 \Sigma^{\frac{3}{2}} M_*^{-\frac{1}{2}}. \tag{51}$$

This gives an estimate for the largest object that can grow in a disk.

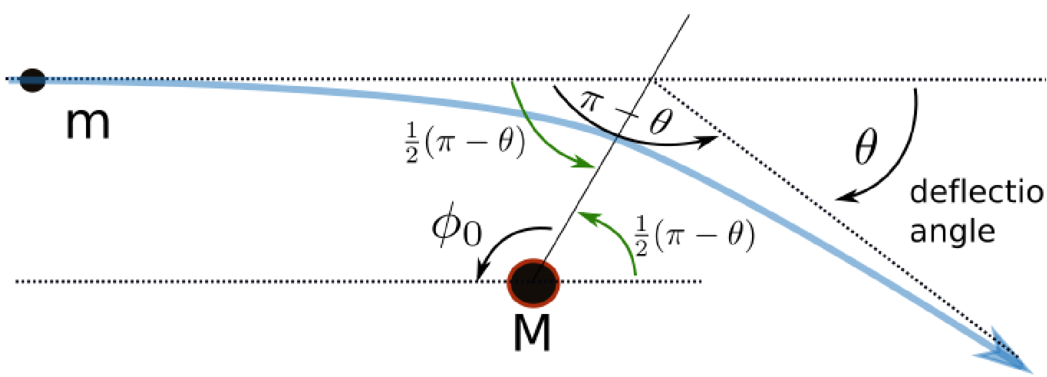

Figure 4: Deflection angle, θ, in terms of the initial true anomaly, ϕ_0, for a hyperbolic orbit. $\theta = 2\phi_0 - \pi$.

1.10 Deflection angle for the hyperbolic orbit

Looking at Figure 4 the deflection angle

$$\theta = 2\phi_0 - \pi$$

where ϕ_0 is the angle measured between initial velocity and pericenter. This angle is equivalent to the initial true anomali. Recall that an angle between the line connecting M to m and a reference direction aligned with M and m at pericenter is the true anomaly. Going back to our orbit equation

$$r = \frac{p}{1 + e\cos f}$$

When $f = 0$ we are at pericenter. So we can take $f = \phi_0$ equal to the initial true anomaly. The radius goes to infinity at an angle where the denominator vanishes or

$$1 + e\cos f = 1 + e\cos\phi_0 = 0$$

or

$$\sec\phi_0 = -e$$

Because $1 + \tan^2\phi = \sec^2\phi$ we find

$$e^2 = 1 + \tan^2\phi_0 \tag{52}$$

and this happens at angles $f = \pm\phi_0$. Using equation 38

$$e^2 = 1 + \tan^2\phi_0 = 1 + \frac{b^2 V_0^4}{G^2(M+m)^2}. \tag{53}$$

We should notice that this implies that

$$\tan\phi_0 = \frac{bV_0^2}{G(M+m)} \tag{54}$$

Inspection of Figure 5 helps us relate the changes in the relative velocity components to the deflection angle.

$$\Delta V_\perp = V_0 \sin\theta_d \tag{55}$$

$$\Delta V_\| = -V_0(1 - \cos\theta_d) \tag{56}$$

Here parallel is along the initial direction of the m and the perpendicular is perpendicular to this direction but in the plane containing the two masses and their trajectories. We can keep the signs straight if we remember that $\Delta V_\|$ must slow down the initially moving mass and $\Delta V_\perp$ is in the direction toward the other mass. With $\mathbf{V} = \mathbf{V}_m - \mathbf{V}_M$, and m the one initially moving with positive V_0 then $\Delta V_\parallel$ is negative and $\Delta V_\perp$ is m moving toward M.

Using some trig identities

$$\begin{aligned} \sin\theta_d &= \sin(2\phi_0 - \pi) \\ &= -\sin(2\phi_0) = -2\sin\phi_0\cos\phi_0 \\ &= -2\tan\phi_0\cos^2\phi_0 \\ &= -\frac{2\tan\phi_0}{1+\tan^2\phi_0} \end{aligned} \tag{57}$$

$$\begin{aligned} 1-\cos\theta_d &= 1+\cos 2\phi_0 \\ &= 2\cos^2\phi_0 \\ &= \frac{2}{1+\tan^2\phi_0} \end{aligned} \tag{58}$$

1.11 Parallel and perpendicular velocity changes

Putting these trig functions (equations 54, 57, 58) together with equation 56 and equation 53, the change in relative velocity components of M caused by an encounter with m (in the center of mass frame)

$$\begin{aligned} \Delta V_\perp &= -V_0\sin\theta_d = -\frac{2bV_0^3}{G(M+m)}e^{-2} \\ \Delta V_\parallel &= V_0(1-\cos\theta_d) = 2V_0 e^{-2} \end{aligned} \tag{59}$$

with

$$e^2 = 1 + \tan^2\phi_0 = 1 + \frac{b^2V_0^4}{G^2(M+m)^2}. \tag{60}$$

The angles are illustrated in Figure 5.

Now we need to transfer out of the center of mass frame using equations 5. So far there is no dependence on which mass is the one initially moving. Suppose we take M initially fixed and m the one that is initially moving. If we want to know the change to M's velocity we need to multiply ΔV in equations 59 by $m/(m+M)$ giving

$$\Delta V_{M\perp} = \frac{2mbV_0^3}{G(M+m)^2}e^{-2} \tag{61}$$

$$\Delta V_{M\parallel} = \frac{2mV_0}{M+m}e^{-2}. \tag{62}$$

With $\Delta V_{M\parallel}$ in the same direction as m's initial velocity (M is sped up) and $\Delta V_{M,\perp}$ in the direction toward m at pericenter.

$$
\begin{aligned}
\Delta V_{\perp} &= V_0 \sin\theta_d \\
\Delta V_{||} &= V_0(1 - \cos\theta_d)
\end{aligned}
$$

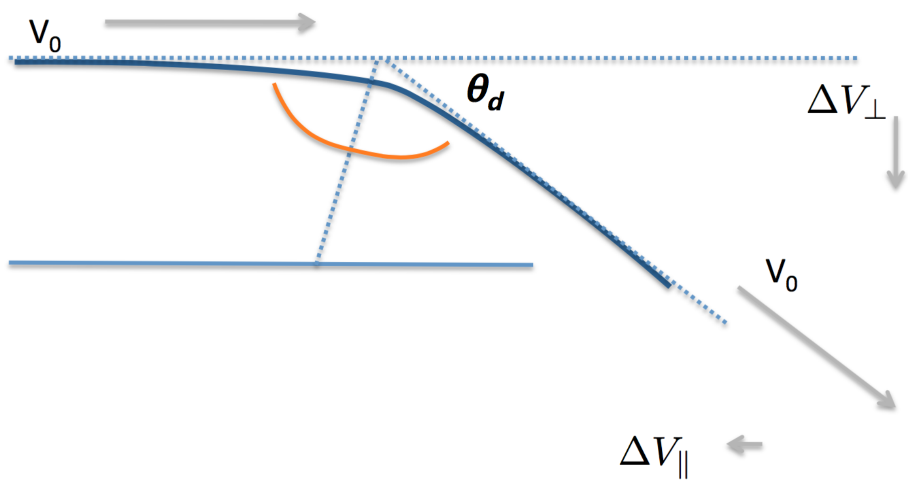

Figure 5: Components of the velocity change due to the encounter in terms of the deflection angle θ_d.

2 Applications

2.1 The impulse approximation

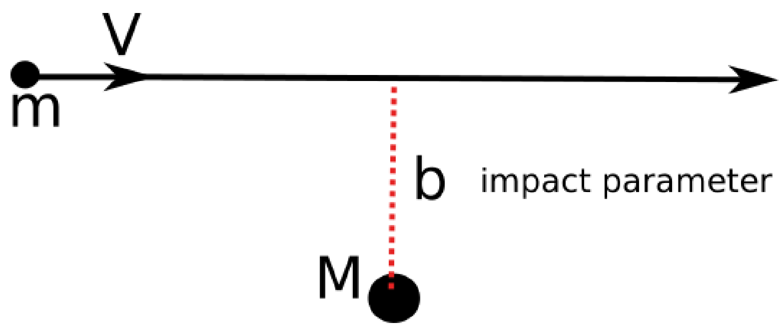

Figure 6: A gravitational encounter with impact parameter b and relative velocity V.

The impulse approximation gives an estimate for a velocity kick caused by a fast encounter. We discussed this in our order of magnitude estimates, but here we show that it is consistent with our estimates for the change in velocity caused by a hyperbolic encounter.

In the high V_0 limit

$$e^2 \to \frac{b^2 V_0^2}{G^2(M+m)^2}$$

(via equation 60). We plug this to the expressions for the velocity changes (equations 62) giving

$$\begin{aligned} \Delta V_{M,\perp} &\to \frac{2Gm}{bV_0} \\ \Delta V_{M,\parallel} &\to \frac{2mG(M+m)}{b^2 V_0^3} \sim 0. \end{aligned} \tag{63}$$

With the expectation that V_0 is large, $\Delta V_{M,\parallel} \sim 0$. We recognize that $\Delta V_{M,\perp}$ is a product of force at pericenter $F \sim Gm/b^2$ times the encounter time $t \sim 2b/V_0$. Thus the changes in velocity computed with a hyperbolic orbit in the limit of high velocity are consistent with the impulse approximate (and vice versa!).

2.2 Dynamical friction

The number density of stars with mass m is f. We assume that M is moving at a relative velocity V_0 with respect to a sea of particles with mass m. The rate that a star with

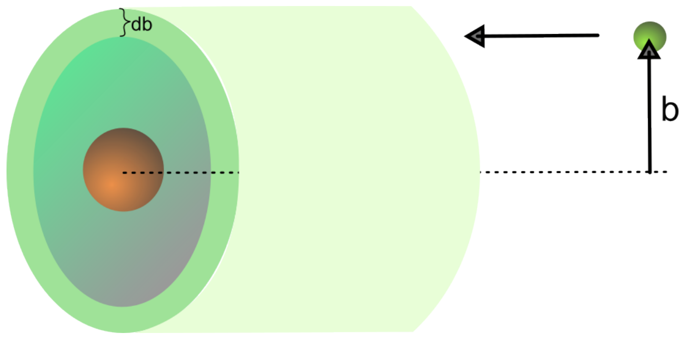

Figure 7: Illustrating the impact parameter b. Dynamical friction is estimated by integrating over different impact parameters.

mass m impact parameter b, range of possible impact parameters db and velocity v has a gravitational encounter with M is

$$2\pi b \; db \; V_0 f$$

as shown in Figure 7.

To find the total rate of change in $\Delta V_{M\parallel}$ we integrate over all impact parameters

$$\frac{d}{dt}\Delta V_{M\parallel} = \int_0^\infty db \; 2\pi b V_0 f \Delta V_{M\parallel}(b) \tag{64}$$

$$= \int_0^\infty db \; 2\pi b V_0 f \frac{2mV_0}{(M+m)} \left[1 + \frac{b^2 V_0^4}{G^2(M+m)^2}\right]^{-1} \tag{65}$$

where I have used equation 62 for $V_{M\parallel}$. If the field of stars is uniform then we can neglect $\Delta V_\perp$ as it should cancel to zero when we integrate.

We notice that the integral in equation 65 is dominated by encounters at large impact parameter b. Let us simplify the integral taking this into account, and setting density $\rho = fm$, the mass density of our sea of particles with mass m;

$$\frac{d}{dt}\Delta V_{M\parallel} \sim \int_0^\infty db \; 2\pi b V_0 \rho \frac{2V_0}{M+m} \frac{G^2(M+m)^2}{b^2 V_0^4} \tag{66}$$

$$= \int_0^\infty db \; \frac{4\pi\rho G^2(M+m)}{bV_0^2} \tag{67}$$

This integral diverges so we can't let impact parameter $b \to \infty$. We can consider a maximum impact parameter b_{max} typical of our system. $G(M+m)/V_0^2$ is in units of length and this is the transition regime were the denominator in equation 65 is 1. Let $u = \frac{bV_0^2}{G(M+m)}$,

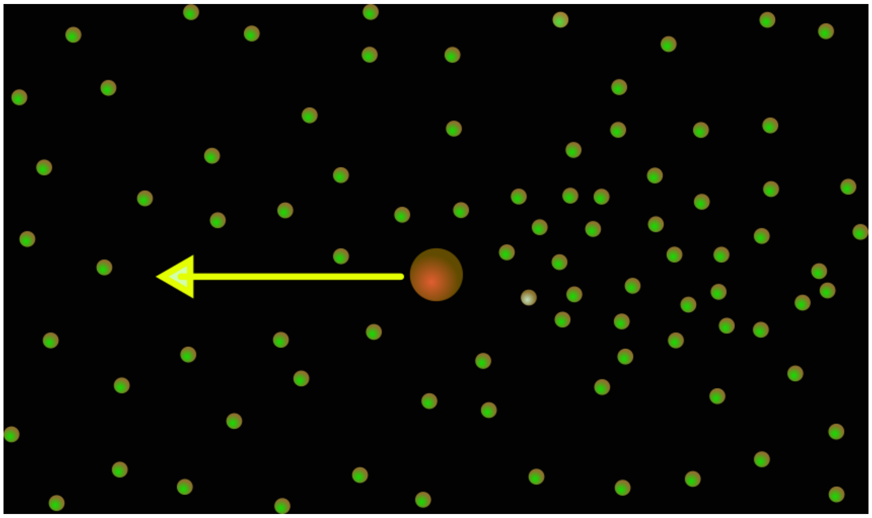

Figure 8: Dynamical Friction. Illustrating how a large mass, moving through a sea of particles accumulates a wake behind it due to gravitational scattering. The wake slows the mass down, causing the frictional force known as dynamical friction.

with $db = du\frac{V_0^2}{G(M+m)}$.

$$\begin{aligned} \frac{d}{dt}\Delta V_{M\|} &\sim \frac{4\pi\rho G^2(M+m)}{V_0^2}\int_0^{b_{max}}\frac{db}{b} \\ &\sim \frac{4\pi\rho G^2(M+m)}{V_0^2}\int_1^{\frac{b_{max}V_0^2}{G(M+m)}}\frac{du}{u} \\ &= \frac{4\pi\rho G^2(M+m)}{V_0^2}\ln\left(\frac{b_{max}V_0^2}{G(M+m)}\right) \end{aligned} \tag{68}$$

It is customary to define a Coulomb log

$$\Lambda \equiv \frac{b_{max}V_0^2}{G(M+m)} \tag{69}$$

The change in velocity is in the same direction as M is moving so

$$\dot{\mathbf{V}}_M \sim -\frac{4\pi\rho G^2(M+m)\ln\Lambda}{V_M^3}\mathbf{V}_M \tag{70}$$

For $M > m$ the change in velocity is proportional to M. As the acceleration is proportional to M the actual force is proportional to M^2. The change in velocity depends on the velocity so this is a dissipative force and that is why it is called *dynamical friction*. The formula diverges for small V_M only because we did not correctly estimate the integral (because we took the limit of large impact parameter).

A similar formula taking into account the integral over relative velocities was derived by Chandrasekkar. In this case we would take $f(\mathbf{v})$ a distribution function and integrate over d^3v. The change in velocity depends on the relative velocity so we would integrate over $v - V_M$.

With an isotropic Maxwellian velocity distribution the integral over all encounter velocities

$$\frac{d\mathbf{V}_M}{dt} = -\frac{4\pi \ln \Lambda \; G^2 M \rho}{V_M^3} \left(\mathrm{erf}(X) - \frac{2X}{\sqrt{\pi}} e^{-X^2} \right) \mathbf{V}_M \tag{71}$$

where σ is the velocity dispersion and $X \equiv \frac{v_M}{2\sigma}$.

As a mass M (a globular cluster or a black hole) passes through a sea of stars, it leaves a gravitational **wake** behind it of focused stars and this wake slowly pulls M backwards slowing it down, as shown in Figure 8.

2.3 Gravitational stirring and heating

With dynamical friction we primarily took into account the drag force from the component of the parallel component of the velocity change in a hyperbolic orbit.

We now think about the other components.

Each encounter gives a random change in velocity. So while perpendicular velocity changes do average to zero, they also cause random motions. The expectation averaged over all encounters $\langle \Delta V_\perp \rangle = 0$. However $\langle \Delta V^2 \rangle$ is not zero. We can describe the behavior with a random walk or with diffusive behavior.

The effect is called **gravitational stirring** or **gravitational heating**.

Diffusion coefficients come from integrating components of ΔV over a distribution of encounters that have a distribution of velocities and impact parameters. $D[\Delta w]$ denotes the expectation of the change in quantity w per unit time. We consider a mass M that is moving with velocity v_M through a sea of particles with mass m. The number of density of particles with mass m is n and the density of particles with m ass m is $\rho = nm$. We assume that the velocity distribution for the m particles is described with a Maxwell Boltzmann distribution function giving a dispersion σ. Drift and diffusion coefficients computed by integrating over the velocity and impact parameter distributions

$$\begin{aligned}
D(\Delta v_\parallel) &= \frac{4\pi G^2\rho(M+m)\ln\Lambda}{\sigma^2}G(X) && (72)\\
D(\Delta v_\parallel^2) &= \frac{4\sqrt{2}\pi G^2\rho m\ln\Lambda}{\sigma}\frac{G(X)}{X} && (73)\\
D(\Delta v_\perp^2) &= \frac{4\sqrt{2}\pi G^2\rho m\ln\Lambda}{\sigma}\left(\frac{\mathrm{erf}(X)-G(X)}{X}\right) && (74)\\
G(X) &= \frac{1}{2X^2}\left[\mathrm{erf}(X)-\frac{2X}{\sqrt{\pi}}e^{-X^2}\right] && (75)\\
X &\equiv \frac{v_M}{2\sigma} && (76)
\end{aligned}$$

Here $D(\Delta v_\parallel)$ gives a drift, whereas $D(\Delta v_\parallel^2), D(\Delta v_\perp^2)$ are diffusive, giving random motions. Statistics can be described in terms of an advective diffusion equation. A star moving through the galaxy is primarily heated by the diffusive terms, whereas a globular cluster moving through a sea of stars would be primarily slowed down by the advective or drift term. A star is drawn from a distribution of possible randomly chosen trajectories. The distribution of trajectories widens due to heating and this is described by the diffusive terms. Gravitational heating is described by the diffusive terms whereas dynamical friction is described by the drift term. When the two balance we have *equipartition*.

We define a timescale and a length scale $t_\rho = (G\rho)^{-1/2}$ and $R_m = \frac{Gm}{\sigma^2}$. Focusing on units only

$$D(\Delta v_\perp^2) \sim \frac{\Delta v_\perp^2}{\Delta t} \sim \frac{G^2\rho m}{\sigma} \sim t_\rho^{-2}\frac{Gm}{\sigma^2}\sigma \sim t_\rho^{-2}R_m\sigma \quad (77)$$

$$D(\Delta v_\parallel^2) \sim D(\Delta v_\perp^2) \sim t_\rho^{-2}R_m\sigma \quad (78)$$

$$D(\Delta v_\parallel) \sim \frac{G^2\rho(M+m)}{\sigma} \sim t_\rho^2 R_m\frac{M+m}{m}. \quad (79)$$

2.4 Equipartition

The kinetic energy of a single particle of mass M

$$E = \sum_i \frac{Mv_i^2}{2}$$

where i is over x,y,z. Diffusion in energy

$$\begin{aligned}
\frac{D(\Delta E)}{M} &= -\sum_i v_i D(\Delta v_i) + \frac{1}{2}\sum_i D(\Delta v_i^2) && (80)\\
&= -vD(\Delta v_\parallel) + \frac{1}{2}\left(D(\Delta v_\parallel^2) + D(\Delta v_\perp^2)\right) && (81)
\end{aligned}$$

where convention is that perpendicular part takes into account both perpendicular components (as we work in 3 dimensions).

The parallel part is negative because this is dynamical friction. This is a cooling term. The other two terms are heating terms.

How does the velocity V_M change?

Looking at equation 72, $D(\Delta v_{\|}) \propto M/\sigma^2$ with σ the velocity dispersion of the masses m. Looking at equation 73, $D(\Delta v_{\|}^2)$ and $D(\Delta v_{\perp}^2) \propto m/\sigma$. Setting a balance with $\frac{D(\Delta E)}{M} = 0$. The two terms in equation 81 are equivalent with

$$\frac{V_M M}{\sigma^2} \sim \frac{m}{\sigma}$$

or when

$$M V_M \sim m\sigma.$$

When two different masses are present the heating and cooling term balance giving what is called **equipartition**.

2.5 Eccentricity and Inclination evolution in a circumstellar disk

We consider two populations of planetesimals that are orbiting a central star M_c. Gravitational heating of planetesimals or dust particles of mass m in a circumstellar disk is due to scattering from masses of mass m_* and m in the disk;

$$\frac{d\langle e^2\rangle}{dt} = \frac{\Omega r^2 \sigma_* M_c^{-2}}{\sqrt{\pi}(\langle e_*^2\rangle - \langle e^2\rangle)^{\frac{1}{2}}(\langle i_*^2\rangle - \langle i^2\rangle)^{\frac{1}{2}}}\left[B J_e m_* + 1.4 A H_e \left(\frac{m_*\langle e_*^2\rangle - m\langle e^2\rangle}{\langle e^2\rangle + \langle e_*^2\rangle}\right)\right] \quad (82)$$

From Stewart and Ida 2000, Icarus 143, 28. Here Ω is angular rotation rate about M_c, σ_* is surface mass density of particles m_*. There are two heating terms and one damping term. Dynamical friction is negative and let's you identify which term it is; it is the cooling or damping term. We could also write a similar equation for the evolution of $\langle e_*^2\rangle$ that is is sensitive to σ, the mass density of particles with mass m.

Inclination evolution is similar except about one half the size and this is consistent with isotropy of the velocity distribution (though the coefficients are different). Here coefficients B, J_e, A, H_e are all terms of order unity. The terms subscripted with e depend on whether the system is in a dispersion or shear dominated regime. Observed debris disks are not in the shear dominated regime, whereas cold ring systems might be, but in that case there would also be cooling due to collisions.

What about encounters from particles with the same mass, m? They are taken into account but are hidden in the coefficients which depend upon ρ, ρ_*.

2.6 Relaxation timescale in a star cluster

Consider a star cluster of mass M with N stars so $M = Nm$ where m is the mass of each star in the cluster. The cluster has a radius R and a typical velocity scale

$$V = \sqrt{\frac{GM}{R}} = \sqrt{\frac{GNm}{R}} \tag{83}$$

which is also approximately the velocity dispersion in the cluster.

What time does it take for a star in the cluster to lose memory of its orbit? This is known as the **relaxation time**.

A gravitational encounter with impact parameter b gives a velocity kick in the perpendicular direction of order

$$\delta v \sim \frac{Gm}{bV}.$$

This is essentially the impulse approximation. Our star undergoes a **random walk** due to these velocity kicks. During a crossing time, our star experiences N kicks and the velocity kicks add in quadrature as they would on average all cancel when added. The number of stars per unit area is N/R^2 but each kick is due to an encounter with a different impact parameter. The total velocity change after **passing through the cluster once** (and having an encounter with every star in the cluster once)

$$\begin{aligned}
\Delta v^2 &\sim \int_{b_{min}}^{b_{max}} 2\pi b \; db \frac{N}{R^2} (\delta v)^2 \\
&\sim \int_{b_{min}}^{b_{max}} 2\pi b \; db \frac{N}{R^2} \left(\frac{Gm}{bV}\right)^2 \\
&\sim \frac{N}{R^2} 2\pi \frac{G^2 m^2}{V^2} \int_{b_{min}}^{b_{max}} \frac{db}{b} \\
&\sim \frac{N}{R^2} 2\pi \frac{G^2 m^2}{V^2} \ln \left| \frac{b_{max}}{b_{min}} \right|
\end{aligned} \tag{84}$$

Here b_{max}, b_{min} cover the range of possible impact parameters.

Define

$$\Lambda \equiv \frac{b_{max}}{b_{min}} \tag{85}$$

and use equation 83 to remove R

$$\frac{\Delta v^2}{V^2} = \frac{2\pi}{N} \ln \Lambda. \tag{86}$$

Losing memory of the initial velocity happens when $\Delta v^2 / V^2 \sim 1$. The number of crossing times required to lose all memory of initial conditions is the inverse of equation 86

$$n_{relax} \sim \frac{N}{6 \ln \Lambda}. \tag{87}$$

where I replaced 2π with 6.

It makes sense that the maximum impact parameter $b_{max} = R$. The minimum impact parameter we can estimate from a gravitational sizescale $b_{min} \sim Gm/V^2$. Taking the ratio

$$\frac{b_{max}}{b_{min}} = \frac{RV^2}{Gm} \sim \frac{M}{m} \sim N$$

and giving

$$n_{relax} \sim \frac{N}{6 \ln N}. \tag{88}$$

To estimate a **relaxation timescale** we multiply this number with the crossing time to find the relaxation time

$$t_{relax} = n_{relax} t_{cross} \tag{89}$$

with the crossing timescale

$$t_{cross} \sim \frac{R}{V}.$$

2.7 Stochastic behavior, ergodicity and chaos

When we discuss gravitational heating in terms of diffusion or gravitational relaxation we assume that gravitational encounters are a *stochastic* phenomena. Stochastic here means involving random behavior. This contrasts with a Keplerian system which is analytically solvable. N-body systems are deterministic in the sense that trajectories are integrated and they are not chosen from a random distribution in any way. However for $N \geq 3$ an N-body system is likely to be chaotic. Our assumption of stochastic behavior rests in the way that N-body systems behave *ergodically*. With the word *ergodic* here meaning acting as if we can model the system as if it were random.

3 Problems

I am listing topics that might be interest to think about or/and research.

1. Disk heating mechanisms for both planetesimal and galactic disks.

2. Tidal stream broadening in the Galaxy.

3. Perturbations of planets and comets by nearby stars.

4. How accretion onto planets or planetary embryos is affected by gravitational focusing.

5. Extending the impulse approximation to cover shocks of the stars in a cluster as it passes through a Galactic disk