# AST233 Lecture notes on Spiral density waves

Alice Quillen

October 9, 2024

## Contents



# 1 Differential rotation, Epicyclic motion and the Shearing sheet

The setting for this section is a disk. The particles in the disk are assumed to be in a single plane and in orbits that are nearly circular.

If the disk were rotating as if it were a solid body then the angular rotation rate of a particle $\Omega$ is in dependent of radius. We first consider this case and then discuss a more generic situation where $\Omega(r)$ is a function of radius, giving differential rotation.

### 1.1 Solid body rotation

What kind of mass distribution could give solid body rotation?

We assume that the mass is mostly invisible (dark matter) and distributed in a spherically symmetric way giving density $\rho(r)$ that is only a function of radius. The disk, we assume is a collection of low mass or tracer particles.

For a uniform shell of mass $M_s$ and radius $R$ at $r > R$, outside the shell, the gravitational force per unit mass from this shell is $\mathbf{F} = \frac{GM_s}{r^2}\hat{\mathbf{r}}$ and is independent of $R$.

For our spherical mass distribution with density $\rho(r)$ we integrate the mass from each shell. The total mass within a radius $r$ is

$$M(r) = \int_0^r dr' \; 4\pi r'^2 \rho(r'). \tag{1}$$

For a particle in a circular orbit, the tangential velocity $v_\theta = r\Omega$. We relate the centripetal acceleration to the mass within radius $r$ to the force from each shell interior to $r$

$$\frac{v_\theta^2}{r} = r\Omega^2 = \frac{GM(r)}{r^2} \tag{2}$$

as $\Omega$ is independent of radius we can solve for $M(r)$

$$M(r) = \frac{r^3\Omega^2}{G}. \tag{3}$$

We set this equal to equation 1 and take a derivative

$$\begin{aligned}
\frac{r^3\Omega^2}{G} &= \int_0^r dr' \; 4\pi r'^2 \rho(r') \\
\frac{3r^2\Omega^2}{G} &= 4\pi r^2 \rho(r) \\
\rho(r) &= \frac{3}{4\pi}\frac{\Omega^2}{G} \qquad \text{is constant!.}
\end{aligned}$$

We find that the density is constant!

What is the gravitational potential? To figure this out we could use Poisson's equation

$$4\pi G\rho = \nabla^2\Phi. \tag{4}$$

We assume that $\Phi(r)$ and is independent of angles $\theta, \phi$ in spherical coordinates. In spherical coordinates Poisson's equation becomes

$$4\pi G\rho = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right). \tag{5}$$

We use the fact that $\rho$ is independent of radius (for uniform body rotation) and integrate once

$$
\begin{aligned}
4\pi G\rho r^2 &= \frac{\partial}{\partial r}\left(r^2\frac{\partial \Phi}{\partial r}\right) \\
\frac{4\pi G\rho r^3}{3} &= r^2\frac{\partial \Phi}{\partial r} + \mathrm{C}_1 \\
\frac{4\pi G\rho r}{3} &= \frac{\partial \Phi}{\partial r} + \frac{C_1}{r^2} \\
\frac{2\pi G\rho r^2}{3} &= \Phi + \frac{C_1'}{r^3} + C_2
\end{aligned}
\tag{6}
$$

with constants $C_1', C_2$. Equivalently (and more quickly and easily) we could use the fact that the force is the gradient of the potential and equation 2

$$
r\Omega^2 = \frac{GM(r)}{r^2} = \frac{d\Phi}{dr} \tag{7}
$$

$$
\frac{1}{2}r^2\Omega^2 = \Phi + \text{constant} \tag{8}
$$

which is consistent with equation 6 using our result for $\rho$ in equation 4.

## 1.2 Differential rotation and the shearing sheet

In most disk settings, galactic and Keplerian, the angular rotation rate $\Omega$ is not a constant, rather it depends on radius. In a Keplerian setting, where the disk mass is low, the angular rotation rate is set by the central mass $M_*$ and $\Omega = \sqrt{\frac{GM_*}{r^3}}$. In a constant density cluster, $\Omega$ could be flat. But as the density drops at large radius, $\Omega$ would start to drop, but less quickly than in the Keplerian setting.

With $\Omega(r)$ dependent upon $r$, there is differential rotation.

It is sometimes convenient numerically to use periodic boundary conditions to example dynamics in a small patch of the disk, as shown in Figure 1.

## 1.3 Epicyclic motion

Spiral density waves involve radial motions for particles that are in a disk. The motions of the particles are close to circular. We use that fact to estimate the frequency of small radial oscillations, known as the **epicyclic frequency**.

We ignore motion out of the disk plane. The energy per unit mass is the sum of kinetic and potential energy

$$
E = \frac{1}{2}v_r^2 + \frac{1}{2}v_\theta^2 + \Phi(r) \tag{9}
$$

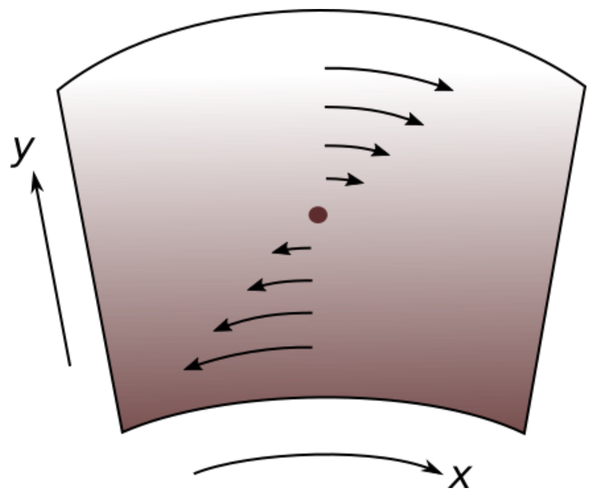


Figure 1: The Shearing Sheet. A central particle remains fixed. Particles with no epicyclic oscillations (in circular orbits) exhibit shear in their horizontal ($x$) velocities as a function of $y$. Often the shearing sheet is simulated with periodic boundary conditions in both $x$ and $y$.

Angular momentum $L = rv_\theta$ is conserved. We replace $v_\theta$ with $L/r$

$$E = \frac{1}{2}v_r^2 + \frac{L^2}{2r} + \Phi(r). \tag{10}$$

We assume that $r = R_0 + w$ and $w$ is small. We expand $\Phi$ to second order around $R_0$.

$$\Phi(r) = \Phi(R_0) + \left.\frac{\partial \Phi}{\partial r}\right|_{r=R_0} w + \left.\frac{\partial^2 \Phi}{\partial r^2}\right|_{r=R_0} \frac{w^2}{2} + ... \tag{11}$$

A particle in a circular orbit has a constant tangential velocity

$$\frac{v_\theta^2}{r} = \frac{\partial \Phi}{\partial r}. \tag{12}$$

Using the fact that $\Omega = v/r$ we find that

$$\Omega^2 = \frac{1}{r}\frac{\partial \Phi}{\partial r}. \tag{13}$$

We denote the angular rotation rate at $r = R_0$ as $\Omega_0$ with

$$\Omega_0^2 = \frac{1}{R_0}\left.\frac{\partial \Phi}{\partial r}\right|_{r=R_0}. \tag{14}$$

Using this expression, equation 11 becomes

$$\Phi(r) = \text{constant} + R_0\Omega_0^2 w + \left.\frac{\partial^2\Phi}{\partial R^2}\right|_{r=R_0} \frac{w^2}{2}. \tag{15}$$

We expand the term containing the angular momentum in equation 10

$$\frac{L^2}{2r^2} = \frac{L^2}{2R_0^2} - \frac{L^2}{R_0^3}w + \frac{3L^2}{R_0^4}\frac{w^2}{2} + .. \tag{16}$$

The angular momentum for a particle in a circular orbit at radius $R_0$ is $L_0 = R_0^2\Omega_0$. This gives

$$\frac{L^2}{2r^2} = \text{constant} - R_0\Omega_0^2 w + 3\Omega_0^2\frac{w^2}{2} + .. \tag{17}$$

Combining equation 10 with equation 15 and equation 17

$$\begin{aligned} E &= \text{constant} + \frac{v_r^2}{2} - R_0\Omega_0^2 w + 3\Omega_0^2\frac{w^2}{2} + R_0\Omega_0^2 w + \left.\frac{\partial^2\Phi}{\partial R^2}\right|_{r=R_0}\frac{w^2}{2} \\ &= \text{constant} + \frac{v_r^2}{2} + \frac{1}{2}\left(3\Omega_0^2 + \left.\frac{\partial^2\Phi}{\partial r^2}\right|_{r=R_0}\right) w^2 \end{aligned} \tag{18}$$

We define a frequency $\kappa$, known as the epicyclic frequency

$$\kappa^2 = 3\Omega^2 + \frac{\partial^2\Phi}{\partial r^2}. \tag{19}$$

With this frequency the energy becomes

$$E = \text{constant} + \frac{v_r^2}{2} + \frac{\kappa_0^2 w^2}{2} \tag{20}$$

where $\kappa_0$ is the epicyclic frequency at radius $R_0$. Notice that the energy per unit mass resembles that of a harmonic oscillator. That implies that $\kappa$ is the frequency of radial oscillations about the circular orbit.

Why is this type of motion called epicyclic motion? We can think of the particle as doing small loops about a circular orbit.

Orbits don't usually close in a galaxy where they could resemble a rosette as shown in Figure 2.

### 1.3.1 Solid body rotation

In this setting $\frac{1}{r}\frac{d\Phi}{dr} = \Omega^2$ is constant. This implies that $r\Omega^2 = \frac{d\Phi}{dr}$ and $\frac{d^2\Phi}{dr^2} = \Omega^2$. Equation 19 gives epicyclic frequency $\kappa = 2\Omega$.

#### 1.3.2 Keplerian orbits

For the Keplerian orbit $\Phi = -\frac{GM}{r}$, and $\frac{d\Phi}{dr} = \frac{GM}{r^2}$. The angular rotation rate $\Omega^2 = \frac{1}{r}\frac{d\Phi}{dr} = \frac{GM}{r^3}$. Furthermore $\frac{d^2\Phi}{dr^2} = -\frac{2GM}{r^3} = -2\Omega^2$. Equation 19 gives epicyclic frequency $\kappa = \Omega$. For Keplerian orbits radial epicycles occur at the same frequency as rotation. This means that nearly circular orbits close and are ellipses.

For an object in orbit outside a planet, the planet's mass breaks the Keplerian symmetry and would induce precession on the object.

#### 1.3.3 A flat rotation curve

If the circular velocity $v_c$ is constant then $r\Omega^2 = \frac{v_c^2}{r} = \frac{d\Phi}{dr}$ and $-\frac{v_c^2}{r^2} = \frac{d^2\Phi}{dr^2} = -\Omega^2$. Equation 19 gives epicyclic frequency $\kappa = \sqrt{2}\Omega$.

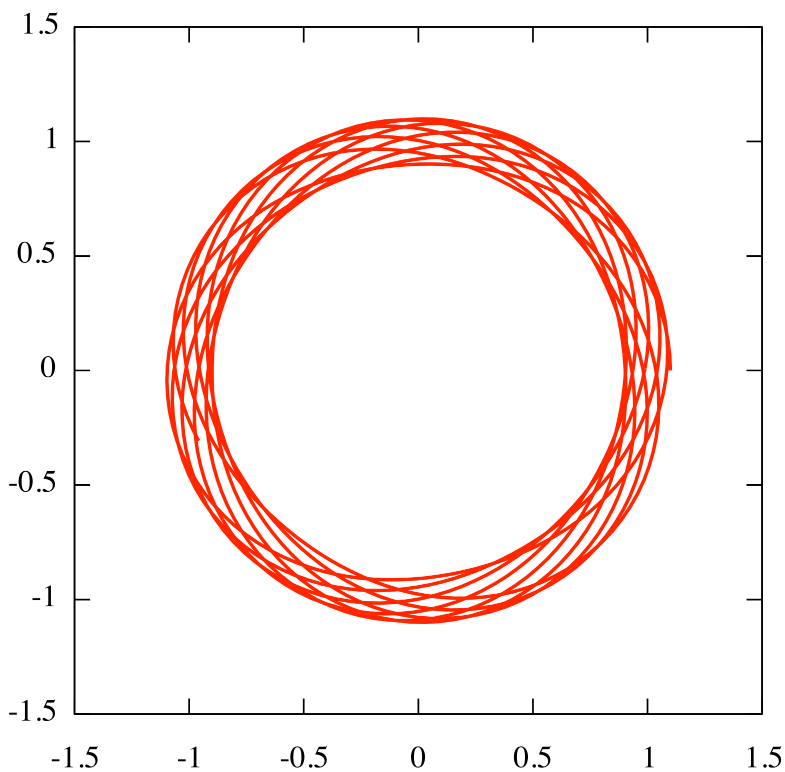


Figure 2: A rosette orbit. The orbit can be described in terms of radial oscillations or epicycles about a circular orbit.

### 1.4 Lindblad resonances

The epicyclic frequency is relevant for accounting for collective motions or patterns that might be stable or long lived in a galactic disk. A pattern that moves in such a way as to be in phase with two epicyclic periods per rotation would look like a galactic bar. A pattern that moves in such a way as to be in phase with a single epicyclic period might give a lopsided or eccentric disk (as in M31). A pattern that moves at a speed that is proportional to a number of epicyclic periods might give a multi-armed spiral pattern.

We consider a disk with a periodic perturbation at a frequency, $\Omega_p$, which we call the pattern frequency, there can be resonances in the disk.

What is a resonance?

A resonance is a location where a sum of different integers times different frequencies is zero.

In this problem we have three frequencies $\Omega, \kappa, \Omega_p$. A resonance would be a location where there is set of integers $i, j, k$ (positive or negative or zero) and

$$i\Omega + j\kappa + k\Omega_p = 0. \tag{21}$$

We have three angles in the problem

$$\begin{array}{lll} \theta & \text{polar coordinate} & \dot{\theta} = \Omega \\ \phi & \text{phase of epicycle} & \dot{\phi} = \kappa \\ \theta_p & \text{phase of pattern} & \dot{\theta}_p = \Omega_p \end{array} \tag{22}$$

We can integrate a condition on frequencies, such as in in equation 21, to find

$$i\theta + j\phi + k\theta_p = \text{constant} \tag{23}$$

Thus the resonance condition is equivalent to an angle that is constant.

Note we are not discussing yet whether the resonance is actually important or strong or how near we need to be to it for it to be important.

A **Lindblad resonance** is at a radius where the condition

$$\kappa^2 - m^2(\Omega - \Omega_p) = 0 \tag{24}$$

is satisfied, for integer $m$. This condition is equivalent to

$$\kappa = \pm m(\Omega - \Omega_p). \tag{25}$$

We can integrate this condition to find

$$\phi \pm m(\theta - \theta_p) = \text{constant} \tag{26}$$

Assume that $\theta_p = \Omega_p t$ describes the pattern peak. Equivalently $\theta_p$ is the angle at which a bar or spiral arm is densest or brightest. We transform into the frame rotating with the pattern via $\theta' = \theta - \theta_p = \theta - \Omega_p t$. The resonant condition is $\kappa = \pm m\dot{\theta}'$. The period of radial oscillations is $P_r = 2\pi/\kappa$ and the rotation period in the rotating frame is $P_{\theta'} = 2\pi/(\Omega - \Omega_p)$. The resonant condition implies that $mP_r = P_{\theta'}$. This means that there are $m$ periods of radial oscillation for each rotation about the origin, as seen in the rotating frame.

# 2 Spiral density waves

## 2.1 A short history

Galactic spiral arms were considered by Lynden-Bell as density waves propagating through a field of stars.

Doug Lin and Frank Shu proposed that spiral density waves in galaxies were long lived modes, similar to vibrational models.

Alar Toomre argued that disks were likely to be unstable to amplification of noise resulting in transient spiral structures.

Peter Goldreich and Scott Tremaine considered how spiral density waves would be driven at resonances by satellites embedded in a ring system, resulting in the *Torque formula*. Despite depending upon many approximations, the torque formula applies in both particulate and gaseous disks.

The torque formula was used by Doug Lin and Jim Papaloizou to derive a gap opening criterion in ring and exoplanet systems and by Bill Ward used it to estimated regimes for planet or satellite migration within disks.

There is a lack of consensus on the pattern speed of the Milky Way's bar. There is also a lack of consensus on the possible pattern speed of local galactic spiral arms. One possibility is that spiral structures are transient.

Galactic disks have Toomre $Q$ close enough to 1 that the disk supports spiral density waves. Gas facilitates spiral arm formation as it aids in maintaining a thin disk. Some planetesimal disks exhibit spiral structure however they may not be massive enough to have $Q$ near 1. In planetesimal disks coupling between gas and pebbles give a different instability, known as the streaming instability (proposed by Andrew Youdin, with many initial simulations by Anders Johansen), and this could trigger planetesimal formation.

## 2.2 Tour of images

For a tour of spiral galaxies see
`https://www.flickr.com/photos/nasahubble/albums/`
For a circumstellar disk with spiral structure
`https://webdisks.jpl.nasa.gov/show.php?id=10` which is AB Aur
For circumstellar disks with rings see this:
`https://webdisks.jpl.nasa.gov/reference.php?id=1812`

Saturn's ring systems

`https://photojournal.jpl.nasa.gov/figures/PIA08389_fig1.jpg`

Newly discovered systems Quaoar, Chariklo, Haumea, Chiron (also shows Uranus and Neptune)

`https://en.wikipedia.org/wiki/Ring_system`

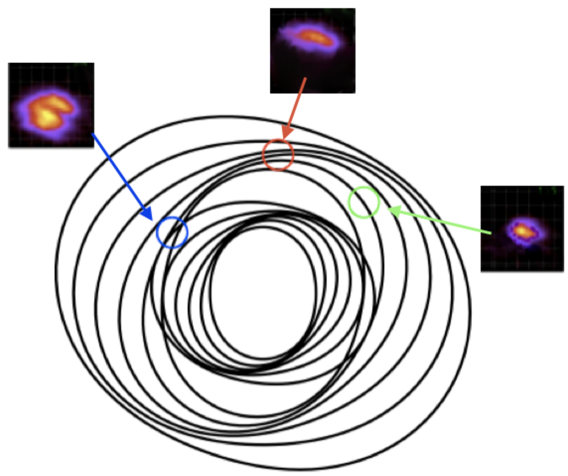

Figure 3: Illustrating how gaps in phase space could be related to overlaps of different pattern. Ellipses are shown with different mean radii but shifted in orientation. There is a jump in rotation angle at an intermediate radius.

### 2.3 The Lin-Shu hypothesis

The Lin-Shu hypothesis is that galactic spiral arms are caused by epicyclic perturbations of populations of stars that are locally in phase. Variations in density and epicycle are coupled and a wave like structure passes through the disk of a galaxy.

### 2.4 Perturbations in a rotating self-gravitating disk

Many instabilities are studied using perturbations near a steady state to derive a set of linear equations. A sinusoidal solution is used to find a relation between frequency and wave number, known as the dispersion relation. The WKB approximation can be used to simplify the equations. The dispersion relation describes either traveling waves or the growth rate of perturbations that can grow.

We assume that the disk is thin and can be approximated in a continuum approximation. In a fluid, conservation of momentum gives Euler's equation

$$\frac{d\mathbf{v}}{dt} + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{\nabla p}{\rho} - \nabla \Phi \tag{27}$$

where $\mathbf{v}$ is velocity, $\rho$ is density, $p$ is pressure, and $\Phi$ is the gravitational potential. We use cylindrical coordinates $r, \phi$, radial velocity component $u$ and tangential velocity component $v$. We also integrate vertically so that $\Sigma$ is the disk density per unit area.

$$\begin{aligned} \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{v}{r}\frac{\partial u}{\partial \phi} - \frac{v^2}{r} &= -\frac{\partial \Phi}{\partial r} - \frac{1}{\Sigma}\frac{\partial p}{\partial r} \\ \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + \frac{v}{r}\frac{\partial v}{\partial \phi} + \frac{uv}{r} &= -\frac{1}{r}\frac{\partial \Phi}{\partial \phi} - \frac{1}{r\Sigma}\frac{\partial p}{\partial \phi} \end{aligned} \tag{28}$$

We assume that the disk is near a steady state with circular rotation and no radial motions. The angular rotation rate satisfies $\Omega^2 = \frac{1}{r}\frac{\partial \Phi}{\partial r}$. The associated tangential velocity $v_0 = r\Omega$. The steady state solution is also called the zero-th order solution as it contains no small perturbations. It is also useful to use the frequency $\kappa$ which is the epicyclic frequency. Zeroth order quantities can be written with a 0 subscript and perturbative quantities with a subscript 1.

We assume that there are small perturbations in $u, v, \Sigma, \Phi$ and all perturbations $\propto e^{i(m\phi - \omega t)}$. We insert the perturbative expressions into Euler's equation (equation 28) and drop all second order terms

$$\begin{aligned} u_1 &= -\frac{i}{D}\left[(m\Omega - \omega)\frac{\partial}{\partial r}(\Phi_1 + h_1) - \frac{2m\Omega}{r}(\Phi_1 + h_1)\right] \\ v_1 &= -\frac{i}{D}\left[-2B\frac{\partial}{\partial r}(\Phi_1 + h_1) + \frac{m(m-\Omega)}{r}(\Phi_1 + h_1)\right] \end{aligned} \tag{29}$$

where

$$D = \kappa^2 - (m\Omega - \omega)^2 \tag{30}$$

describes the distance to Lindblad resonances and

$$B = \frac{\kappa^2}{4\Omega} \tag{31}$$

is an Oort constant. The enthalpy $h$ is a convenient function that satisfies

$$\nabla h = \frac{\nabla P}{\Sigma}. \tag{32}$$

With sound speed $c_s^2 = \frac{dP}{d\Sigma}$,

$$h_1 = c_s^2 \frac{\Sigma_1}{\Sigma_0}. \tag{33}$$

We use the WKB approximation. We assume the perturbations are $\propto e^{ikr}$ or $e^{i\alpha \ln r}$ where $k$ is a radial wave vector and $\alpha$ describes a winding angle. We assume that $kr > 1$ or $\alpha > 1$ so that terms with radial derivatives that don't involve $k$ or $\alpha$ are dropped.

We use Poisson's equation

$$\nabla^2 \Phi = 4\pi G \rho \tag{34}$$

and a thin disk approximation to relate the potential perturbation to the density perturbation. We assume that the density is zero above the plane. That means $\nabla^2\Phi = 0$ everywhere but in the plane. This means that

$$\Phi_1 \propto e^{ikr - k|z|}. \tag{35}$$

With the WKB approximation, slow radial perturbations are ignored in both $\Sigma$ and $\Phi$.

A Gaussian pillbox, centered on the galactic plane, with base area $A$ and height $z$ gives

$$\begin{aligned} \int_V 4\pi G \rho dV = \int_V \nabla^2 \Phi dV &= \int_S \nabla \Phi \cdot dA \\ 4\pi G \Sigma A \sim 2\frac{\partial \Phi}{\partial z} A. & \end{aligned} \tag{36}$$

With the potential perturbation in the form of equation 35

$$\Phi_1 = \frac{2\pi G \Sigma_1}{|k|}. \tag{37}$$

Conservation of mass is

$$\frac{\partial \rho}{\partial t}\rho + \nabla \cdot (\mathbf{v}\rho) = 0. \tag{38}$$

In cylindrical coordinates and integrating vertically

$$\frac{\partial \Sigma}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(ur\Sigma) + \frac{1}{r}\frac{\partial}{\partial \phi}(v\Sigma) = 0. \tag{39}$$

To first order in perturbations

$$i(m\Omega - \omega)\Sigma_1 + \frac{1}{r}\frac{\partial}{\partial r}(r\Sigma_0 u_1) + \frac{im\Sigma_0}{r} v_1 = 0. \tag{40}$$

Using the WKB approximation, we ignore the last term giving

$$(m\Omega - \omega)\Sigma_1 + k\Sigma_0 u_1 = 0. \tag{41}$$

Applying the WKB approximation to equations 29,

$$\begin{aligned} u_1 &\sim \frac{(m\Omega - \omega)k(\Phi_1 + h_1)}{D} \\ v_1 &\sim -\frac{2Bik(\Phi_1 + h_1)}{D}. \end{aligned} \tag{42}$$

## 2.5 Dispersion relation

Combining equations 42, 41, 37 and 33 gives a relation between frequency $\omega$ and wave number $k$ known as a **dispersion relation**.

$$(m\Omega - \omega)^2 = \kappa^2 - 2\pi G\Sigma|k| + c_s^2 k^2. \tag{43}$$

At large $k$ (short wavelength) we have sound waves. Small $k$, giving very open waves, are found near Lindblad resonances invaliding the WKB approximation. Waves can reflect off or be absorbed at boundaries or at resonances such as the Lindblad resonances. Waves can be driven at Lindblad resonances. Waves carry angular momentum. Angular momentum is a second order quantity in perturbation strength and so is more difficult to estimate, and here we have used a linear and first order approximation!

## 2.6 Toomre Q and instability

We set $m = 0$, describing an axisymmetric perturbation, and the dispersion relation becomes

$$\omega^2 = \kappa^2 - 2\pi G\Sigma_0|k| + c_s^2 k^2. \tag{44}$$

This gives a function $\omega^2(k)$. The right hand side is a quadratic equation. It looks like a parabola except shifted upward by $\kappa^2$ and shifted horizontally by the middle term. The frequency $\omega^2 > 0$ when the right hand side is positive. We can find the location of the minimum value for $\omega^2$, find $\omega^2$ at this minimum value and set it to zero to find critical values of $\kappa, c_s, \Sigma_0$ ensuring that $\omega^2 > 0$. The square $\omega^2 > 0$ is satisfied at all $k$ only if

$$Q \equiv \frac{c_s \kappa}{\pi G \Sigma_0} > 1. \tag{45}$$

This parameter is known as the Toomre $Q$ parameter. If $Q < 1$ then $\omega^2 < 0$ and that means we could have a perturbation that grows exponentially quickly rather than travels as a wave.

For $Q \lesssim 1$, a galaxy or protoplanetary disk is unstable and would spontaenously form clumps, spiral arms or a bar.

The Toomre $Q$ parameter is estimated for axisymmetric disturbances but it turns out to be relevant for non-axisymmetric disturbances too.

For a stellar or non-collisional disk, sound waves do not propagate, but short wavelength waves are damped via Landau damping, in direct analogy to the same mechanism in plasma physics. The $Q$ parameter is approximately the same in a particular disk as for a gaseous disk but with sound speed replaced by the velocity dispersion.

In most settings, disk thickness $h$ is related to the velocity dispersion $\sigma$ or sound speed $c_s$, setting

$$h \sim c_s\Omega \qquad \text{or} \qquad h \sim \sigma\Omega \tag{46}$$

where $\Omega$ is the angular rotation of a particle in a circular orbit. This can be derived assuming hydrostatic equilibrium or one of Jean's equations.

For a circumstellar disk $Q \sim \frac{M_*}{M_d}\frac{h}{r}$ where $M_d$ is the mass of the disk, $M_*$ the mass of the star and $h/r$ is the ratio of disk thickness to radius. Estimates of circumstellar disk masses and thickness imply that $Q > 1$, though a very thin layer of solids in the midplane could be gravitationally unstable.

Charles Gammie extended the theory of stability to spiral density waves to include cooling, and found that optical thick disks should not clump unless their cooling time is shorter than $\Omega^{-1}$ where $\Omega$ is the angular rotation rate. Hence if you see a simulation of a disk that forms many small clumps, probably it is able to cool off rapidly. Often assumed is an isothermal gas, with a constant temperature which acts as if it cools infinitely quickly as it stays the same temperature!

## 3 Torque Formula

Planets embedded within a disk or a moon embedded in a ring system drive density waves. An example is shown in Figure 4 which is from `https://photojournal.jpl.nasa.gov/catalog/PIA21627`.

Example equations 29 again. The term $D$ becomes large near a Lindblad resonance. Instead of taking the gravitational potential from self-gravity, assume that it is driven by a planet. Decompose the planet's gravitational perturbation in a Fourier series

$$\Phi_p(r,\theta,t) = \sum_m W_m(r)(m\theta - \Omega_p t)$$
$$W_m(r) = -\frac{GM_p}{r_p} b^m_{\frac{1}{2}}\left(\frac{r_p}{r}\right) \tag{47}$$

where $b^j_s()$ is a Laplace coefficient, $M_p, r_p$ are the mass and orbital radius of the planet. Here we assume the planet is in a circular orbit with mean motion $\Omega_p$.

The angular momentum flux at the $m$-th Lindblad resonance is

$$T_m = -\frac{m\pi^2\Sigma|\Psi_{GT,m}|}{r\frac{dD}{dr}} \tag{48}$$
$$\Psi_{GT,m} = r\frac{dW_m}{dr} + \frac{2\Omega}{\Omega - \Omega_p}W_m \tag{49}$$
$$D = \kappa^2 - m^2(\Omega - \Omega_p). \tag{50}$$

Waves are only driven at resonances. Otherwise perturbations don't add in phase with motions of the disk.

Waves become more open and stronger the closer to resonance.

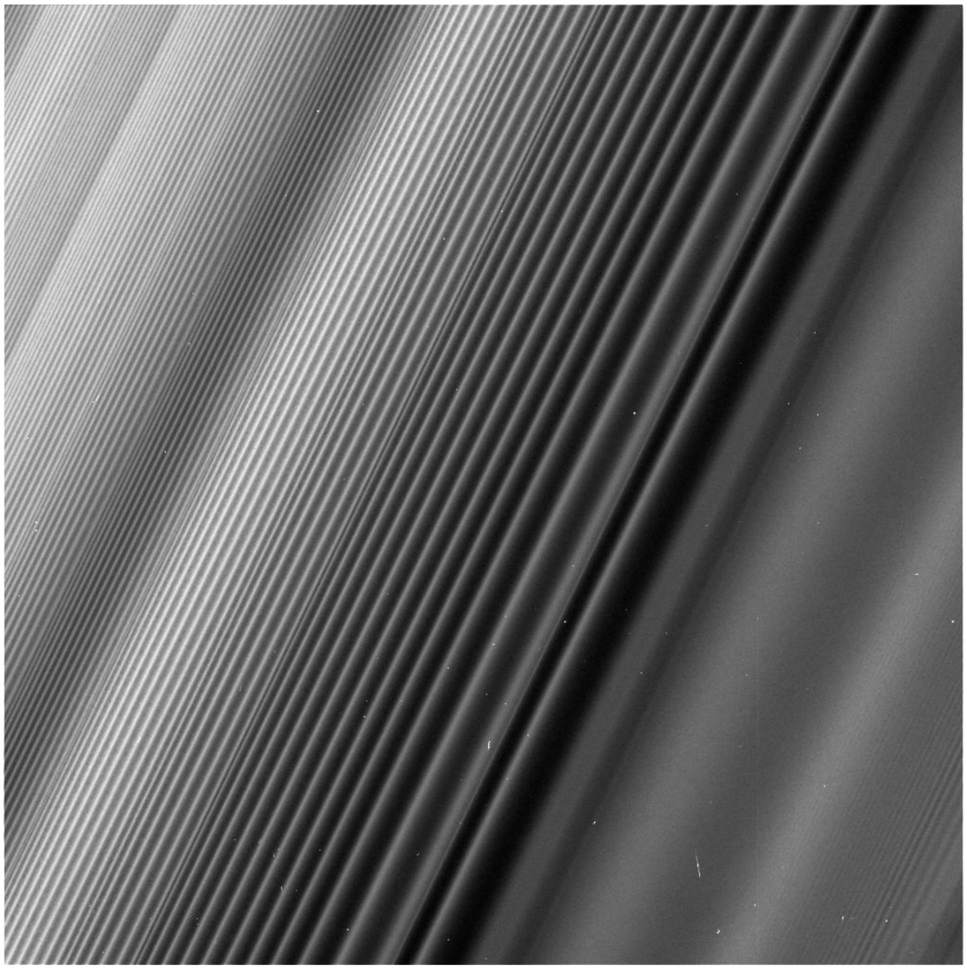

Figure 4: This is PIA21627 (PIA stands for planetary imaging atlas). Cassini spacecraft shows a wave structure in Saturn's B-ring ring known as the Janus 2:1 spiral density wave.

Waves are rapidly damped away from resonance.

High $m$ Lindblad resonances are found close to the planet's orbital radius. As $m$ can go to infinity, this would imply an infinitely large torque. However, because of pressure in the disk, the locations of the resonances are slightly further away than predicted directly from Keplerian motion. There is a cutoff known as the **torque cutoff**. Resonances with $m > h/r$ are ineffective at driving spiral density waves.

### 3.1 Gap opening

Spiral density waves driven by a planet carry angular momentum, pushing the disk away from the planet's orbital radius. If a gap is in a disk, viscosity in the disk would slowly fill in the gap. We can estimate a condition for gap opening based on an estimate for when the torque from spiral density waves overcomes viscous torque.

Viscous torque in an accretion disk depends on viscosity $\nu$

$$T_\nu = 3\pi\nu\Sigma r^2\Omega. \tag{51}$$

We define $q$ as the ratio of planet mass to stellar mass

$$q \equiv \frac{M_p}{M_*}. \tag{52}$$

We use equation 48 to estimate the torque from a spiral density wave driven at a Lindblad resonance with index $m$. Near the planet the term in the planet's potential (from equation 47)

$$W_m \sim q r_p^2 \Omega^2 \ln x_m \tag{53}$$

where $r_p$ is the radius of the planet's orbit and $x_m$ is the distance from the planet's orbital radius to the $m$ resonance. We have used an asymptotic limit for the Laplace coefficient. This gives (from equation 49)

$$\Psi_{GT,m} \sim \frac{r_p^3\Omega^2 q}{x}. \tag{54}$$

Since the torque $T_m \propto \frac{m|\Psi_{GT,m|^2}}{dD} dr$ (equation 48 and the distance to resonance $D \propto x$, and resonance index $m \propto 1/x$, the torque

$$T_m \sim q^2 \Sigma r_p^4 \Omega^2 \left(\frac{r_p}{x}\right)^4. \tag{55}$$

Summing all resonances to a distance $\Delta$ we find a total torque of

$$T \sim \Sigma r_p^4 \Omega^2 q^2 \left(\frac{r}{\Delta}\right)^3. \tag{56}$$

Set $\Delta = R_H \sim q^{\frac{1}{3}} r_p$ the planet's Roche radius to find

$$T \sim \Sigma r_p^4 \Omega^2 q. \tag{57}$$

We set equation 57 equal to the torque from viscosity (equation 51) so we can determine whether the planet can overcome viscous infill by driving density waves that push away the disk. We solve for the mass ratio giving a planet that can open a gap

$$q \gtrsim 40 Re^{-1} \tag{58}$$

where Reynolds number

$$Re = \frac{r_p^2 \Omega}{\nu}. \tag{59}$$

This set of arguments is following works by Lin & Papaloizou, Geof Bryden, and Aurelien Crida.

## 3.2 Planet migration

Planet migration is divided into two regimes, depending upon whether a gap is opened in the disk or not.

### 3.2.1 Type I migration

The setting is a planet that does not open a gap. The drift rate is set by the torque on the planet's orbit that is generated by driving spiral density waves into the disk. The planet's orbit has angular momentum

$$L_p = M_p \sqrt{G M_*} r_p^{\frac{1}{2}}. \tag{60}$$

If the planet's orbit drifts then the torque

$$\begin{aligned} \frac{dL_p}{dt} &= M_p \sqrt{G M_*} r_p^{-\frac{1}{2}} \frac{dr_p}{dt} \\ &= \frac{1}{2} M_p r_p \Omega_p \frac{dr_p}{dt}. \end{aligned} \tag{61}$$

To estimate the migration rate, we need to estimate the the difference between torque from driven by waves in one direction and that driven by waves going in the other direction. The result (modifying equation 57) gives a drift rate

$$\frac{dr_p}{dt} \sim r_p \Omega_p \left( \frac{M_p}{M_*} \right) \left( \frac{r_p^2 \Sigma}{M_*} \right) \left( \frac{h}{r} \right)^{-2}. \tag{62}$$

Note that the drift rate depends on the planet mass, so large planets drift more quickly.

### 3.2.2 Type II migration

In this setting the planet opens a gap and the planet drifts at a rate set by the viscous drift in the disk

$$\frac{dr_p}{dt} \sim \frac{\nu}{r} \tag{63}$$

The drift rate is independent of everything but disk viscosity.

Many gaps in Saturn's rings don't have a large single central body responsible for their opening. Recent observations of circumstellar disk have revealed gaps, but they are not necessarily opened by planets. Planetary migration seems inevitable and is an ingredient of many explorations for planetary system formation. Likewise Saturn's rings are likely dynamic in the sense that its structure has changed over the millennia (or billions of years) that they have been present. The exact age of Saturn's rings has long been debated, with spreading timescales suggesting youth, and potential sources of material suggesting age.