AST233 Lecture notes - Some Celestial Mechanics

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September 25, 2024

Contents

1 Why study Celestial Mechanics?

Celestial mechanics is an analytical setting with rich complexity that has continued fascinate scientists for centuries. Powerful numerical tools for integration of orbits have been complimentary to theoretical developments. For the study of chaotic systems, celestial mechanics reigns supreme in the breadth of analytical techniques that have been developed.

With increased observational measurements and characterization of numerous bodies in our solar system and the complex and different dynamics that likely occurs in recently discovered exoplanetary systems, celestial mechanics remains current and exciting.

The three body problem cannot be solved analytically because it can be chaotic. If a system is chaotic, that does not necessarily mean you cannot calculate anything. Chaotic behavior can sometimes be delineated in specific regions of parameter space, and timescales for evolution could be characterized.

Sometimes complicated problems can be elegantly and accurately modeled with toy models that capture the important dynamics, for example by focusing on resonant behavior or averaging over fast and unimportant angles.

1.1 Lagrangians, Hamiltonians and Newton's equations

This section is a lightening fast introduction to Lagrange's equations and Hamiltonian's equations and how they are consistent with Newton's equations.

Newton's equations give a relation between force and acceleration $\mathbf{F} = m\mathbf{a}$. In the Newtonian limit the gravitational force is a gradient of a potential $\mathbf{F} = -\nabla U$.

Newton's equations are consistent with minimization of the integral of a Lagrangian along a trajectory where Lagrangian

$$\mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}, t) = T(\dot{\mathbf{q}}) - U(\mathbf{q}) \tag{1}$$

is a function of coordinates $\mathbf{q}$ and their time derivatives $\dot{\mathbf{q}}$. Here the kinetic energy $T = \frac{m\dot{q}^2}{2}$. Using calculus of variations, we can show that the action on a trajectory $S = \int \mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}, t) ds$ is minimized when Lagrange's equations are obeyed on the trajectory. Lagrange's equations are

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} = \frac{\partial \mathcal{L}}{\partial \mathbf{q}}. \tag{2}$$

We show that Lagrange's equations for the Lagrangian in equation 1 are consistent with Newton's equation.

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} &= m\dot{\mathbf{q}} \\ \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} &= m\ddot{\mathbf{q}} \\ \frac{\partial \mathcal{L}}{\partial \mathbf{q}} &= -\frac{\partial U}{\partial \mathbf{q}} = -\nabla U = \mathbf{F} \end{aligned}$$

Equating the last two of these (using equation 2 which is Lagrange's equation) gives $\mathbf{F} = m\mathbf{a} = m\ddot{\mathbf{q}}$ which is Newton's equation.

Newton's equations are also consistent with Hamilton's equations. Instead of using a Lagrangian, we construct a Hamiltonian function $H(\mathbf{p}, \mathbf{q}, t) = T + U$ with $\mathbf{p} = m\dot{\mathbf{q}}$ the momentum. Hamilton's equations are

$$\frac{\partial H}{\partial \mathbf{p}} = \dot{\mathbf{q}} \qquad \frac{\partial H}{\partial \mathbf{q}} = -\dot{\mathbf{p}}. \tag{3}$$

We show that Hamilton's equations are consistent with Newton's equation. In terms of momentum, the kinetic energy $T = \frac{p^2}{2m}$ and the Hamiltonian

$$H(\mathbf{p}, \mathbf{q}) = \frac{p^2}{2m} + U(\mathbf{q}). \tag{4}$$

Hamilton's equations (equations 3)

$$\frac{\partial H}{\partial \mathbf{p}} = \frac{\mathbf{p}}{m} = \dot{\mathbf{q}} \qquad \rightarrow \dot{\mathbf{p}} = m\ddot{\mathbf{q}} \tag{5}$$

$$\frac{\partial H}{\partial \mathbf{q}} = \nabla U = -\mathbf{F} \tag{6}$$

Using the second of Hamilton's equations gives $\mathbf{F} = m\ddot{\mathbf{q}}$ which again is Newton's equation.

I have mentioned a Lagrangian viewpoint which tends to give equations of motion that involve accelerations of variables. After a series of variable transformations, it is possible to find equations of motion for orbital elements instead of cartesian coordinates in the Lagrangian viewpoint.

The Hamiltonian viewpoint has the advantage that momenta that are missing from the Hamiltonian are conserved quantities. Furthermore the Hamiltonian viewpoint illustrates that that volume in phase space is conserved. Integrators developed with this view (symplectic or more recently obeying time reversal symmetry) are often much better behaved during long integrations than integrators that are developed simply to approximate the solutions to a high order of accuracy.

2 Celestial mechanics

2.1 The Keplerian orbit

We consider the orbit of a very low mass point particle around a massive object of mass M_c. It is conventional to divide by the mass of the particle so that potential energy is potential energy per unit mass and kinetic energy is that per unit mass. Using this convention the gravitational potential energy $U = -\frac{k}{r}$ where r is the radius from the massive object and $k = GM_c$. With this convention momenta are also divided by mass.

We can describe the dynamics with a Lagrangian

$$\mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}) = \frac{1}{2}\dot{\mathbf{q}}^2 + \frac{k}{|\mathbf{q}|}.$$

Equivalently we could describe the dynamics with a Hamiltonian

$$H(\mathbf{p},\mathbf{q}) = \frac{1}{2}\mathbf{p}^2 - \frac{k}{|\mathbf{q}|}.$$

(Note we have neglected the mass of the particle m).

If we restrict motion to a plane, then it is convenient to use polar coordinates. The velocity in polar coordinates can be decomposed in to a radial component $v_r = \dot{r}$ and a tangential component $v_\theta = r\dot{\theta}$. In terms of the angular momentum per unit mass $L = r^2\dot{\theta} = v_\theta/r$

The square the velocity in Cartesian coordinates is related to that in polar coordinates

$$\dot{x}^2 + \dot{y}^2 = \dot{r}^2 + (r\dot{\theta})^2 = \dot{r}^2 + \frac{L^2}{r^2}.$$

The kinetic energy can be split into two pieces, a radial component and a tangential component. In cylindrical coordinates we obtain a Hamiltonian

$$H(p_r, L; r, \theta) = \frac{1}{2}p_r^2 + \frac{L^2}{2r^2} - \frac{k}{r}. \tag{7}$$

Here $p_r = \dot{r}$ is the radial component of momentum per unit mass and L is the angular momentum per unit mass. I have neglected motion out of the plane but we could include it by adding $p_z^2/2$.

Because the Hamiltonian does not depend upon θ, the angular momentum is conserved.

The radial degree of freedom gives an equation of motion (using Hamilton's equations for the radial degrees of freedom)

$$-\dot{p}_r = \frac{\partial H}{\partial r} = -\frac{L^2}{r^3} + \frac{k}{r^2} = -\ddot{r}$$

Because the Hamiltonian is independent of θ, angular momentum per unit mass (in the z direction), is conserved. Using Hamilton's equations, but this time using the angle, we can check that our system gives

$$L = r^2\dot{\theta}$$

as expected.

It is convenient to use a variable inverse radius $u = 1/r$ with

$$\dot{u} = -\frac{\dot{r}}{r^2}$$

$$\frac{du}{dt} = \frac{du}{d\theta}\frac{d\theta}{dt}$$

using $L = r^2\dot{\theta}$

$$\frac{du}{dt} = \frac{du}{d\theta}\frac{L}{r^2}$$

Putting this together with our previous expression for $\dot{u}$

$$-\dot{r} = \frac{du}{d\theta} L \tag{8}$$

Taking the time derivative of this

$$\begin{aligned} -\ddot{r} &= \frac{d}{dt}\frac{du}{d\theta}L = \dot{\theta}\frac{d}{d\theta}\frac{du}{d\theta}L \\ &= \frac{d^2u}{d\theta^2}L^2u^2 \end{aligned}$$

The equations of motion are

$$\ddot{r} = \frac{L^2}{r^3} - \frac{k}{r^2} = L^2u^3 - ku^2$$

Putting these together

$$\begin{aligned} \frac{d^2u}{d\theta^2}L^2u^2 &= -L^2u^3 + ku^2 \\ u'' + u &= kL^{-2}. \end{aligned}$$

This has solution for inverse radius

$$u = (1 + e\cos\theta)p^{-1} \tag{9}$$

with

$$p = L^2/k \tag{10}$$

and free parameter e known as the eccentricity. Inverting this for radius

$$r = \frac{p}{1 + e\cos f}$$

and we have replaced θ with angle f called the true anomaly. For $f = 0$ the orbit is a pericenter. The minimum and maximum radius are $r_{min} = p/(1+e)$ and $r_{max} = p/(1-e)$ giving a semi-major axis a

$$2a = \frac{p}{1+e} + \frac{p}{1-e} = \frac{2p}{1-e^2}$$

so that

$$p = a(1-e^2)$$

The orbit is then

$$r(f) = \frac{a(1-e^2)}{1+e\cos f} \tag{11}$$

We have found the orbit as a function of true anomaly f. It is much hard to find $r(t)$, or radius as a function of time or $f(t)$ true anomaly as a function of time.

With some manipulation it is possible to show that energy per unit mass and angular momentum per unit mass are

$$E = -\frac{k}{2a} \tag{12}$$

$$L = \sqrt{ka(1-e^2)}. \tag{13}$$

These are appropriate for elliptical orbits.

With some generalization a similar description covers parabolic and hyperbolic orbits. Hyperbolic orbits have e greater than 1, $a < 0$ and parabolic orbits have $e = 1$, energy $E = 0$, and $L = \sqrt{2GMq}$ where q is the radius of pericenter.

2.2 Eccentric anomaly

In a coordinate system defined from the ellipse focal point, a point on the orbit

$$\begin{aligned} x &= r\cos f \\ y &= r\sin f \end{aligned} \tag{14}$$

Here f is the true anomaly and r the radius. This coordinate system uses as origin an ellipse focal point which is also the location of the Sun for the orbit of a planet in motion around the Sun.

In a coordinate system with origin at the center of the ellipse, the orbit defines an ellipse obeying (see Figures 1)

$$\left(\frac{\bar{x}}{a}\right)^2 + \left(\frac{\bar{y}}{b}\right)^2 = 1 \tag{15}$$

with semi-major axis a and semi-minor axis $b = a\sqrt{1-e^2}$. The coordinates for a point on the orbit can be written in terms of an angle called the eccentric anomaly E

$$\begin{aligned} \bar{x} &= a\cos E \\ \bar{y} &= b\sin E = a\sqrt{1-e^2}\sin E = y \end{aligned} \tag{16}$$

with origin the center of the ellipse rather than an ellipse focal point. Note that this definition for $\bar{x}, \bar{y}$ ensures that equation 15 is obeyed. Also helpful is the x distance from the focal point in terms of the eccentric anomaly

$$x = a(\cos E - e). \tag{17}$$

These relations can be read off Figure 1 showing the orbit.

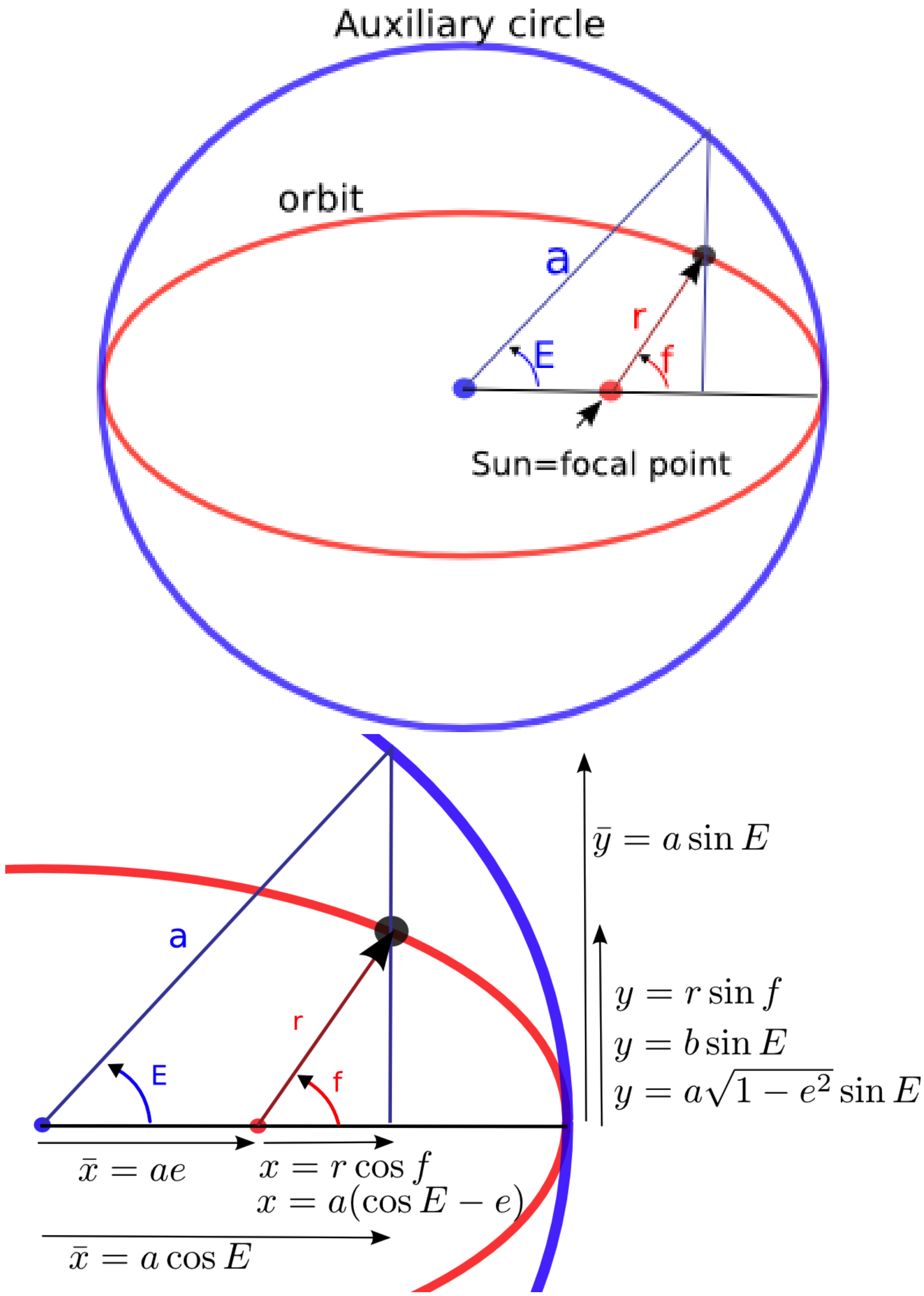

Figure 1: a) One focal point of an elliptic orbit is the location of the Sun. Also drawn is the auxiliary circle with radius a, the angle known as the true anomaly f, and the eccentric anomaly E. The true anomaly f is defined with respect to origin at the Sun and the focal point of the ellipse. The Eccentric anomaly is defined with origin at the center of the ellipse and the auxiliary circle. b) Coordinate relations. x, y are positions with origin at the Sun. $\bar{x}, \bar{y}$ are coordinates with origin at the center of the ellipse.

We have not actually shown that the orbit equation is consistent with being an ellipse but you can turn equation 11 into an equation only involving x^2, x and y^2 and constants and this can be turned into the equation for an ellipse by shifting the origin.

Because $x = a(\cos E - e)$ and $y = b \sin E$

$$\begin{aligned} r^2 &= x^2 + y^2 = a^2(\cos E - e)^2 + a^2(1 - e^2)\sin^2 E \\ \frac{r^2}{a^2} &= \cos^2 E - 2e\cos E + e^2 + (1 - e^2)(1 - \cos^2 E) \\ &= \cos^2 E - 2e\cos E + e^2 + 1 - e^2 - \cos^2 E + e^2\cos^2 E \\ &= 1 - 2e\cos E + e^2\cos^2 E = (1 - e\cos E)^2 \end{aligned}$$

This is consistent with

$$r = a(1 - e\cos E). \tag{18}$$

Useful is a relation between true and eccentric anomaly

$$\tan(f/2) = \tan(E/2)\sqrt{\frac{1+e}{1-e}}. \tag{19}$$

By differentiating equation 18 w.r.t time

$$\dot{r} = ae\sin E\dot{E}. \tag{20}$$

It can be handy to find the velocity in terms of $\dot{r}$ and $\dot{f}$. Because the tangental velocity component $v_\theta = r\dot{f}$ and $L = rv_\theta$,

$$(r\dot{f})^2 = \frac{L^2}{r^2} \tag{21}$$

$$v^2 = (r\dot{f})^2 + (\dot{r})^2. \tag{22}$$

Using the orbit equation for $r(f)$ (equation 11)

$$e\cos f = \frac{a(1-e^2)}{r} - 1. \tag{23}$$

Using equation 8 which is $\dot{r} = -\frac{du}{d\theta}L$,

$$\begin{aligned} u &= \frac{1 + e\cos f}{p} = \frac{1 + e\cos f}{L^2/(GM)} \\ \frac{du}{df} &= -\frac{e\sin f}{L^2/(GM)} \\ \dot{r} &= \frac{e\sin f}{L^2/(GM)}L \end{aligned}$$

$$e\sin f = \dot{r}\frac{L}{GM}. \tag{24}$$

These relations are useful for finding the true anomaly f given Cartesian positions and velocities.

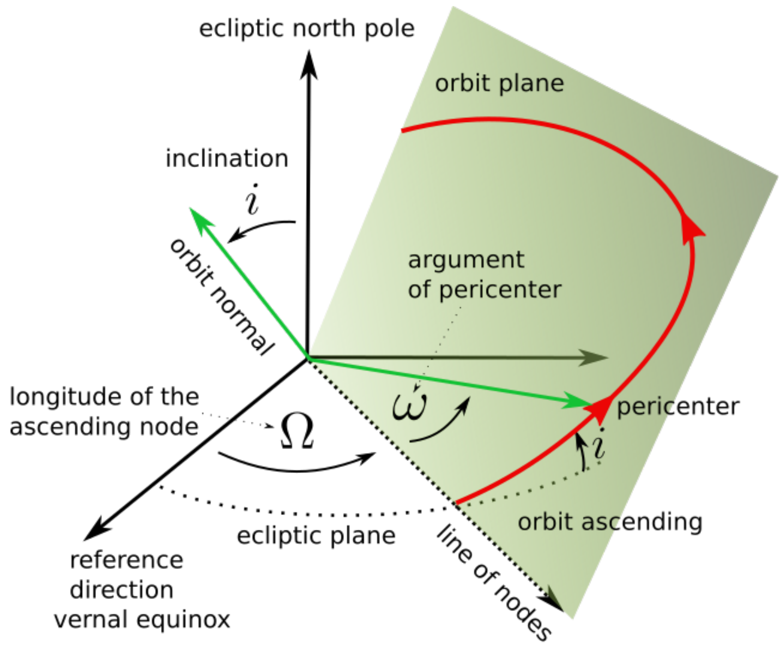

Figure 2: Angles describing orbit orientation and the orbital elements.

2.3 Orbital elements

Orbital elements are the following quantities

1. The semimajor axis, a. Set by the energy of the orbit.

2. The orbital eccentricity, e, describing the ellipticity of the orbit. Depends on the magnitude of the orbital angular momentum.

3. The inclination, I or i. Describes the tilt of the orbit. In the solar system an orbit with $I = 0$ lies in the ecliptic. Depends on the orientation of the angular momentum vector.

4. The argument of pericenter, ω. Determines the location of pericenter.

5. The longitude of the ascending node, Ω. Determines where the orbit crosses a reference plane such as the ecliptic plane. Is sensitive to the orientation of the angular momentum vector.

6. The mean anomaly M. Describes how the orbit advances in time. Is not an angle on a plot.

In a purely Keplerian system (the two body problem), all orbital elements are constant (conserved quantities) except M which continuously advances at the mean motion. The orbital elements are all angles except for a and e. Only semi-major axis a has dimensions (of length). The orbital elements are illustrated in Figure 2.

Sometimes it is useful to group the orbital elements in two groups of three (a, e, I) and (M, ω, Ω). Or they can be grouped pairs as (a, M), (e, ω), (I, Ω).

The true and eccentric anomalies are not orbital elements.

2.4 Anomalies, arguments and longitudes

Any angle can be called an *argument*. Example: the argument of pericenter ω.

An angle measured from pericenter is often called an *anomaly*. We have three of them:

1. M, the mean anomaly, (not an angle you can identify on a map, serves only to be related to time with mean motion $n = \dot{M}$.)

2. f, the true anomaly (measured from the ellipse focal point which is the same as the star).

3. E, the eccentric anomaly. Measured from the center of the ellipse.

These anomalies are computed using the position of the body in its orbit and they are all equal to zero at pericenter.

Angles measured from a specific reference direction, such as the Sun/Earth line at the vernal equinox, are called *longitudes*. Examples are

1. The longitude of the ascending node Ω.

2. The longitude of pericenter $\varpi = \Omega + \omega$.

3. The true longitude $\theta = \varpi + f = \Omega + \omega + f$.

4. The mean longitude $\lambda = \varpi + M = \Omega + \omega + M$.

Note that these sums involve angles that are not necessarily measured in a single plane.

2.5 The mean anomaly and Kepler's equation

We take the time derivative of $r = a(1 - \cos E)$

$$\dot{r} = ae \sin E \dot{E}. \tag{25}$$

We recognize that $y = r\sin f = a\sqrt{1-e^2}\sin E$ and use equation 24

$$\begin{aligned} e\sin f &= \dot{r}\frac{L}{k} \\ \frac{e}{r}a\sqrt{1-e^2}\sin E &= \dot{r}\frac{\sqrt{ka(1-e^2)}}{k} \\ \frac{e}{a(1-e\cos E)}a\sin E &= \dot{r}\sqrt{\frac{a}{k}} = \frac{\dot{r}}{na} \\ \dot{r} &= na\frac{e\sin E}{1-e\cos E} \end{aligned}$$

with $n = \sqrt{k/a^3}$. We set the two equations for $\dot{r}$ (eqns 25, 26) to be equal to find

$$\dot{E} = \frac{n}{1-e\cos E}. \tag{26}$$

The frequency n is called the **mean motion**

$$n = \sqrt{\frac{GM_c}{a^3}} \tag{27}$$

with $k = GM_c$ and M_c the central mass. We are purposely not using the M symbol as central mass because we need M as the symbol for a new angle!

We assume that there is an angle M, known as the **mean anomaly**, that advances with constant angular rotation rate given by the mean motion n

$$M = M_0 + nt$$

so that

$$\dot{M} = n. \tag{28}$$

(The mean motion $n = \dot{M}$ should not be confused with a mass accretion rate!) The mean anomaly is not a physical angle on the sky or any geometric drawing. Now insert this into our equation for $\dot{E}$

$$\dot{E} = \frac{\dot{M}}{1-e\cos E}$$

we can integrate this equation (with respect to time) finding

$$M = E - e\sin E. \tag{29}$$

This equation is known as **Kepler's equation**. Kepler's equations implies that given E we can find M and vice versa. While integrating, we dropped a constant which is removed with the requirement that $M = 0$ when $E = 0$. In other words, the mean anomaly is defined so that $M = 0$ at pericenter.

Kepler's equation cannot be solved analytically. In other words, given M, it is not possible to solve analytically for E. However extremely rapid numerical techniques that converge to third order are known (like Laguerre's method). To converge to a solution to a precision of order 10^{-16} (double precision floating point) I find that it takes less than 6 or 7 iterations of the Laguerre method, even at high eccentricity.

To advance an orbit in time, M is advanced, then E computed. From E, the position in the orbit can be computed. Then the orbit is rotated according to its longitude of perihelion ω, inclination, I, and longitude of the ascending node Ω. The reverse procedure is done to convert a cartesian positions and velocities to orbital elements. The orbital elements are a, e, i and angles M, ω, Ω. The ordering in my two lists is chosen because canonical momenta depending primarily on a, e, i are conjugate to canonical angles either equal to or related to the angles M, ω, Ω, respectively.

2.6 Finding orbital elements from Cartesian coordinates

We start with a position $\mathbf{r} = (x, y, z)$ and a velocity $\mathbf{v}$ in a coordinate system with the xy plane set to the ecliptic plane. We first compute the angular momentum vector $\mathbf{L} = \mathbf{r} \times \mathbf{v}$. The sign of L_z is important. The angular momentum vector

$$\begin{aligned} L_x &= \pm L \sin I \cos \Omega \\ L_y &= \mp L \sin I \sin \Omega \\ L_z &= L \cos I \end{aligned} \tag{30}$$

defines two angles, the inclination I and the longitude of the ascending node Ω. The signs depend on wether L_z is positive or negative. If $L_z > 0$ then take the positive L_x sign and the negative L_y sign. The inclination by convention $I \in [0, \pi)$ with $[\pi/2, \pi)$ corresponding to retrograde orbits.

Here the positive x axis I think refers to the direction from Sun to the Earth at the vernal equinox which is the March equinox (spring in the northern hemisphere).

The line of nodes is the intersection of the orbital plane and the reference plane or ecliptic. A vector pointing along the line of nodes can be constructed with $\hat{\mathbf{n}} = \mathbf{L} \times \hat{\mathbf{e}}/L$ where $\hat{\mathbf{e}}$ is perpendicular to the ecliptic plane. Longitude of ascending node Ω is the angle between line of nodes (the ascending side) and the reference line (vernal equinox).

The **argument** of pericenter ω is the angle between the line of nodes and pericenter. Position of the star is the origin.

The **longitude** of pericienter $\varpi = \Omega + \omega$. Note that Ω and ω do not usually lie in the same plane.

With the angular momentum vector you can compute the two orbital elements Ω, I. Using $L = \sqrt{GMa(1 - e^2}$ and orbital energy $E_o = -GM/(2a)$ the position and velocity vectors can be used to compute orbital elements a, e.

Compute the right hand sides of these two relations

$$e \cos f = \frac{L^2}{GMr} - 1.0 \tag{31}$$
$$e \sin f = \frac{\dot{r} L}{GM} \tag{32}$$

that are from equations 23, 24, and then from their ratio compute the true anomaly f.

We can calculate the right hand sides of the two equations

$$\begin{aligned} x' = r\cos u &= x \cos\Omega + y \sin\Omega \\ y' = r \sin u &= \frac{(y\cos\Omega - x\sin\Omega)}{\cos I} \end{aligned} \tag{33}$$

where u is known as the argument of latitude. From the ratio of these two we can compute u. Then we can compute

$$\omega = u - f$$

This arises from the following: In the orbital plane $(x', y', z') = (r\cos u, r\sin u, 0)$ with orbit

$$r = \frac{a(1-e^2)}{1 + e\cos(u-\omega)}$$

with $u - \omega = f$. The argument of latitude seems to be the angle in the orbital plane taking into account the argument of pericenter ω. Two rotations relate (x', y', z') with (x, y, z), one involving Ω and the other involving inclination I. Starting with x, y, z we first rotate the orbit with Ω in the x, y plane. The orbit is then tilted in y, z by the inclination I. The two rotations should give equations 33.

After computing true anomaly f, the mean anomaly E is computed using equation 19 and then by solving Kepler's equation (equation 29) iteratively, we can find M, the true anomaly.

The orbital plane x_o, y_o, z_o with x axis aligned with perihelion and the references ones x, y, z. They are related via rotation matrices

$$P_\omega = \begin{pmatrix} \cos\omega & -\sin\omega & 0 \\ \sin\omega & \cos\omega & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{about } \hat{\mathbf{z}} \tag{34}$$
$$P_I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos I & -\sin I \\ 0 & \sin I & \cos I \end{pmatrix} \quad \text{about } \hat{\mathbf{x}} \tag{35}$$
$$P_\Omega = \begin{pmatrix} \cos\Omega & -\sin\Omega & 0 \\ \sin\Omega & \cos\Omega & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{about } \hat{\mathbf{z}} \tag{36}$$

$$
\begin{aligned}
\mathbf{x}_o &= P_\Omega P_I P_\omega \mathbf{x} &(37)\\
\mathbf{x} &= P_\omega^{-1} P_I^{-1} P_\Omega^{-1} \mathbf{x}_o &(38)\\
\mathbf{x}' &= P_I^{-1} P_\Omega^{-1} \mathbf{x}_o &(39)
\end{aligned}
$$

As

$$\mathbf{x}_o = r(\cos f, \sin f, 0)$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = r \begin{pmatrix} \cos\Omega\cos(\omega+f) - \sin\Omega\sin(\omega+f)\cos I \\ \sin\Omega\cos(\omega+f) + \cos\Omega\sin(\omega+f)\cos I \\ \sin(\omega+f)\sin I \end{pmatrix} \tag{40}$$

2.7 Finding Cartesian coordinates from orbital elements

1. Compute the eccentric anomaly using eccentricity e and mean anomaly M.
2. Compute true anomaly f using the relation between f and E (equation 19).
3. Compute x, y using true anomaly in orbital plane. Set $z = 0$.
4. Rotate in xy plane using longitude of pericenter.
5. Rotate using inclination.
6. Rotate using longitude of the ascending node.

Question: Are the orbital elements ω, Ω, M well defined for an orbit with zero eccentricity and inclination?

The answer is No.

If you are specifying a circular orbit, it is a good idea to use the true longitude to specify the position of a particle within the orbit. If the inclination is zero, then it is a good idea to use the longitude of pericenter and true longitude to specify the position in the orbit.

2.8 f and g functions

Numerical integrations leverage the knowledge of Keplerian orbits to improve the accuracy of the integration. In a numerical integration, it is not necessarily to follow the entire procedure of finding every orbital element during every integration step. Instead two functions, known as f and g functions are used to compute new functions of position and time. The f and g functions are functions of position, velocity and time

$$
\begin{aligned}
\mathbf{r}(t) &= f(t, t_0, \mathbf{r}_0, \mathbf{v}_0)\mathbf{r}_0 + g(t, t_0, \mathbf{r}_0, \mathbf{v}_0)\mathbf{v}_0 \\
\mathbf{v}(t) &= \dot f(t, t_0, \mathbf{r}_0, \mathbf{v}_0)\mathbf{r}_0 + \dot g(t, t_0, \mathbf{r}_0, \mathbf{v}_0)\mathbf{v}_0 &(41)
\end{aligned}
$$

where $\mathbf{r}_0$ and $\mathbf{v}_0$ are a current position and velocity. During an integration you would like to know $\mathbf{r}, \mathbf{v}$ at a later time, with $\Delta t = t - t_0$. The functions f, g can be written in terms of the eccentric anomaly

$$\begin{aligned} f &= 1 - \frac{a}{r_0}[1 - \cos(E - E_0)] \\ g &= (t - t_0) - \frac{1}{n}[(E - E_0) - \sin(E - E_0)] \\ \dot{f} &= -\frac{na^2}{r - r_0}\sin(E - E_0) \\ \dot{g} &= 1 - \frac{a}{r}[1 - \cos(E - E_0)]. \end{aligned} \tag{42}$$

Subtracting Kepler's equation at two different times we find that

$$\begin{aligned} \Delta M &= n\Delta t \\ &= \Delta E - e\cos E_0 \sin \Delta E + e \sin E_0 (1 - \cos \Delta E) \end{aligned} \tag{43}$$

This is a differential form of Kepler's equation and can be solved in a similar way as solving Kepler's equation.

To do an integration from $\mathbf{r}_0, \mathbf{v}_0$ and advance to a new position and velocity with a time-step Δt:

1. Compute a, e from energy and angular momentum.
2. Compute E_0 from current position and velocity.
3. Compute ΔE from solving numerically the differential form of Kepler's equation.
4. Compute f, g functions, use them to find the new position $\mathbf{r}$.
5. Compute $\dot{f}, \dot{g}$, find the new velocity vector!

Note, Δt need not be small. Large steps can be taken accurate as long as Keplerian motion is a good approximation.

There is a way to generalize the f and g functions so that they work for hyperbolic and parabolic orbits. For hyperbolic orbits, cosine is replaced by cosh and sine by sinh. The technique is described by Prussing and Conway in their book "Orbital Mechanics", where they refer to a formala due to Battin.

2.9 The parabolic orbit

Consider our equation for the orbit (equations 9, 10

$$r(f) = \frac{L^2/GM}{1 + e\cos f} \tag{44}$$

With $e < 1$, the orbit is an ellipse and for a hyperbolic orbit, $e > 1$. The transition is an parabolic orbit which we can mimic with $e = 1$. The minimum radius is at $f = 0$ where the pericenter distance

$$2q = \frac{L^2}{GM}. \tag{45}$$

The orbit

$$r(f) = \frac{2q}{1 + \cos f}. \tag{46}$$

The orbit energy per unit mass $E = 0$ and the angular momentum per unit mass $L = \sqrt{2qGM}$.

2.10 Canonical Variables — Poincaré and Delaunay Variables

While with orbital elements we have numerous conserved quantities the orbital elements are not a set of canonical momenta and coordinates.

Using Hamilton-Jacobi equations it is possible to take the the Keplerian Hamiltonian in polar coordinates and convert it into action angle variations leaving it as a function of a single momentum. Hence there are 5 conserved quantities. We knew that already as of the orbital elements, all are conserved except for the mean anomaly, M. These new variables can be written in terms of the orbital elements.

An example of canonical coordinates for the Keplerian system are the Poincaré variables

$$\begin{array}{lllllll} \lambda & = & M + \omega + \Omega, & \quad & \Lambda & = & \sqrt{ka} \\ \gamma & = & -\varpi = -\omega - \Omega, & & \Gamma & = & \sqrt{ka}(1 - \sqrt{1-e^2}) \\ z & = & -\Omega, & & Z & = & \sqrt{ka(1-e^2)}(1 - \cos I) \end{array}, \tag{47}$$

where angles λ is the mean longitude, γ is the negative of the longitude of pericenter, and z is the negative of the longitude of the ascending node. These angles are conjugate to the momenta Λ, Γ, Z in action angle pairs

$$(\Lambda, \lambda), (\Gamma, \gamma), (Z, z)$$

In Poincaré coordinates the Keplerian Hamiltonian is

$$H_{kep}(\lambda, \gamma, z; \Lambda, \Gamma, Z) = -\frac{k^2}{2\Lambda^2} \tag{48}$$

Let us quickly check that this is consistent with the mean motion

$$\begin{aligned} n &= \dot{\lambda} = \frac{\partial H}{\partial \Lambda} = \frac{k^2}{\lambda^3} \\ &= k^{\frac{1}{2}} a^{-\frac{3}{2}}, \end{aligned}$$

as we expect!

A closely related set of canonical coordinates are called the Delaunay variables:

$$\begin{array}{rclrcl} l &=& M, & L &=& \sqrt{ka} \\ g &=& \omega, & G &=& \sqrt{ka}(1-\sqrt{1-e^2}) \\ h &=& \Omega, & H &=& \cos I\sqrt{ka(1-e^2)}. \end{array} \tag{49}$$

With the Hamiltonian in canonical coordinates the equations of motion are straightforward. However if there is a perturbation (such as arises from a planet) we can write the Hamiltonian as

$$H = H_0 + H_1$$

where $H_0 = H_{kep}$ from equation 48 and H_1 is -1 times a function known as the *disturbing function*. Working in a heliocentric coordinate system the disturbing function contains two terms often called direct and indirect. The indirect term arises because a heliocentric coordinate system is not an inertial one (the Sun moves).

The equations of motion now depend on derivatives of the disturbing function with respect to canonical coordinates. When written in terms of orbital elements the equations of motion are known as *Lagrange's equations*.

2.11 What is the disturbing function?

The disturbing function is equal to -1 times the potential energy and it is per unit mass for a particular particle that one is studying. Often the disturbing function is given in a heliocentric coordinate system, with origin at the location of the Sun. This coordinate system is not an inertial one if there is a planet in the system. This means there is a term in the disturbing function caused by the recoil of the Sun due to the perturbing planet.

Consider a system with two planets with masses m_i, m_j. The disturbing function when studying planet m_i would be

$$\mathcal{R}_i = \frac{Gm_j}{|\mathbf{r}_i - \mathbf{r}_j|} - \frac{Gm_j}{r_j^3}\mathbf{r}_i \cdot \mathbf{r}_j. \tag{50}$$

The first term is -1 times interaction potential term between m_j and m_j and because it is per unit mass of m_i it only contains the mass m_j in it. The first term is called the **direct** term. The second term in equation 50 is caused by the recoil of the Sun because of m_j and it is called the **indirect term**.

The disturbing function when studying m_j looks the same except both terms are proportional to m_i and the indirect term depends on r_i^3 instead of r_j^3,

$$\mathcal{R}_j = \frac{Gm_i}{|\mathbf{r}_i - \mathbf{r}_j|} - \frac{Gm_i}{r_i^3}\mathbf{r}_i \cdot \mathbf{r}_j. \tag{51}$$

A disturbing function can be expanded or averaged. When expanded as a function of eccentricity and inclination and with angles put in terms of orbital elements, often two cases are distinguished by the ratio of the two object's semi-major axes. Laplace coefficients are written in terms of α, the ratio of the two semi-major axes, but requiring that $\alpha < 1$. That means you would first decide whether the perturber has orbit external or internal to the object you are studying before doing the expansion.

If you take a look the appendix of Murray and Dermott's book which gives a 4th order expansion of the disturbing function due to a single point mass (either internal or external to the orbit of the object you are studying), the notation should make more sense with these definitions in mind. However, I find that I am often confused on which object is given the ' in the orbital elements in the expansion.

2.12 Low eccentricity expansions

For a setting where the planets are in nearly circular orbits with low inclination $e, i \ll 1$. The dynamics can be approximated with a low inclination and eccentricity expansion. In this section we illustrate some expansions that are common in celestial mechanics. We expand $e \sin E$ in a Fourier series of the mean anomaly M. In other words, we find coefficients a_s, b_s such that

$$e \sin E = a_0 + \sum_{s=1}^{\infty} (a_s \sin M + b_s \cos M) \tag{52}$$

where s are integers. We can show that the a_0 term is zero by integrating with Kepler's equation. The coefficients a_s for $s > 0$ terms are zero based on a symmetry argument. To find the b_s coefficients, we integrate

$$b_s = \frac{1}{\pi} \int_0^{2\pi} e \sin E \sin sM \; dM. \tag{53}$$

This is equivalent to finding a Fourier coefficient leveraging the orthogonality of the functions $\sin sM$ and $\cos sM$ when integrated. The result (following a series of manipulations that involve Kepler's equation and integrating by parts) is that

$$b_s = \frac{2}{s} J_s(se) \tag{54}$$

where J_s is a Bessel function of the first kind. The expansion is

$$e \sin E = \sum_{s=1}^{\infty} b_s(e) \sin(sM). \tag{55}$$

We insert this into Kepler's equation to find

$$E = M + \sum_{s=1}^{\infty} \frac{2}{s} J_s(se) \sin(sM). \tag{56}$$

To third order (inserting the actual Bessel function polynomials) this is

$$E = M + e \sin M + \frac{e^2}{2} \sin(2M) + e^3 \left(\frac{3}{8} \sin(3M) - \frac{1}{8} \sin M \right) + ...$$

With a series of similar manipulations (integrating by parts, and using Kepler's equation, discarding terms that integrate to zero by symmetry arguments) the following expansions can be derived

$$\cos(nE) = -\frac{e}{2}\delta_{n,1} + \sum_{k=1}^{\infty} \frac{n}{k} \left[J_{k-n}(ke) - J_{k+n}(ke) \right] \cos(kM) \tag{57}$$

$$\sin(nE) = \sum_{k=1}^{\infty} \frac{n}{k} \left[J_{k-n}(ke) + J_{k+n}(ke) \right] \sin(kM). \tag{58}$$

For more details see on how these are derived see books on Celestial mechanics, such as Valtonen and Karttunen's book or Murray and Dermott's book. Restating equation 18

$$r = a(1 - e \cos E)$$

the radius can be expanded in terms of M by expanding the cosine with equation 57

$$\begin{aligned} \frac{r}{a} &= 1 - e \cos E \\ \frac{r}{a} &= 1 + e^2 - \sum_{k=1}^{\infty} \frac{e}{k} \left[J_{k-1}(ke) - J_{k+1}(ke) \right] \cos(kM). \end{aligned} \tag{59}$$

The gravitational potential involves $1/r$ we would also like to expand this. We starting with Kepler's equation

$$\begin{aligned} M &= E - e \sin E \qquad \text{Kepler's equation} \\ \frac{dM}{dE} &= 1 - e \cos E \qquad \text{take derivative} \end{aligned} \tag{60}$$

$$\begin{aligned} \frac{a}{r} &= \frac{1}{1 - e \cos E} \qquad \text{from Equation 18} \\ &= \frac{dE}{dM} \qquad \text{using equation 60} \\ &= 1 + \sum_{s=1}^{\infty} 2 J_s(se) \cos sM \qquad \text{taking derivative of equation 56} \end{aligned} \tag{61}$$

We use the orbit and equation 61 to find an expression for $\cos f$

$$\begin{aligned} r &= \frac{a(1-e^2)}{1+e\cos f} \\ e\cos f &= \frac{a}{r}(1-e^2) - 1 \\ &= (1-e^2)(1+\sum_{s=1}^{\infty} 2J_s(se)\cos sM) - 1 \qquad (62) \\ &= (1-e^2)\sum_{s=1}^{\infty} 2J_s(se)\cos sM - e^2. \qquad (63) \end{aligned}$$

We now work on $\sin f$. Starting with equation 24

$$\begin{aligned} e\sin f &= \dot{r}\frac{L}{GM} \\ &= \dot{r}\sqrt{GMa(1-e^2)}\frac{1}{GM} = \dot{r}\sqrt{1-e^2}\sqrt{\frac{a}{GM}} \\ &= \frac{\dot{r}}{na}\sqrt{1-e^2}. \qquad (64) \end{aligned}$$

Using equation 59

$$\frac{\dot{r}}{a} = \sum_{k=1}^{\infty} \frac{e}{k}\left[J_{k-1}(ke) - J_{k+1}(ke)\right]\sin(kM)kn \qquad (65)$$

This and equation 64 give

$$\sin f = \sqrt{1-e^2}\sum_{k=1}^{\infty}\left[J_{k-1}(ke) - J_{k+1}(ke)\right]\sin(kM) \qquad (66)$$

These expressions illustrate how powers of r can be turned into sums of cosine terms each with their own power of eccentricity.

2.13 Expansion of an interaction term

The potential energy between two point masses

$$U = \frac{Gm_1m_2}{|\mathbf{r}_1 - \mathbf{r}_2|} = \frac{Gm_1m_2}{\Delta}$$

$$\Delta = |\mathbf{r}_1 - \mathbf{r}_2| = (r_1^2 + r_2^2 - 2r_1r_2\cos\psi)^{\frac{1}{2}} \qquad (67)$$

with angle

$$\cos\psi \equiv \hat{\mathbf{r}}_1 \cdot \hat{\mathbf{r}}_2 \tag{68}$$

We define two angles θ_1, θ_2 known as **true longitudes**.

$$\begin{aligned}\theta_1 &= \varpi_1 + f_1 = \omega_1 + \Omega_1 + f_1 \\ \theta_2 &= \varpi_2 + f_2 = \omega_2 + \Omega_2 + f_2.\end{aligned} \tag{69}$$

Recall $\varpi = \omega + \Omega$ is the longitude of pericenter whereas ω is the argument of pericenter and Ω is the longitude of the ascending node.

The angle ψ is related to the true longitudes via

$$\Psi = \cos\psi - \cos(\theta_1 - \theta_2) \tag{70}$$

The reason for using Ψ is to separate terms that depend upon inclination with those that depend upon eccentricity. If the inclination is small, then Ψ would be small. The interaction term depends on Δ which is expanded in a Taylor series of Ψ

$$\frac{1}{\Delta} = \sum_{i=0}^{\infty} \frac{(2i)!}{(i!)^2} \left(\frac{rr'\Psi}{2}\right)^i \Delta_0^{-2i+1} \tag{71}$$

where

$$\Delta_0 = \sqrt{r^2 + r'^2 - 2rr'\cos(\theta - \theta')}. \tag{72}$$

Then two small parameters are defined

$$\varepsilon = \frac{r}{a} - 1 \qquad \varepsilon' = \frac{r'}{a'} - 1 \tag{73}$$

and Δ_0 can be expanded assuming that ε and ε' are small. The inclinations are only contained within Ψ and the eccentricities are only contained in ε, ε'.

It is useful to define

$$\rho_0 = (a^2 + a'^2 - 2aa'\cos(\theta - \theta'))^{\frac{1}{2}} \tag{74}$$

Note that $r = a(1+\varepsilon)$ and $r' = a'(1+\varepsilon')$. We insert these into Δ_0 giving

$$\Delta_0 = \sqrt{a^2(1+\varepsilon)^2 + a'^2(1+\varepsilon')^2 - 2a(1+\varepsilon)a'(1+\varepsilon')\cos(\theta - \theta')}. \tag{75}$$

We expand $\Delta_0^{-(2i+1)}$ (in equation 71) assuming that $\varepsilon, \varepsilon'$ are small

$$\begin{aligned}\Delta_0^{-(2i+1)} &= \rho_0^{-(2i+1)} + (r-a)\frac{\partial}{\partial a}\rho_0^{-(2i+1)} + (r'-a')\frac{\partial}{\partial a'}\rho_0^{-(2i+1)} \\ &\quad + \frac{1}{2}(r-a)^2\frac{\partial^2}{\partial a^2}\rho_0^{-(2i+1)} + \frac{1}{2}(r'-a')^2\frac{\partial^2}{\partial a'^2}\rho_0^{-(2i+1)} \\ &\quad + (r-a)(r'-a')\frac{\partial^2}{\partial a \partial a'}\rho_0^{-(2i+1)} + \ldots\end{aligned} \tag{76}$$

We define

$$D_{m,n} \equiv a^m a'^n \frac{\partial^{m+n}}{\partial a^n \partial a'^m} \tag{77}$$

$$\Delta_0^{-(2i+1)} = \left[1 + \varepsilon D_{1,0} + \varepsilon' D_{0,1} + \frac{1}{2!}\left(\varepsilon^2 D_{2,0} + \varepsilon'^2 D_{0,2} + 2\varepsilon\varepsilon' D_{1,1}\right) + ...\right]\rho_0^{-(2i+1)} \tag{78}$$

2.13.1 Laplace coefficients

The coefficients for $\rho_0^{-(2i+1)}$ are written in terms of **Laplace coefficients** which are

$$b_s^{(j)}(\alpha) \equiv \frac{1}{\pi}\int_0^{2\pi} \frac{\cos j\psi \; d\psi}{(1 - 2\alpha\cos\psi + \alpha^2)^s} \tag{79}$$

for non-negative half integer s, integer j and positive real number $\alpha < 1$. These are Fourier coefficients of the function $(1 - 2\alpha\cos\psi + \alpha^2)^{-s}$ which is related to powers of ρ_0 using α, a ratio of the semi-major axes. The ratio $\alpha = a/a'$ or $\alpha = a'/a$ depending upon which is larger since we require $\alpha < 1$.

What is the result of all of this? Products of cosines and sines can be manipulated to be a single cosine of an angle that is a sum of orbital elements. The result is that the expansion can be written to look like

$$\Delta^{-1} = \sum_{integers} f(\alpha, e, s, e', s') \cos\phi_{integers} \tag{80}$$

where f is a polynomial of e, s, e', s' and a function of α, the variables $s = \sin(I/2), s' = \sin(I'/2)$, the angle

$$\phi = k_\lambda \lambda + k_{\lambda'}\lambda' + k_\varpi \varpi + k_{\varpi'}\varpi' + k_\Omega \Omega + k_{\Omega'}\Omega' \tag{81}$$

and the k's are integers.

An appendix by Murray and Dermott lists all the terms to fourth order (in e and s).

There is a nice new python package for calculating Laplace coefficients: `https://pylaplace.readthedocs.io/en/latest/` (though I have not yet tried it out) This package will calculate one of the derivatives. If you need a second or higher order derivative you will need to use recursion relations that are available (their eqn 6.71) in Murray + Dermott's book and taken from the classic book by Brouwer and Clemence.

2.14 Heliocentric Coordinates

Orbital elements for objects in the solar system are conventionally given in heliocentric coordinates.

For N massive bodies with masses m_i and coordinates $\mathbf{r}_i$ and interaction via gravity alone

$$H(\mathbf{p}_i, \mathbf{r}_i, \text{all i}) = \sum_{i=0}^{N-1} \frac{p_i^2}{2m_i} - \sum_{i>j} \frac{G m_i m_j}{|\mathbf{r}_i - \mathbf{r}_j|}$$

we take a generating function

$$F_2(\mathbf{r}_i, \mathbf{P}_i, \text{all i}) = \sum_{i>0} (\mathbf{r}_i - \mathbf{r}_0) \cdot \mathbf{P}_i + \mathbf{r}_0 \cdot \mathbf{P}_0$$

giving new coordinates

$$\mathbf{Q}_{i\neq 0} = \frac{\partial F_2}{\partial \mathbf{P}_i} = \mathbf{r}_i - \mathbf{r}_0$$

Coordinates are now heliocentric in that they are with respect to body with index 0.

$$\mathbf{p}_{i\neq 0} = \frac{\partial F_2}{\partial \mathbf{r}_i} = \mathbf{P}_i$$

New momenta are the same as old momenta for $i \neq= 0$.

$$\mathbf{Q}_0 = \frac{\partial F_2}{\partial \mathbf{P}_0} = \mathbf{r}_0$$

The new coordinate of the central body is unchanged.

$$\mathbf{p}_0 = \frac{\partial F_2}{\partial \mathbf{r}_0} = \mathbf{P}_0 - \sum_{i>0} \mathbf{P}_i$$

The new momenta for the central mass is with respect to the centre of mass of all the other bodies.

The new Hamiltonian, in heliocentric coordinates, looks like

$$H(\mathbf{P}, \mathbf{Q}, \text{all i}) = \sum_{i>0} \left(\frac{P_i^2}{2m_i} - \frac{G m_i m_0}{Q_i} \right) - \sum_{i>j, j>0} \frac{G m_i m_k}{|\mathbf{r}_i \mathbf{r}_j|} + \frac{1}{2m_0} \left(P_0 - \sum_{i>0} P_i \right)^2$$

We notice that the first term is a Keplerian term. This form is convenient if we want to construct an integrator, leveraging our ability to integrate exact Keplerian systems and considering the other terms as perturbations. The second term is a list of potential interaction terms. The odd new term is the last one, containing the momenta, that acts like a drift. It acts like a drift because terms in the Hamiltonian only containing momenta and so cause only changes in position.

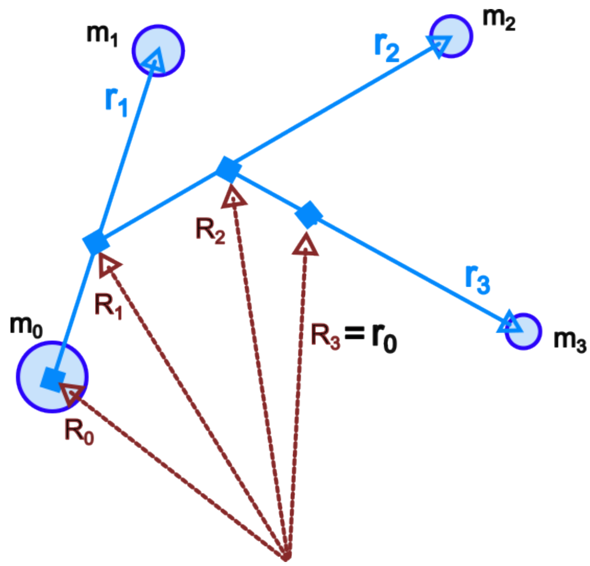

Figure 3: A set of masses m_j with $j \in \{0, ..., N-1\}$ have coordinates $\mathbf{x}_j$ with respect to an origin. A consecutive set of centers of mass $\mathbf{R}_j$ (for masses up to and including m_j, shown as blue squares and defined in equation 83) are shown with respect to the origin. Jacobi coordinates are $\mathbf{r}_0, ...\mathbf{r}_{N-1}$ and are with respect to the centers of mass $\mathbf{R}_j$.

2.15 Jacobi Coordinates

Jacobi coordinates are used in some integrators and are the default in the *rebound* simulation code.

Using *Jacobi coordinates* the drift term in the Hamiltonian can be eliminated entirely. Firstly particles are ranked in order (usually of mass). The coordinates of each mass depends on the center of mass of the previous ones. For N masses with masses m_j and positions $\mathbf{x}_j$ with index $j \in \{0, 1, ..., N-1\}$,

$$\eta_j \equiv \sum_{k=0}^{j} m_k \tag{82}$$

is the sum of all masses up to and including j's. The center of mass of the masses up to and including j's is

$$\mathbf{R}_j = \frac{1}{\eta_j} \sum_{k=0}^{j} \mathbf{x}_j. \tag{83}$$

The Jacobi coordinates are with respect to the different centers of mass

$$\mathbf{r}_j = \mathbf{x}_j - R_{j-1} \qquad j > 0 \tag{84}$$

$$\mathbf{r}_0 = R_{N-1}. \tag{85}$$

There are variants in the coordinate definition with different choices in the signs and indexing.

The use of Jacobi and Heliocentric coordinates in the development of symplectic integrators is attributed to Matt Holman, Jack Wisdom, and Jihad Touma in the 80's and early 90's.

2.16 Lagrange's Planetary equations of motion

The mean longitude

$$\lambda = M + \varpi = M + \omega + \Omega = n(t - \tau) + \varpi = nt + \epsilon$$

with

$$\epsilon = \omega + \Omega - n\tau$$

and τ is a particular reference time chosen to give the orbital elements.

Taking a Hamiltonian equal to the Keplerian one plus a perturbation $\mathcal{R}$ known as the disturbing function, the equations of motion (for the orbital elements) are

$$\frac{da}{dt} = \frac{2}{na}\frac{\partial \mathcal{R}}{\partial \epsilon} \tag{86}$$

$$\frac{de}{dt} = -\frac{\sqrt{1-e^2}}{na^2 e}(1 - \sqrt{1-e^2})\frac{\partial \mathcal{R}}{\partial \epsilon} - \frac{\sqrt{1-e^2}}{na^2 e}\frac{\partial \mathcal{R}}{\partial \varpi} \tag{87}$$

$$\frac{d\epsilon}{dt} = -\frac{2}{na}\frac{\partial \mathcal{R}}{\partial a} + \frac{\sqrt{1-e^2}}{na^2 e}(1 - \sqrt{1-e^2})\frac{\partial \mathcal{R}}{\partial e} + \frac{\tan(I/2)}{na^2\sqrt{1-e^2}}\frac{\partial \mathcal{R}}{\partial I} \tag{88}$$

$$\frac{d\Omega}{dt} = \frac{1}{na^2\sqrt{1-e^2}\sin I}\frac{\partial \mathcal{R}}{\partial I} \tag{89}$$

$$\frac{d\varpi}{dt} = \frac{\sqrt{1-e^2}}{na^2 e}\frac{\partial \mathcal{R}}{\partial e} + \frac{\tan(I/2)}{na^2\sqrt{1-e^2}}\frac{\partial \mathcal{R}}{\partial I} \tag{90}$$

$$\frac{dI}{dt} = -\frac{\tan(I/2)}{na^2\sqrt{1-e^2}}\left(\frac{\partial \mathcal{R}}{\partial \epsilon} + \frac{\partial \mathcal{R}}{\partial \varpi}\right) - \frac{1}{na^2\sqrt{1-e^2}\sin I}\frac{\partial \mathcal{R}}{\partial \Omega} \tag{91}$$

Usually $\frac{\partial}{\partial \epsilon}$ is replaced by $\frac{\partial}{\partial \lambda}$. There is a technical difference as mean motion n depends on a and if you take a partial derivative with respect to λ you need to assume that a is held fixed.

If one transfers the Keplerian Hamiltonian into action angle coordinates (using canonical transformations), deriving Poincaré or Delaunay variables, then Lagrange's equations follow from Hamilton's equations.