1. On Pitch-Fork Bifurcations

Consider the dynamical system

\[ \dot{x} = rx - 2 \sinh x \]

Show that a pitchfork bifurcation occurs and find the value of \( r \) and \( x \) at which it occurs. Classify the bifurcation as super-critical or sub-critical.

2. On Classifying Bifurcations

In each case find the value of \( r \) and \( x^* \) at which a bifurcation occurs and classify the type of bifurcation. Draw a bifurcation diagram.

(a) \[ \dot{x} = 4 - re^{-x^2} \]

(b) \[ \dot{x} = rx - \frac{x}{1+x^2} \]

(c) \[ \dot{x} = rx - \frac{3x}{1+x^2} \]

This problem is much easier to do if you plot the function for different values of \( r \) and look at where it crosses the \( x \) axis.

3. On Cobweb Plots

Consider the map

\[ x_{n+1} = \mu \sin(\pi x_n) \]

that has a similar shape to the logistic map.

The routine given here http://astro.pas.rochester.edu/~aquillen/phy411/pylab/cobweb_sin.py plots a cobweb plot for this function for \( \mu = 0.81 \). The cobweb plot shows a period two orbit (see figure ??).

(a) Explore what the cobweb plot looks like for smaller values of \( \mu \) than 0.81.

(b) Increase \( \mu \) until you find a value where the two cycle becomes unstable and you can see a period four orbit in the cobweb plot.
(c) Increase \( \mu \) further until you see a chaotic orbit.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{Graphical analysis of \( x_{n+1} = \mu \sin(\pi x_n) \) with \( \mu = 0.83 \). The orbit is a two-cycle. The green line shows \( y = x \), the red-line \( f(x) = \mu \cos(\pi x) \).}
\end{figure}

4. The logistic map, period doubling and self-similarity

The logistic map,
\[ f_{\mu}(x) = \mu x (1 - x) \]

You can modify the python code “logistic.py” to make plots of the attracting orbits as a function of \( \mu \). See:

http://astro.pas.rochester.edu/~aquillen/phy411/pylab/logistic.py

Consider the properties of \( f_\mu^3(x) \) that is \( f(f(f(x))) \) for a given \( \mu \) value.

(a) For which parameter values of \( \mu \) is there an attracting prime period 3 point for \( f_\mu^3 \)? Illustrate the range graphically.

(b) Find a box where the attracting points of \( f_\mu^3(x) \) resembles that of \( f_s(x) \) for some \( s \).

(c) Identify a region where there is period doubling in either map.
5. Conjugacy between the Logistic and Tent maps and computing a Lyapunov Exponent

The logistic map (for $0 \leq x_n \leq 1$) is

$$x_{n+1} = rx_n(1 - x_n)$$

The tent map (for $0 \leq y_n \leq 1$) is

$$y_{n+1} = \begin{cases} 
R y_n & 0 \leq y_n \leq \frac{1}{2} \\
R(1 - y_n) & \frac{1}{2} \leq y_n \leq 1 
\end{cases}$$

In this problem we relate the logistic map with $r = 4$ to the tent map with $R = 2$.

Consider the transformation

$$x_n = \sin^2\left(\frac{\pi y_n}{2}\right)$$

that takes $0 \leq y_n \leq 1$ to $0 \leq x_n \leq 1$.

(a) Show that the logistic map with $r = 4$ transforms into the tent map with $R = 2$.

(b) Compute the Lyapunov exponent of the tent map with $R = 2$.

(c) Using part b, find the Lyapunov exponent of the logistic map with $r = 4$.


Consider the map

$$x_{n+1} = r \sin (\pi x_n)$$

that is similar to the logistic map.

Show the periodic attracting orbits and compute the Lyapunov exponent as a function of $r$.

You can modify the routines in

http://astro.pas.rochester.edu/~aquillen/phy411/pylab/lyap.py