AST233 Lecture notes

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1 Eulerian and Lagrangian views

We view the system from a fixed coordinate system and describe each variable as a function of $(\mathbf{x}, t)$. The partial time derivative

$$\frac{\partial}{\partial t}$$

describes how variables change in time from the point of view of a fixed point in space attached to a coordinate system or an inertial frame. This is the Eulerian viewpoint.

We could also describe the system from the view point of particles moving with the fluid. Suppose we have a scalar quantity like T. We would like to predict what would cause a small change δT as our fluid element moves. Over a small change in time δt and with small changes in coordinates $\delta x, \delta y, \delta z$.

$$\delta T = \frac{\partial T}{\partial t}\delta t + \frac{\partial T}{\partial x}\delta x + \frac{\partial T}{\partial y}\delta y + \frac{\partial T}{\partial z}\delta z$$

We now divide by δt.

$$\frac{\delta T}{\delta t} = \frac{\partial T}{\partial t} + \frac{\partial T}{\partial x}\frac{\delta x}{\delta t} + \frac{\partial T}{\partial y}\frac{\delta y}{\delta t} + \frac{\partial T}{\partial z}\frac{\delta z}{\delta t} \tag{1}$$

If we chose $\delta x, \delta y, \delta t$ to be an element of the fluid that is moving along with the fluid then $\frac{\delta \mathbf{x}}{\delta t} = \mathbf{u}$ and we can write the above as

$$\frac{\delta T}{\delta t} = \frac{\partial T}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla} T$$

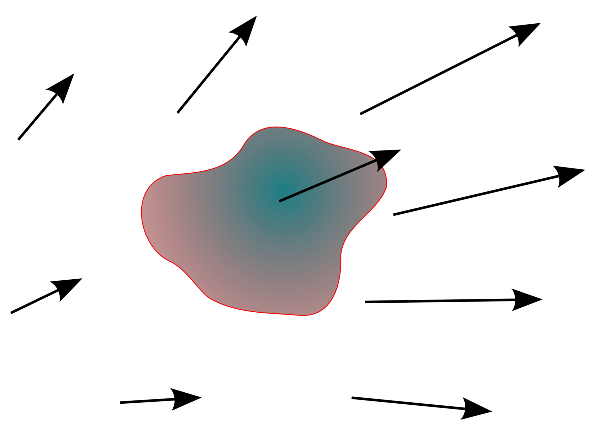

Figure 1: A fluid element moving within a larger flow.

If we consider derivatives from the point of view of particles moving with the fluid then we can describe changes with the Lagrangian time-derivative or

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla}$$

Let us write this out in terms of components

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \sum_i u_i \frac{\partial}{\partial x_i}$$

as we had done in equation (1). With summation notation it is understood that any repeated index is summed. With summation notation we would write

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u_i \frac{\partial}{\partial x_i}.$$

The index $i = 1$ gives x, $i = 2$ gives y and $i = 3$ gives the z coordinate.

Another way to think about this is to consider a fluid element at $\mathbf{x}$ that has moved by $\mathbf{u}\delta t$ in a time δt. If we consider T for that fluid element we can write T as

$$T(\mathbf{x} + \mathbf{u}\delta t, t + \delta t)$$

so the change in T moving with the fluid element

$$\begin{aligned} \frac{DT}{Dt} &= \lim_{\delta t \to 0} \left(\frac{T(\mathbf{x} + \mathbf{u}\delta t, t + \delta t) - T(\mathbf{x}, t)}{\delta t} \right) \\ &= \left[\frac{\partial}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla} \right] T \end{aligned}$$

If we write equations from the view point of fluid elements that are moving we say we are using the Lagrangian view point.

Consider traffic flow. We can describe traffic flow in terms of density, ρ, (cars per unit length) and a velocity, u, the speed of cars on the road. If we describe ρ and u as a function of position on the road we are using the Eulerian view point. If we describe ρ and u in terms of those seen by individual drivers we say we are using the Lagrangian viewpoint.

Numerical methods that use fixed grids work in the Eulerian view point. Numerical methods that allow particles to move in the simulation and compute forces on these particles work in the Lagrangian viewpoint. Smooth Particle Hydrodynamics (SPH) codes use the Lagrangian viewpoint.

2 The collisionless Boltzman equation

We call $f(\mathbf{x}, \mathbf{v})$ the phase space distribution function. A volume element in real space

$$d\mathbf{x}^3 = dx\ dy\ dz$$

A volume element in velocity space

$$d\mathbf{v}^3 = dv_x dv_y dv_z$$

The distribution function $f()$ is the number of stars (or particles) per unit volume in space per unit volume in velocity space. For a specific phase space volume element the number of stars in it is

$$f(\mathbf{x}, \mathbf{v}, t) d\mathbf{x}^3 d\mathbf{v}^3$$

What is the number of stars per unit volume?

$$n(\mathbf{x},t) = \int_{-\infty}^{\infty} dv_x \int_{-\infty}^{\infty} dv_y \int_{-\infty}^{\infty} dv_z \; f(\mathbf{x},\mathbf{v},t) = \int d^3\mathbf{v} f(\mathbf{x},\mathbf{v},t) \tag{2}$$

If all the particles have the same mass m then the density at position $\mathbf{x}$ is

$$\rho(\mathbf{x},t) = mn(\mathbf{x},t)$$

What is the mean velocity at a position $\mathbf{x}$?

$$\langle \mathbf{v}\rangle(\mathbf{x},t) = \mathbf{u}(\mathbf{x},t) = \frac{1}{n(\mathbf{x},t)} \int \mathbf{v} f(\mathbf{x},\mathbf{v},t) d^3\mathbf{v} \tag{3}$$

This is similar to the expression for an expectation value where f gives a probability distribution.

Conservation of mass for a fluid gives

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{u}) = 0$$

where density $\rho(\mathbf{x},t)$.

If stars are not born and do not disappear then similarly

$$\frac{\partial n}{\partial t} + \boldsymbol{\nabla} \cdot (n \mathbf{u}) = 0$$

This can be written in index form and using summation notation as

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x_i}(n u_i) = 0$$

Stars can change velocity. If stars are not born and do not die then $Df/dt = 0$. We can take $f(\mathbf{x},\mathbf{v},t)$ and differentiate all variables w.r.t. to time

$$\begin{aligned} \frac{Df}{Dt} &= \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x_i}\frac{dx_i}{dt} + \frac{\partial f}{\partial v_i}\frac{dv_i}{dt} = 0 \\ &= \frac{\partial f}{\partial t} + \boldsymbol{\nabla} f \cdot \mathbf{v} + \boldsymbol{\nabla}_v f \cdot \dot{\mathbf{v}} = 0, \end{aligned} \tag{4}$$

In the first line I used summation notation. I am using gradient operators

$$\begin{aligned} \boldsymbol{\nabla} &= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \\ \boldsymbol{\nabla}_v &= \left(\frac{\partial}{\partial v_x}, \frac{\partial}{\partial v_y}, \frac{\partial}{\partial v_z}\right) \end{aligned}$$

Equation 4 is the **collisionless Boltzmann equation**. The acceleration is related to the gradient of the gravitational potential

$$\dot{\mathbf{v}} = -\boldsymbol{\nabla}\Phi$$

so the **collisionless Boltzmann equation** can also be written as

$$\frac{\partial f}{\partial t} + \boldsymbol{\nabla} f \cdot \mathbf{v} - \boldsymbol{\nabla}_v f \cdot \boldsymbol{\nabla}\Phi = 0. \tag{5}$$

Collisions and birth and death of stars would add terms to the collisionless Boltzmann equation.

We are only keeping track of the position and velocity of stars. We could also take into account more degrees of freedom, such as mass or age or metallicity.

2.1 The particle distribution function

To describe a distribution of particles we can consider a particle distribution function that depends on position, velocity and time, $f(\mathbf{x}, \mathbf{v}, t)$. Here $f(\mathbf{x}, \mathbf{v}, t)d^3\mathbf{x}d^3\mathbf{v}$ represents the number of particles found in a volume element of volume $d^3\mathbf{x}$ and in a velocity bin of size $d^3\mathbf{v}$ at time t. Here volume elements

$$d^3\mathbf{x} = dx\ dy\ dz \qquad d^3\mathbf{v} = dv_x\ dv_y\ dv_z$$

in Cartesian coordinates. The number density (number of particles per unit volume) at position $\mathbf{x}$ and at time t would be

$$n(x, t) = \int_{-\infty}^{\infty} f(\mathbf{x}, \mathbf{v}, t) d^3\mathbf{v}$$

where we perform the integral in 3 dimensions. If each particle has mass m then the density

$$\rho(\mathbf{x}, t) = mn(\mathbf{x}, \mathbf{t}). \tag{6}$$

We can consider the average of any function $Q(\mathbf{v})$ as

$$\langle Q \rangle(\mathbf{x}, t) = n^{-1} \int Q(\mathbf{v}) f(\mathbf{x}, \mathbf{v}, t) d^3\mathbf{v}. \tag{7}$$

For example the bulk or average velocity would be

$$\mathbf{u}(\mathbf{x}, t) = \langle \mathbf{v} \rangle = n^{-1} \int \mathbf{v} f(\mathbf{x}, \mathbf{v}, t) d^3\mathbf{v} \tag{8}$$

and

$$\int v_i v_j f(\mathbf{x}, \mathbf{v}, t) d^3\mathbf{v} = n \langle v_i v_j \rangle \qquad \text{for} \qquad i \neq j$$

For a single component (like v_x^2 or v_y^2) we can write

$$\int v_i^2 f(\mathbf{x}, \mathbf{v}, t) d^3\mathbf{v} = n\langle v_i^2 \rangle$$

But this is not necessarily the same as $nu_i^2 = n(\langle v_i \rangle)^2$ which depends on the square of the average velocity. Usually

$$\langle v_i^2 \rangle \neq u_i^2 \qquad \langle v_i v_j \rangle \neq u_i u_j$$

We can define a total *velocity dispersion*, σ_a, **averaged over all directions**, as

$$\begin{aligned}
\sigma_a^2 &\equiv \frac{1}{3}\left(\langle (v_x - u_x)^2 \rangle + \langle (v_y - u_y)^2 \rangle + \langle (v_z - u_z)^2 \rangle\right) \\
&= \frac{1}{3n} \int |\mathbf{v} - \mathbf{u}|^2 f d^3\mathbf{v}
\end{aligned}$$

Evaluating σ_a^2

$$\begin{aligned}
\sigma_a^2 &= \frac{1}{3n} \int (v^2 + u^2 - 2\mathbf{u} \cdot \mathbf{v}) d^3\mathbf{v} \\
&= \frac{1}{3}(\langle v^2 \rangle + u^2) - \frac{2}{3n}\mathbf{u} \cdot \int \mathbf{v} d^3\mathbf{v} \\
&= \frac{1}{3}(\langle v^2 \rangle + u^2) - \frac{2}{3}u^2 \\
&= \frac{1}{3}(\langle v^2 \rangle - u^2)
\end{aligned} \tag{9}$$

so we can write

$$n\langle v^2 \rangle = \int v^2 f d^3\mathbf{v} = n(u^2 + 3\sigma_a^2)$$

We can think about the velocity v_i as a sum of the mean velocity u_i plus a random component. Let us consider a velocity dispersion tensor

$$w_{ij} \equiv \langle (v_i - u_i)(v_j - u_j) \rangle = \langle v_i v_j \rangle - u_i u_j$$

Here w_{ij} is a symmetric dispersion tensor with two indexes where each index can assume one of three values (x, y, z). When w_{ij} contains off diagonal components or its diagonal components are not equal we say the dispersion tensor is "anisotropic." If the system is "isotropic" then the diagonal components would all be the same and the off diagonal components would be zero.

We can write the trace of w as $\mathrm{tr} w = \sum_j w_{jj}$. The dispersion averaged over all directions (using equation 9)

$$\sigma_a^2 = \frac{1}{3}(\langle v^2 \rangle - u^2) = \frac{1}{3}\text{trace } w$$

If $w_{xx} = w_{yy} = w_{zz}$ then $\sigma_a^2 = w_{xx}$. The dispersion tensor is symmetric. We can decompose the dispersion tensor, w_{ij}, into the sum of a trace component that has zeros off the diagonal and a *symmetric traceless component*, y_{ij};

$$y_{ij} = \frac{w_{ij} + w_{ji}}{2} - \text{trace } w\frac{\delta_{ij}}{3} = \frac{w_{ij} + w_{ji}}{2} - \sigma_a^2 \delta_{ij}$$

Note that y_{ij} can contain components on the diagonal but their sum would be zero. If the system is isotropic then all components of y_{ij} would be zero.

We can associate pressure in a fluid or gas with the trace of the dispersion tensor $\sum_j w_{jj}$ or σ_a^2.

2.2 Collisionless Boltzmann equation

In the absence of collisions the collisionless Boltzmann equation describes the evolution of the density distribution.

$$\frac{Df}{Dt} = \frac{\partial f(\mathbf{x}, \mathbf{v}, t)}{\partial t} + \frac{\partial f(\mathbf{x}, \mathbf{v}, t)}{\partial \mathbf{x}} \cdot \frac{d\mathbf{x}}{dt} + \frac{\partial f(\mathbf{x}, \mathbf{v}, t)}{\partial \mathbf{v}} \cdot \frac{d\mathbf{v}}{dt} = 0.$$

The derivative here is done with respect to all degrees of freedom of the distribution function. As $\mathbf{v} = d\mathbf{x}/dt$ and $d\mathbf{v}/dt = -\boldsymbol{\nabla}\Phi$ for a force field with potential Φ we can write

$$\frac{\partial f(\mathbf{x}, \mathbf{v}, t)}{\partial t} + \boldsymbol{\nabla} f(\mathbf{x}, \mathbf{v}, t) \cdot \mathbf{v} - \boldsymbol{\nabla}_v f(\mathbf{x}, \mathbf{v}, t) \cdot \boldsymbol{\nabla}\Phi = 0. \tag{10}$$

Using summation notation this equation is

$$\frac{\partial f(\mathbf{x}, \mathbf{v}, t)}{\partial t} + \frac{\partial f(\mathbf{x}, \mathbf{v}, t)}{\partial x_i} v_i - \frac{\partial f(\mathbf{x}, \mathbf{v}, t)}{\partial v_i} \frac{\partial \Phi(\mathbf{x}, t)}{\partial x_i} = 0. \tag{11}$$

Equation 10 (or 11) is known as the collisionless Boltzmann equation. It is used to study the kinetic theory of gases, atomic nuclei and for stellar dynamical systems such as galaxies and globular clusters. The collisionless Boltzmann equation is sufficiently complex that it is usually difficult to solve. Equation 10 is sometimes written

$$\frac{Df}{Dt} = 0$$

where the Lagrangian derivative is

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \boldsymbol{\nabla} - \boldsymbol{\nabla}\Phi \cdot \boldsymbol{\nabla}_v$$

Here the Lagrangian derivative describes a small element moving in *phase space* or $(\mathbf{x}, \mathbf{v})$. Previously we used a Lagrangian derivative for a small element moving only in Cartesian space.

When collisions are important we can use the full Boltzmann equation by adding a source term that is due to collisions

$$\frac{Df}{Dt} = \left(\frac{\partial f}{\partial t}\right)_C$$

where the term on the right hand side depends on the cross sections of particles and their velocity differences. In many situations collisions conserve mass, momentum and kinetic energy. When these are conserved

$$\int m \left(\frac{\partial f}{\partial t}\right)_C d^3\mathbf{v} = 0$$
$$\int m\mathbf{v} \left(\frac{\partial f}{\partial t}\right)_C d^3\mathbf{v} = 0$$
$$\int mv^2 \left(\frac{\partial f}{\partial t}\right)_C d^3\mathbf{v} = 0.$$

Connections between different equations		
No collisions	in phase space	Collisionless Boltzmann equation
With collisions	in phase space	Fokker-Planck equation
Averaging over velocity and space		Tensor Virial equations
Taking the trace		Virial equation
Velocity moments	in real space	Jeans equations

The Fokker-Planck equation can used within what is known as **kinetic theory** to derive quantities like viscosity and thermal conductivity. The collisionless Boltzmann equation can be relevant in the context of rarefied gases where the mean free path is long compared to sizes of interest, for example a particle embedded in the outer parts of a protostellar disk.

2.3 Galactic Observables

If $\hat{\mathbf{n}}_{los}$ is a unit vector giving the direction of the line of sight, the line of sight component of the mean velocity

$$\langle \mathbf{v} \rangle \cdot \hat{\mathbf{n}}_{los} = \mathbf{u} \cdot \hat{\mathbf{n}}_{los} = u_{los} \tag{12}$$

is an observable. This would be a mean velocity measured from a spectrum at a particular position.

The velocity dispersion component in the same direction is would be

$$\sigma^2_{los}(\mathbf{x}, t) = \langle (\mathbf{v} \cdot \hat{\mathbf{n}}_{los} - u_{los})^2 \rangle = \frac{1}{n} \int d^3\mathbf{v} f(\mathbf{x}, \mathbf{v}, t)(v_z - u_z)^2 \tag{13}$$

Suppose the line of sight is in the z direction and the sky is in the x, y plane. We can integrate along the line of sight

$$g(x, y, \mathbf{v}, t) = \int dz f(x, y, z, v_x, v_y, v_z, t). \tag{14}$$

This is a distribution function as a function of velocity and position on the sky. The surface brightness on the sky

$$\mu(x, y, t) \propto \int dz d^3\mathbf{v} f(x, y, z, v_x, v_y, v_z, t) \tag{15}$$

The constant of proportionality depends on the brightnesses of the stars. A spectrum would be sensitive to Doppler shifts in the z direction giving

$$h(x, y, v_z, t) = \int dz dv_x dv_y f(x, y, z, v_x, v_y, v_z, t). \tag{16}$$

which is proportional to the brightness as a function of velocity at position x, y. This is what would be measured from an integral field spectrograph at different positions x, y on the sky. The mean velocity in the z direction would be the average velocity at a particular position

$$u_z = \frac{\int h(x, y, z, v_z, t) v_z dv_z}{\int h(x, y, v_z, t) dv_z} \tag{17}$$

In a galaxy absorption lines seen in stars are broadened by the different Doppler shifts caused by the motions of the stars. The velocity dispersion along the line of sight direction at different positions on the sky can be observed spectroscopically

$$\sigma_z^2(x, y, t) = \frac{\int dz d^3\mathbf{v} (v_z - u_z)^2 f(x, y, z, v_x, v_y, v_z, t)}{\int dz d^3\mathbf{v} f(x, y, z, v_x, v_y, v_z, t)}.$$

2.4 Conservation of mass

The simplest continuum equation can be made by integrating the Boltzmann equation over all possible velocities. The first term in the collisionless Boltzmann equation ($\partial f/\partial t$) gives us the time derivative of the particle density. Integrating the first term in the collisionless Boltzmann equation over velocity space

$$\int_{-\infty}^{\infty} \frac{\partial f(\mathbf{x}, \mathbf{v}, t)}{\partial t} d^3\mathbf{v} \approx \frac{\partial}{\partial t} \int_{-\infty}^{\infty} f(\mathbf{x}, \mathbf{v}, t) d^3\mathbf{v} = \frac{\partial}{\partial t} n(\mathbf{x}, t)$$

The second term in the collisionless Boltzmann equation is $\mathbf{v} \cdot \boldsymbol{\nabla} f$. Both $\mathbf{x}$ and $\mathbf{v}$ are arguments of the distribution function f so their derivatives commute. We can integrate the second term

$$\int_{-\infty}^{\infty} \boldsymbol{\nabla} f(\mathbf{x}, \mathbf{v}, t) \cdot \mathbf{v}\, d^3\mathbf{v} = \boldsymbol{\nabla} \cdot \int f \mathbf{v} d^3\mathbf{v} = \boldsymbol{\nabla} \cdot (n\mathbf{u})$$

where we have rewritten the last term in terms of the average velocity $\mathbf{u}$. The last term in the collisionless Boltzmann equation is $-\boldsymbol{\nabla}_v f \cdot \boldsymbol{\nabla}\Phi(\mathbf{x})$. We integrate this over velocity space

$$-\boldsymbol{\nabla}\Phi(\mathbf{x},t) \cdot \int d^3\mathbf{v} \boldsymbol{\nabla}_v f(\mathbf{x},\mathbf{v},t)$$

Consider one part of the sum

$$-\frac{\partial \Phi(\mathbf{x},t)}{\partial x} \int dv_x dv_y dv_z \frac{\partial f}{\partial v_x} = -\frac{\partial \Phi(\mathbf{x},t)}{\partial x} \int dv_y dv_z f(\mathbf{x},\mathbf{v},t) \Big]_{v_x=-\infty}^{v_x=\infty} = 0$$

This vanishes as long as we assume that the numbers of stars is small at large velocity, or $f \to 0$ as $v_i \to \pm\infty$.

Putting these together with the integral of the collision term (also zero) we find

$$\frac{\partial n}{\partial t} + \boldsymbol{\nabla} \cdot (n\mathbf{u}) = 0 \tag{18}$$

To summarize: the integral over velocity space of the Boltzmann equation gives an equation that looks just like the equation for conservation of mass for a fluid.

2.5 Conservation of momentum and Jeans equations

To derive an equation similar to Euler's equation (which is a result of conservation of momentum) we multiply the Boltzmann equation by $\mathbf{v}$ and then again integrate over velocity space. Taking the i-the component of the velocity and using summation notation for the other indices

$$\int \left(\frac{\partial f}{\partial t} v_i + \frac{\partial f}{\partial x_j} v_j v_i - \frac{\partial f}{\partial v_j} \frac{\partial \Phi}{\partial x_j} v_i \right) d^3\mathbf{v} = \int \left(\frac{\partial f}{\partial t} \right)_C v_i d^3\mathbf{v} = 0 \tag{19}$$

Consider the first term

$$\int \frac{\partial f}{\partial t} v_i d^3\mathbf{v} = \frac{\partial}{\partial t} \int f v_i d^3\mathbf{v} = \frac{\partial}{\partial t}(n\langle v_i \rangle) = \frac{\partial (n u_i)}{\partial t}$$

Consider the second term of equation 19. This can be written

$$\int \frac{\partial f}{\partial x_j} v_j v_i d^3\mathbf{v} = \frac{\partial}{\partial x_j} \int f v_j v_i d^3\mathbf{v} = \frac{\partial}{\partial x_j}[n\langle v_j v_i \rangle].$$

We can decompose this in terms of the dispersion tensor ($\mathbf{w}$) and then the traceless component of the dispersion tensor ($\mathbf{y}$) and the average dispersion (σ_a^2)

$$\begin{aligned} \frac{\partial}{\partial x_j}[n\langle v_j v_i \rangle] &= \frac{\partial}{\partial x_j}[n(u_i u_j + w_{ij})] \\ &= \frac{\partial}{\partial x_j}[n(u_i u_j + \sigma_a^2 \delta_{ij} + y_{ij})] \\ &= \frac{\partial}{\partial x_j}[n(u_i u_j + y_{ij}) + P\delta_{ij}], \end{aligned} \tag{20}$$

where we define a pressure in terms of the trace of the dispersion tensor

$$P \equiv n\sigma_a^2 = \frac{nw_{ii}}{3}.$$

Altogether the second term in the momentum equation (19) becomes

$$\frac{\partial}{\partial x_j}(nu_iu_j + P\delta_{ij} + ny_{ij}).$$

Within the context of hydrodynamics, we would call the term nu_iu_j *ram pressure*. The first two terms inside the derivative, $nu_iu_j + P\delta_{ij}$ contribute to the **stress tensor**. The last term ny_{ij} depends in the traceless component of the dispersion tensor and is only non-zero when the velocity distribution is *anisotropic*.

The third term in the momentum equation (19) can be integrated by parts. The term is

$$\frac{\partial \Phi}{\partial x_j}\int \frac{\partial f}{\partial v_j} v_i \, d^3\mathbf{v}.$$

First consider the case $i \neq j$ and let k be the third index

$$\frac{\partial \Phi}{\partial x_j}\int dv_k \int dv_i v_i \int dv_j \frac{\partial f}{\partial v_j} = \frac{\partial \Phi}{\partial x_j}\int dv_k \int dv_i v_i \; f(\mathbf{x},\mathbf{v},t)\,\Big|_{v_j=-\infty}^{v_j=\infty} = 0.$$

Now consider the case $i = j$. We integrate by parts

$$\begin{aligned}\int \frac{\partial f}{\partial v_i} v_i dv_i &= f v_i\Big|_{-\infty}^{\infty} - \int f dv_i \\ &= -\int f dv_i.\end{aligned}$$

Insert this back into the full term for $i = j$,

$$\begin{aligned}\frac{\partial \Phi}{\partial x_j}\int \frac{\partial f}{\partial v_j} v_i \, d^3\mathbf{v}\delta_{ij} &= -\frac{\partial \Phi}{\partial x_i}\int f d^3\mathbf{v} \\ &= -n\frac{\partial \Phi}{\partial x_i}.\end{aligned}$$

This is the integrated third term of equation (19)

Altogether (19) becomes

$$\frac{\partial}{\partial t}(nu_j) + \frac{\partial}{\partial x_j}(nu_iu_j + P\delta_{ij} + ny_{ij}) + n\frac{\partial \Phi}{\partial x_i} = 0$$

This is an equation for momentum conservation. Except for the term associated with anisotropy this looks just like that derived in hydrodynamics but with n replaced by mass density ρ.

By making use of the equation of continuity we can manipulate this equation so that it becomes an equation for acceleration that resembles Euler's equation

$$\frac{D\mathbf{u}}{Dt} = -\frac{1}{n}\boldsymbol{\nabla} P - \boldsymbol{\nabla}\Phi - \frac{1}{n}\boldsymbol{\nabla}\cdot(n\mathbf{y})$$

where the last term is a divergence of the traceless component of the dispersion tensor. If the velocity dispersion is isotropic then $\mathbf{y} = 0$ and we recover Euler's equation. To summarize: by multiplying the Boltzmann equation by velocity and integrating over all velocities we recover an equation that looks remarkably like Euler's equation.

Here we have integrated over velocity. We have taken the first "moment" of the collisionless Boltzmann equation. If one also integrates over all space one can derive tensor "virial" equations. Integrating only over velocity and working in cylindrical or spherical coordinates the equations, and in the setting of stellar dynamics, the equations are called the *Jeans equations*.

Using equation 20 and not trying to use a pressure like term we can also write the momentum equation as

$$\frac{\partial}{\partial t}(nu_j) + \frac{\partial}{\partial x_i}(nu_iu_j + nw_{ij}) + n\frac{\partial\Phi}{\partial x_j} = 0 \tag{21}$$

and using summation notation.

Then combined with the equation of continuity (equation 42) this becomes

$$n\frac{\partial u_j}{\partial t} + nu_i\frac{\partial u_j}{\partial x_i} + n\frac{\partial\Phi}{\partial x_j} + \frac{\partial(nw_{ij})}{\partial x_i} = 0 \tag{22}$$

These equations are known as the **Jeans equations**.

Jeans equations in vector form is

$$n\frac{\partial\mathbf{u}}{\partial t} + n(\mathbf{u}\cdot\nabla)\mathbf{u} + n\nabla\Phi + \nabla\cdot(n\mathbf{w}) = 0. \tag{23}$$

In the last term $\mathbf{w}$ is a 2 index tensor. In this form the Jeans equations resemble the Navier Stokes equation, as expected, since they are consistent with conservation of mass and momentum.

The Jeans equations contain 9 unknowns (3 average velocities and 6 dispersion tensor terms), but are only 3 equations. Jeans equations cannot be solved unless additional assumptions are made, or by leveraging symmetry. It is often convenient to work in a coordinate system in which the dispersion tensor is diagonal or assume that the system is isotropic, which means that the dispersion tensor is diagonal in Cartesian coordinates and all components are the same.

2.6 Jeans equation for a spherically symmetric isotropic system

Consider a system that is spherically symmetrical and in equilibrium. Since the system obeys rotational symmetry, $\mathbf{u} = 0$ and the velocity dispersion tensor must be diagonal $w_{ij} \propto \delta_{ij}$. If we in addition assume an isotropic velocity dispersion, then $w_{ij} = \sigma^2 \delta_{ij}$. If the system is static then $\frac{\partial \mathbf{u}}{\partial t} = 0$. Because of the spherical symmetry, $\sigma(r), n(r), \Phi(r)$ are functions of radius. We relate the number density to the mass density with a mean stellar mass m.

We apply Jean's equation (equation 22) along the x direction,

$$n\frac{\partial \Phi}{\partial x} + \frac{\partial (n\sigma^2)}{\partial x} = 0. \tag{24}$$

We can consider Jeans equation along the x axis where $x = r$ and

$$n\frac{\partial \Phi}{\partial r} + \frac{\partial (n\sigma^2)}{\partial r} = 0. \tag{25}$$

We relate the derivative of the potential to the mass density n and the mass $M(r)$ within radius r

$$\begin{aligned}
M(r) &= \int_0^r dr' 4\pi r'^2 n(r') m \\
\frac{dM}{dr} &= 4\pi r^2 n(r) m \\
\frac{d\Phi}{dr} &= \frac{GM(r)}{r^2}
\end{aligned}$$

If you know the mass distribution, then you know $n(r)$ from the second equation and you know $\frac{d\Phi}{dr}$ from the third equation. These can be inserted into equation 25 to solve for $\sigma^2(r)$.

2.7 The singular isothermal sphere

What would be the velocity dispersion $\sigma(r)$ of a spherically symmetric isotropic system with a flat rotation curve that has velocity v_c? By isotropic we mean

$$\langle (v_x - u_x)^2 \rangle = \langle (v_y - u_y)^2 = \langle (v_z - u_z)^2 = \sigma^2. \tag{26}$$

Because the mass distribution is spherically symmetric

$$\begin{aligned}
\frac{GM(r)}{r^2} &= \frac{v_c^2}{r} = \frac{d\Phi}{dr} \\
\Phi(r) &= v_c^2 \ln r \\
n(r) &= \frac{1}{4\pi r^2 m}\frac{dM}{dr} = \frac{v_c^2}{Gm4\pi r^2}.
\end{aligned}$$

We plug these into Jeans equation (equation 25)

$$\frac{v_c^2}{Gm4\pi r^2} \times \frac{v_c^2}{r} + \frac{\partial}{\partial r}\left(\frac{v_c^2}{Gm4\pi r^2}\sigma^2\right) = 0$$
$$\frac{v_c^2}{r^3} + \frac{\partial}{\partial r}\left(\frac{\sigma^2}{r^2}\right) = 0$$
$$-2\frac{v_c^2}{r^2} + \frac{\sigma^2}{r^2} = \text{constant.}$$

If we desire a finite σ^2 at large r, we neglect the constant and the solution is

$$\sigma = \sqrt{2}v_c. \tag{27}$$

Because the rotational velocity is constant, so is the velocity dispersion. In a gas, the velocity dispersion is directly related to temperature. Because the velocity dispersion is constant, this mass distribution is called **isothermal**. Because the density $n(r) \propto r^{-2}$, the mass distribution is called **singular**.

3 Moments of the Collisionless Boltzmann equation

3.1 The tensor virial equations

We will integrate the collisionless Boltzmann equation over all space.

We define something that is *like* a moment of inertia tensor

$$I_{ij} \equiv \int d^3\mathbf{x}\rho(\mathbf{x})x_i x_j = m \int d^3\mathbf{x}\ x_i x_j \int d^3\mathbf{v} f(\mathbf{x}, \mathbf{v}, t)$$

This is to be compared to the actual moment of inertia tensor for a rigid body about the origin which is the sum over mass elements inside the rigid body

$$I_{ij,actual} = \sum_k m_k(r^2\delta_{ij} - x_i x_j) = \int d^3\mathbf{x}\rho(\mathbf{x})(r^2\delta_{ij} - x_i x_j)$$

where r is the distance to the origin for each particle in the sum and x_i is x, y or z depending upon the index.

Kinetic energy per unit volume

$$\sum_i \frac{1}{2}\int d^3\mathbf{v} v_i^2 f(\mathbf{x}, \mathbf{v}, t)m = \sum_i \frac{1}{2} n(\mathbf{x}, t) m\langle v_i^2\rangle = \sum_i \frac{1}{2}\rho(\mathbf{x}, t)\langle v_i^2\rangle$$

The total kinetic energy

$$K = \sum_i \frac{1}{2}\int d^3\mathbf{x} n(\mathbf{x}, t) m\langle v_i^2\rangle = \sum_i \frac{1}{2}\int d^3\mathbf{x}\rho(\mathbf{x}, t)\langle v_i^2\rangle$$

A more general total kinetic energy tensor we define as

$$K_{ij} \equiv \frac{1}{2} \int d^3\mathbf{x} \rho(\mathbf{x}, t) \langle v_i v_j \rangle \tag{28}$$

$$= \frac{1}{2} m \int d^3\mathbf{x} \int d^3\mathbf{v} f(\mathbf{x}, \mathbf{v}, t) v_i v_j. \tag{29}$$

The trace of this

$$\sum_i K_{ii} = K$$

is the total kinetic energy.

A velocity tensor can be defined as

$$T_{ij} = \frac{1}{2} \int d^3\mathbf{x} \rho(\mathbf{x}, t) u_i u_j. \tag{30}$$

Here $\mathbf{u}$ is the local average velocity (defined in equation 8) and this vector is a function of position. This tensor describes streaming or rotational motions.

A tensor describing random rather than streaming motions is an integral of the velocity dispersion

$$\Pi_{ij} \equiv \int d^3\mathbf{x} \rho(\mathbf{x}) w_{ij}^2 \tag{31}$$

$$= \int d^3\mathbf{x} \rho(\mathbf{x}) (\langle v_i v_j \rangle - u_i u_j)$$

$$= 2K_{ij} - 2T_{ij}.$$

This gives a relation between the total kinetic energy tensor, the order velocity tensor and the the random velocity tensor

$$K_{ij} = T_{ij} + \frac{1}{2} \Pi_{ij}. \tag{32}$$

Lastly we create a tensor for the gravitational energy. We tentatively define a **gravitational potential energy tensor** as

$$W_{jk} \equiv - \int d^3\mathbf{x} \rho(\mathbf{x}) x_j \frac{\partial \Phi(\mathbf{x})}{\partial x_k} \tag{33}$$

This is also known as the *Chandrasekkar potential energy tensor*. The gravitational potential

$$\Phi(\mathbf{x}) = G \int d^3\mathbf{x}' \frac{\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}$$

The gradient of the gravitational potential

$$\frac{\partial \Phi(\mathbf{x})}{\partial x_k} = -G \int d^3\mathbf{x}' \frac{\rho(\mathbf{x}')(x_k - x'_k)}{|\mathbf{x} - \mathbf{x}'|^3}$$

This gives an alternative form for W_{jk}

$$W_{jk} = G\int d^3\mathbf{x}\int d^3\mathbf{x}'\rho(\mathbf{x})\rho(\mathbf{x}')\frac{x_j(x_k - x'_k)}{|\mathbf{x}-\mathbf{x}'|^3}$$
$$= -\frac{1}{2}G\int\int d^3\mathbf{x}d^3\mathbf{x}'\rho(\mathbf{x})\rho(\mathbf{x}')\frac{(x_j - x'_j)(x_k - x'_k)}{|\mathbf{x}-\mathbf{x}'|^3}$$

In the last step we infer that we can flip the indices to rewrite the integral in such a way that it is clear that it is symmetric. The trace of this

$$W = \sum_j W_{jj} = \frac{1}{2}\int d^3\mathbf{x}\rho(\mathbf{x})\Phi(\mathbf{x})$$

is equal to the total gravitational potential energy.

Now that we have a few definitions, we go back to the first moment of the collisionless Boltzmann equation (equation 5, or 21) which I repeat here:

$$\frac{\partial}{\partial t}(nu_j) + \frac{\partial}{\partial x_i}(nu_iu_j + nw_{ij}) + n\frac{\partial\Phi}{\partial x_j} = 0 \tag{34}$$

We multiply this by mx_k and integrate this over all space

$$\int d^3\mathbf{x}x_k\frac{\partial}{\partial t}(\rho u_j) + \int d^3\mathbf{x}x_k\frac{\partial}{\partial x_i}(\rho u_iu_j + \rho w_{ij}) + \int d^3\mathbf{x}x_k\rho\frac{\partial\Phi}{\partial x_j} = 0 \tag{35}$$

The last term on the right is equal to the potential energy tensor $-W_{jk}$ from the definition in equation 33. The second term can be integrated by parts

$$\int d^3\mathbf{x}x_k\frac{\partial}{\partial x_i}(\rho u_iu_j + \rho w_{ij}) = \delta_{ik}\left[x_k(\rho u_iu_j + \rho w_{ij})|_{x_k=-\infty}^{\infty} - \int d^3\mathbf{x}(\rho u_iu_j + \rho w_{ij})\right]$$
$$= 0 - \delta_{ik}\left(2T_{ij} + \Pi_{ij}\right)$$
$$= -(2T_{jk} + \Pi_{jk})$$

where we have neglected a term on the second line with the assumption that the density is zero at infinity, $\rho \to 0$ at $x \to \pm\infty$. This neglect means the outer boundaries might affect the results. The first term can be written in terms of

$$\int d^3\mathbf{x}x_k\frac{\partial}{\partial t}(nu_j) = \int d^3\mathbf{x}d^3\mathbf{v}x_kv_j\frac{\partial f}{\partial t}$$
$$= \int d^3\mathbf{x}d^3\mathbf{v}x_kv_j(-\frac{\partial f}{\partial x_i}v_i + \frac{\partial f}{\partial v_i}\frac{\partial\Phi}{\partial x_i})$$

using $\frac{df}{dt} = 0$. The second term is removed by integrating by parts in space. The first term, after integrating by parts becomes

$$\int d^3\mathbf{x}x_k\frac{\partial}{\partial t}(nu_j) = \int d^3xd^3vfv_jv_k. \tag{36}$$

Equation 35 becomes

$$\frac{d}{dt}\int d^3\mathbf{x} x_k \rho u_i = 2T_{jk} + \Pi_{jk} + W_{jk}. \tag{37}$$

With a bit more effort, the left hand side can be related to the moment of inertia tensor.

$$\begin{aligned}
\frac{d}{dt}I_{jk} &= \frac{d}{dt}\int d^3x d^3v f x_j x_k \\
&= \int d^3x d^3v \left(f v_j x_k + f x_j v_k + x_j x_k \frac{df}{dt} \right) \\
&= \int d^3x d^3v \left(f v_j x_k + f x_j v_k \right) \qquad \text{because } \frac{df}{dt} = 0 \\
\frac{d^2}{dt^2}I_{jk} &= \int d^3x d^3v f(\dot{v}_j x_k + x_j \dot{v}_k + 2v_j v_k).
\end{aligned}$$

We use the fact that $\dot{v}_j = -\frac{\partial \Phi}{\partial x_j}$ and integrate by parts (in space) to show that the terms with accelerations are zero. This leaves

$$\frac{1}{2}\frac{d^2}{dt^2}I_{jk} = \int d^3x d^3v \; f v_j v_k.$$

Which is the same thing as in equation 36.

The resulting **tensor virial equation** is

$$\frac{1}{2}\frac{d^2 I_{jk}}{dt^2} = 2T_{jk} + \Pi_{jk} + W_{jk}. \tag{38}$$

In steady state, there is a relationship between the gravitational potential energy which depends on shape, the velocity dispersion, with could be anisotropic, and the bulk motion or rotation. Elongated non rotating (elliptical) galaxies tend to have anisotropic velocity dispersions. Rotating galaxies tend to be flat (they are disky). Elliptical galaxies tend to be supported by their velocity dispersion rather than by rotation.

Using equation 32 for the kinetic energy tensor, the tensor virial equation (equation 38) becomes

$$\frac{1}{2}\frac{d^2 I_{jk}}{dt^2} = 2K_{jk} + W_{jk}. \tag{39}$$

Taking the trace of the steady state equation, the tensor virial theorem becomes

$$2K + W = 0 \tag{40}$$

which we recognize as the **scalar** version of the virial theorem.

3.2 Applications of Jean's equations

The velocity moments of the collisionless Boltzmann equation are called Jeans equations.

One application is known as *asymmetric drift*. Consider a disk of stars all in circular orbits about the center a galaxy and all confined to a single plane. The velocity dispersion is small. In a local region the average velocity is tangential and is equal to the circular velocity.

Now consider a similar disk of stars but the stars have some ellipticity to their orbits and undergo radial oscillations. The orbits have random phases so the stars do not move in and out together. The velocity dispersion arises from the radial oscillations of the orbits. What is the mean tangential velocity component? It must be slightly lower than the rotation velocity. This makes sense looking at the tensor virial equations. The difference between the mean tangential velocity and that of a star in a circular orbit is known as *asymmetric drift*.

Using Jeans equation in polar coordinates, it is possible to show that

$$v_a \equiv \langle v_\phi \rangle - v_c \approx \frac{\langle v_R^2 \rangle}{2v_c} \left[\frac{\sigma_\phi^2}{\langle v_R^2 \rangle} - 1 - \frac{\partial \ln(n \langle v_R^2 \rangle)}{\partial \ln R} - \frac{R}{\langle v_R^2 \rangle} \frac{\partial(\langle v_R v_z \rangle)}{\partial z} \right]$$

Another application of Jean's equations is similar to hydrostatic equilibrium giving a relation between the velocity dispersion and density in the z direction and the gradient of the potential. Repeating Jeans equations (equation 22)

$$\frac{\partial u_j}{\partial t} + u_i \frac{\partial u_j}{\partial x_i} + \frac{\partial \Phi}{\partial x_k} + \frac{1}{n} \frac{\partial (n w_{ij})}{\partial x_i} = 0 \tag{41}$$

We assume steady state and drop the first term. We assume symmetry about the galactic plane, no vertical bulk or average motion and take the z component. The result is this:

$$\frac{1}{n} \frac{\partial (n \langle v_z^2 \rangle)}{\partial z} = -\frac{\partial \Phi}{\partial z}$$

Using Poisson's equation

$$\frac{\partial^2 \Phi}{dz^2} = 4\pi G \rho$$

Putting these two together we find

$$\frac{\partial}{\partial z} \left[\frac{1}{n} \frac{\partial (n \langle v_z^2 \rangle)}{\partial z} \right] = -4\pi G \rho$$

The left hand side can be measured using vertical velocity measurements for stars as a function of distance above and below the Galactic plane. The stellar tracers, giving n, need not necessarily represent the total mass whereas ρ on the right would contain contributions from both stellar mass and dark matter. Thus measurements of the vertical stellar velocity distribution as a function of distance above the Galactic plane can be used to estimate the fraction of dark matter in the vicinity of the Sun.

3.3 Tremaine-Weinberg method for measuring pattern speeds

The continuity equation in Cartesian coordinates

$$\frac{\partial n}{\partial t} + \boldsymbol{\nabla} \cdot (n\mathbf{u}) = 0$$

where $n(x, y, z, t)$ is the stellar number density. Assume that the density of a flat galaxy in 2D rotates at a fixed and steady pattern speed Ω_p, $n(r, \theta - \Omega_p t)$ in polar coordinates. We assume that the density distribution does not vary in a frame rotating with the pattern.

The continuity equation in 2D Cartesian coordinates becomes

$$-\Omega_p \left(x\frac{\partial n}{\partial y} - y\frac{\partial n}{\partial x} \right) + \frac{\partial (nu_x)}{\partial x} + \frac{\partial (nu_y)}{\partial y} = 0. \quad (42)$$

Consider integrating the continuity equation (equation 42) along the y axis. This is as if we are integrating along a slit that is oriented along the y axis. The first term

$$\int dy\ \Omega_p x \frac{\partial n}{\partial y} = 0$$

because $n \to 0$ at large y. The second term is

$$\int dy\ \Omega_p y \frac{\partial n}{\partial x} = \Omega_p \frac{\partial}{\partial x} \int dy\ yn(x, y).$$

The third term

$$\int dy\ \frac{\partial (nu_x)}{\partial x} = \frac{\partial}{\partial x} \int dy\ nu_x(x, y).$$

The fourth term

$$\int dy\ \frac{\partial (nu_y)}{\partial y} = 0.$$

because $n \to 0$ at large y. Putting this together

$$\frac{\partial}{\partial x} \left(\Omega_p \int dy\ yn(x, y) + \int dy\ nu_x(x, y) \right) = 0.$$

Integrating this

$$\Omega_p \int dy\ yn(x, y) + \int dy\ nu_x(x, y) = C$$

where C is a constant. This relation must be true for any x value and C cannot depend on x. This means that it must be true at large x and we can let the constant C be zero. This gives the relation

$$\Omega_p = -\frac{\int dy\ nu_x(x, y)}{\int dy\ yn(x, y)}.$$

The estimate for the pattern speed depends on the mean velocity component in the direction perpendicular to the slit, u_x. The denominator weights the stellar density by the distance along the slit. The estimate for the pattern speed is also valid if the number density is replaced by the light density. The light density would also be a conserved quantity, but again we assume that the density is fixed in a frame rotating with the pattern.

The galaxy is likely inclined with respect to the viewer. When measuring the mean velocity component u_x with a spectrum and using a Doppler shift, you would need to correct for galaxy inclination to get the full size of the in-plane velocity component.

This technique has been used to measure bar pattern speeds in some barred galaxies. We made a few assumptions. There is only a single pattern speed and the galaxy is nearly steady state. Both of these might be violated as galaxies can be changing shape and barred galaxies often also host spiral arms which may move at different or even varrying pattern speeds. Bars tend to have high surface brightness compared to spiral arms, making it easier to measure a mean velocity from a spectrum.

4 Some additional topics

4.1 Jeans Theorem

It is possible to switch variables $f(L, E)$ for example, depending upon quantities that are conserved in a spherically symmetric gravitational potential, angular momentum L and energy E. Alternatively one can write or $f(\mathbf{I}, \boldsymbol{\theta}, t)$ where $\mathbf{I}, \boldsymbol{\theta}$ are pairs of action angle variables. The collisionless Boltzmann can be evaluated similarly with advective derivatives. If the potential is fixed and the system *relaxed*, the phase space distribution function only depends on the actions.

4.2 Core Collapse

Up to this time we have primarily considered static or steady state systems. Stars in the outer parts of clusters tend to expand due to encounters. Some stars can be evaporated from the cluster. As energy is conserved, this means the center of the cluster increases in density. Numerical integrations (without binaries) find that after about 16 relaxation times, the density of the core increases without bound, causing the integrations to fail. This phenomenon is called **core collapse**. Some globular clusters are though to have experienced episodes where the central density increased. The integrations that saw rapidly growing core density, lacked binaries. As a core density increases, encounters of single stars can cause binaries to tighten. The binaries serve as an energy source. Consequently the formation and evolution of binaries in the cluster can halt core collapse. Because binaries can tighten, a prediction is that phenomena associated with close binaries is seen in globular clusters. Examples of exotic objects associated with cluster evolution would be X-ray binaries (tight enough to exhibit mass transfer), revived pulsars (pulsars that are spun up by mass transfer in a binary) and blue stragglers. A blue straggler is an apparently young star that seems younger than the other stars in the cluster. An apparently young star can be the result of two low mass stars, containing unburnt hydrogen, that merge to form a hydrogen rich and bright main sequence star that appears to have an age that is younger than the cluster itself.

4.3 Some dynamics in a galaxy center

Binaries can be disrupted near the black hole leaving a tightly bound star in an eccentric orbit around the black hole. In our Galaxy these stars are called **S-stars**. The other stars in the original binary can be ejected at high velocity giving what is called a **hypervelocity star**.

M31 hosts a lopsided stellar eccentric disk within its bulge. The eccentric disk remains coherent because stars near the black hole are on nearly Keplerian orbits that have pericenters that precess slowly. Formation mechanisms include a young stellar disk that forms and then becomes lopsided through instability. Alternatively there might have been a minor merger leaving a dense galaxy nucleus that disrupts near the black hole somehow leaving a remnant that remains coherently lopsided.

Stars near a black hole can vary in eccentricity and inclination due to interactions with each other. This is sometimes called *resonant relaxation* as the encounters are sometimes computed by averaging over the orbits. Stars that can be disrupted by the black hole are said to be *the loss cone*.

Galaxy centers could host binary black holes which merge due to scattering with stars. If stars are scattered, the black hole binary would tighten and the stellar core would flattened as stars are kicked away from the center. Massive elliptical galaxies tend to have low surface brightness near their galaxy centers, suggesting that stars have been scattered by a black hole.

If star is disrupted by a black hole, then a bright transient could occur which could be detected in a photometric survey.

The blackhole can become an AGN if fed with gas, but that same gas could also form stars.

Galaxies centers are dynamic and complex. Of recent interest is a new class of galactic center transients called quasi-periodic eruptions.

5 Problems

- **Problem 1**

  Show that in a frame that rotates with constant angular velocity $\mathbf{\Omega}$ the collisionless

Boltzmann equation is

$$\frac{\partial f}{\partial t} + (\mathbf{v} \cdot \boldsymbol{\nabla}) f - \left[\boldsymbol{\nabla} \left(\Phi - \frac{1}{2} (\boldsymbol{\Omega} \times \mathbf{r})^2 \right) + 2\boldsymbol{\Omega} \times \mathbf{v} \right] \cdot \boldsymbol{\nabla}_v f = 0$$

Note that acceleration $\mathbf{a}' = \dot{\mathbf{v}}$ in a rotating frame is

$$\mathbf{a}' = \mathbf{a} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}) + 2\boldsymbol{\Omega} \times \mathbf{v}$$

It is helpful to use vector identities to evaluate the gradient operator.

- **Problem 2:** Averaging over z

The collisionless Boltzmann equation in cylindrical coordinates R, ϕ, z is

$$\frac{\partial f}{\partial t} + v_R \frac{\partial f}{\partial R} + v_z \frac{\partial f}{\partial z} + \frac{v_\phi}{R} \frac{\partial f}{\partial \phi} + \left(\frac{v_\phi^2}{R} - \frac{\partial \Phi}{\partial R} \right) \frac{\partial f}{\partial v_R} - \left(\frac{v_R v_\phi}{R} + \frac{1}{R} \frac{\partial \Phi}{\partial \phi} \right) \frac{\partial f}{\partial v_\phi} - \frac{\partial \Phi}{\partial z} \frac{\partial f}{\partial v_z} = 0 \tag{43}$$

a. Consider integrating the collisionless Boltzmann equation over v_z. Why would this be true?

$$\int dv_z \frac{\partial \Phi}{\partial z} \frac{\partial f}{\partial v_z} = 0$$

b. Consider integrating the collisionless Boltzmann equation over z. Why would this be true?

$$\int dz v_z \frac{\partial f}{\partial z} = 0$$

c. In two dimensions we can describe the problem in terms of a distribution function $f(x, y, v_x, v_y, t)$ or in polar coordinates $f(R, \phi, v_R, v_\phi, t)$. The collisionless Boltzmann equation in 2D polar coordinates is the same as equation 43 except lacking those terms that depend on z, v_z or their gradients.

$$\frac{\partial f}{\partial t} + v_R \frac{\partial f}{\partial R} + \frac{v_\phi}{R} \frac{\partial f}{\partial \phi} + \left(\frac{v_\phi^2}{R} - \frac{\partial \Phi}{\partial R} \right) \frac{\partial f}{\partial v_R} - \left(\frac{v_R v_\phi}{R} + \frac{\partial \Phi}{\partial \phi} \right) \frac{\partial f}{\partial v_\phi} = 0 \tag{44}$$

Using parts a, b, argue that by integrating in z and v_z we derive the same equation. In other words if $f_3(R, \phi, z, v_R, v_\phi, v_z, t)$ satisfies equation 43 then

$$f(R, \phi, v_R, v_\phi, t) = \int dz \; dv_z f_3(R, \phi, z, v_R, v_\phi, v_z, t)$$

satisfies equation 44.