PHY411 Lecture notes Part 8

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Figure 1: A string!
1 Continuum Mechanics

1.1 Strings

Consider a string of linear mass \( \mu(x) \) and under tension \( \tau(x) \) where \( x \) is horizontal distance along the string when it is at rest. Its vertical displacement we describe with a function \( y(x,t) \) that is also a function of time. When the string is at rest \( y(x,t) = 0 \). A length of the string due to displacement is

\[
 ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + \left( \frac{\partial y}{\partial x} \right)^2} \ dx 
\]

\[
 \sim dx \left( 1 + \frac{1}{2} \left( \frac{\partial y}{\partial x} \right)^2 \right) \tag{1}
\]

The change in length locally is

\[
 dl = ds - dx = \frac{dx}{2} \left( \frac{\partial y}{\partial x} \right)^2 
\]

The differential potential energy (force times distance)

\[
 dU = \tau(x) dl = \frac{1}{2} \tau(x) \left( \frac{\partial y}{\partial x} \right)^2 \ dx 
\]

The differential kinetic energy

\[
 dT = \frac{1}{2} \mu(x) \left( \frac{\partial y}{\partial t} \right)^2 \ dx 
\]

We find the total Lagrangian by integrating over \( x \),

\[
 L = \int dx \ L(y, \dot{y}, y'; x, t) 
\]

and \( L \) we call the Lagrangian density

\[
 L(y, \dot{y}, y'; x, t) = \frac{1}{2} \mu(x) \left( \frac{\partial y}{\partial t} \right)^2 - \frac{1}{2} \tau(x) \left( \frac{\partial y}{\partial x} \right)^2 \tag{2}
\]

The action is a double integral

\[
 S = \int_{t_a}^{t_b} dt \int_{x_a}^{x_b} dx L(y, \dot{y}, y'; x, t) = \int_{t_a}^{t_b} dt \int_{x_a}^{x_b} dx \left[ \mu \dot{y}^2 - \tau y'^2 \right]
\]
When we minimize the action we can consider minimizing it for paths \( y(x, t) \to y(x, t) + h(x, t) \).

\[
\begin{align*}
\partial S &= \int_{t_a}^{t_b} dt \int_{x_a}^{x_b} dx \left[ \mathcal{L}(y + h, \dot{y} + \dot{h}, y' + h'; x, t) - \mathcal{L}(y, \dot{y}, y'; x, t) \right] \\
&= \int_{t_a}^{t_b} dt \int_{x_a}^{x_b} dx \left[ \frac{\partial \mathcal{L}}{\partial y} h + \frac{\partial \mathcal{L}}{\partial \dot{y}} \dot{h} + \frac{\partial \mathcal{L}}{\partial y'} h' \right]
\end{align*}
\]

We integrate by parts

\[
\begin{align*}
\int_{x_1}^{x_2} dx \int_{t_1}^{t_2} dt \frac{\partial \mathcal{L}}{\partial \dot{y}} h &= \int_{x_1}^{x_2} dx \left[ - \int_{t_1}^{t_2} dt \left( \frac{\partial \mathcal{L}}{\partial y} \right) h + \frac{\partial \mathcal{L}}{\partial \dot{y}} h \right]_{t_1}^{t_2} \\
\int_{x_1}^{x_2} dx \int_{t_1}^{t_2} dt \frac{\partial \mathcal{L}}{\partial y'} h' &= \int_{t_1}^{t_2} dt \left[ - \int_{x_1}^{x_2} dx \left( \frac{\partial \mathcal{L}}{\partial y'} \right) h + \frac{\partial \mathcal{L}}{\partial y'} h \right]_{x_1}^{x_2}
\end{align*}
\]

Inserting these into equation 3

\[
\begin{align*}
\partial S &= \int_{t_a}^{t_b} dt \int_{x_a}^{x_b} dx \left[ \frac{\partial \mathcal{L}}{\partial y} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{y}} - \frac{d}{dx} \frac{\partial \mathcal{L}}{\partial y'} \right] h + \int_{x_1}^{x_2} dx \left. \frac{\partial \mathcal{L}}{\partial \dot{y}} \right|_{t_1}^{t_2} + \int_{t_1}^{t_2} dt \left. \frac{\partial \mathcal{L}}{\partial y'} h \right|_{x_1}^{x_2}
\end{align*}
\]

With suitable boundary conditions

\[
\frac{\partial \mathcal{L}}{\partial \dot{y}} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{y}} - \frac{d}{dx} \frac{\partial \mathcal{L}}{\partial y'} = 0
\]

Evaluating the derivatives with \( \tau, \mu \) independent of \( x \) and \( t \) for the Lagrangian in equation 2

\[
\frac{\partial \mathcal{L}}{\partial \dot{y}} = \mu \dot{y} \quad \frac{\partial \mathcal{L}}{\partial y'} = -\tau y'
\]

giving

\[
\mu \frac{\partial^2 y}{\partial t^2} - \tau \frac{\partial^2 y}{\partial x^2} = 0
\]

which is the wave equation.

We can perform a Legendre transformation (at each value of \( x \))

\[
\pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{y}(x)} = \mu \dot{y}(x)
\]

Inverting this

\[
\dot{y}(\pi(x)) = \pi(x) / \mu
\]

and we find a Hamiltonian or energy density

\[
H(\pi(x), y(x), y'(x)) = \pi \dot{y}(\pi) - \mathcal{L}(y, \dot{y}(\pi), y', x) = \frac{\pi(x)^2}{2\mu} + \frac{y'(x)^2}{2}
\]
The total energy is integrated over $x$

$$E = \int dx H(\pi(x), y(x), y'(x))$$

and because the Hamiltonian does not depend on time, the total integrated energy should be a constant.

It is confusing to see our functions depend on $y'$ here. It is helpful to think in the limit of a discrete system as the number of discrete items becomes large. The string can be considered as a large $N$ limit of many coupled bodies, where the coupling depends on the differences between nearby bodies that is here represented by the spatial derivative. Each discrete body is at a different $x$, so we can think of $N$ different bodies at $N$ different $x$ positions. In the large $N$ limit the number of bodies becomes infinite. In this sense $\pi(x)$ is an infinite dimensional vector of momenta conjugate to $y(x)$, an infinite dimensional vector of coordinates. Our Hamiltonian is really depend on momenta $\pi(x)$ and coordinates $y(x)$ and $y'(x)$ is a function of nearby coordinates.

2 Infinite dimensional Classical Hamiltonian systems

2.1 The functional derivative

Instead of just conjugate vectors $p, q$ we can consider conjugate fields $\rho(x)$ and $\phi(x)$. In the continuum limit the Hamiltonian is an integral over all space.

$$H[\rho, \phi] = \int dx f(\rho(x), \phi(x))$$

and $f(\rho, \phi)$ a function that depends on the fields $\rho(x), \phi(x)$. The value of the fields at each position $x$ serve as coordinates. These are functions of a continuous variable, effectively like an infinite number of coordinates.

How do we take derivatives with respect to fields instead of with respect to coordinates? We use a functional derivative

$$\frac{\delta F[u(x)]}{\delta u(y)} \equiv \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left( F[u(x) + \epsilon \delta(x - y)] - F[u(x)] \right)$$

Here, as is common in physics, the delta function is used. Above $F$ is a functional (operating on the function or field $u$) and $u$ is a function. More rigorously the functional derivative can be defined

$$\int \frac{\delta F}{\delta \rho}(x)g(x)dx = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left[ F[\rho + \epsilon g] - F[\rho] \right]$$

$$= \frac{d}{d\epsilon} \left[ F[\rho + \epsilon g] \right]_{\epsilon=0} \quad (4)$$
where $F$ is a functional (function of functions) and $\rho$ is a function and $g$ is an arbitrary function. We can think of $\delta F / \delta \rho$ as the gradient of $F$ at the point $\rho$. In analogy to

$$dF = \sum_i \frac{\partial F}{\partial \rho_i} d\rho_i$$

$$\delta F = \int \frac{\delta F[\rho]}{\delta \rho(x)} g(x) dx$$

with $g(x) dx$ serving as $d\rho_i$.

A function can be written as a functional using an integral and a delta function

$$F_x[\phi] = \int dy \delta(y - x) \phi(y) = \phi(x)$$

For this point case the functional derivative

$$\frac{\delta F_x[\phi]}{\delta \phi(y)} = \lim_{\epsilon \to 0} \frac{1}{\epsilon} [\phi(x) + \epsilon \delta(x - y) - \phi(x)]$$

$$= \delta(x - y)$$

This follows as the value of the derivative can only change at the location of $x$. Consider a slightly more difficult but point case

$$F_x[\phi] = \nabla \phi(x)$$

The functional derivative

$$\frac{\delta F_x[\phi]}{\delta \phi(y)} = \lim_{\epsilon \to 0} \frac{1}{\epsilon} [\nabla_x(\phi(x) + \epsilon \delta(x - y)) - \phi(x)]$$

$$= \nabla_x \delta(x - y)$$

The functional derivative satisfies the product rule

$$\frac{\delta (FG)[\rho]}{\delta \rho(x)} = \frac{\delta F[\rho]}{\delta \rho(x)} G[\rho] + F[\rho] \frac{\delta G[\rho]}{\delta \rho(x)}$$

If $g$ is a function then the chain rule is

$$\frac{\delta F[g(\rho)]}{\delta \rho(y)} = \frac{\delta F[g(\rho)]}{\delta g(\rho(y))} \frac{dg(\rho)}{d\rho(y)}$$

If $G$ is an operator or a functional then the chain rule is

$$\frac{\delta}{\delta \rho(y)} F[G[\rho]] = \int dx \frac{\delta F[G]}{\delta G(x)} \bigg|_{G=G[\rho]} \frac{\delta G[\rho]}{\delta \rho(y)}$$
Suppose we have a function

\[ F[\rho] = \int f(x, \rho(x), \nabla \rho(x))dx \]

The functional derivative

\[ \frac{\delta F[\rho]}{\delta \rho(x)} = \frac{\partial f}{\partial \rho} - \nabla \cdot \frac{\partial f}{\partial \nabla \rho} \] (5)

We now show this using the definition for the functional derivative in equation 4

\[
\int \frac{\partial F[\rho]}{\delta \rho(x)} g(x) dx = \frac{d}{d\epsilon} \left[ \int f(x, \rho(x) + \epsilon g(x), \nabla \rho + \epsilon \nabla g) dx \right]_{\epsilon=0}
\]

\[ = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int dx \left[ f(x) + \epsilon \frac{\partial f}{\partial \rho} g(x) + \epsilon \frac{\partial f}{\partial \nabla \rho} \nabla g(x) - f(x) \right] \]

\[ = \int \left( \frac{\partial f}{\partial \rho} g + \frac{\partial f}{\partial \nabla \rho} \cdot \nabla g \right) dx \]

\[ = \int \left( \frac{\partial f}{\partial \rho} g + \nabla \cdot \left( \frac{\partial f}{\partial \nabla \rho} g \right) - \left( \nabla \cdot \frac{\partial f}{\partial \nabla \rho} \right) g \right) dx \]

\[ = \int \left( \frac{\partial f}{\partial \rho} g - \nabla \cdot \frac{\partial f}{\partial \nabla \rho} \right) dx \]

where we have assumed that \( \phi \) or \( f \) are zero on the boundary of the integral.

For a Hamiltonian \( H[\rho, \phi] \), with \( \rho \) acting like a momentum and \( \phi \) acting like a coordinate, we can write Hamilton’s equations as

\[ \frac{\partial \rho(x)}{\partial t} = -\frac{\delta H[\phi, \rho]}{\delta \phi(x)} \]

\[ \frac{\partial \phi(x)}{\partial t} = \frac{\delta H[\phi, \rho]}{\delta \rho(x)} \]

in analogy to Hamilton’s equations for discrete or finite systems.

What is the Poisson bracket? The equation of motion should be consistent with

\[ \dot{\phi}(x) = \{\phi(x), H[\phi, \rho]\} \quad \dot{\rho}(x) = \{\rho(x), H[\phi, \rho]\} \]

In the finite dimensional setting, the Poisson bracket is summed over derivatives with respect to all pairs of momenta and coordinates. We expect that in the continuum limit the Poisson bracket will involve an integral and instead of derivatives of functional \( H \) we will be taking functional derivatives of \( H \). As \( \rho(x), \phi(x) \) are both fields and canonical coordinates the Poisson bracket with \( \rho \) will only involve an integral of a functional derivative.
of $H$ with respect to $\phi$ and the Poisson bracket with $\phi$ would be the integral of a functional derivative of $H$ with respect to $\rho$. We expect something like this

$$
\{\rho(y), H[\phi, \rho]\} = \int dx (-\frac{\delta \rho(y)}{\delta \rho(x)} \frac{\delta H}{\delta \phi(x)})
$$

$$
= -\int dx \delta(y - x) \frac{\delta H}{\delta \phi(x)}
$$

$$
\{\phi(y), H[\phi, \rho]\} = \int dx \frac{\delta \phi(y)}{\delta \phi(x)} \frac{\delta H}{\delta \rho(x)}
$$

$$
= \int dx \delta(y - x) \frac{\delta H}{\delta \rho(x)}
$$

That the fields are like canonical coordinates is seen with the delta function in the integrals.

3 Examples with canonical coordinates

3.1 Wave equation

We consider a field $\phi(x)$. The time derivative $\dot{\phi}(x)$. We can assume that the Lagrangian can be written as a sum of kinetic and potential energy terms

$$
\mathcal{L}(\phi, \dot{\phi}) = T[\dot{\phi}] - W[\phi]
$$

The total kinetic energy is an integral over space of the kinetic term

$$
T[\dot{\phi}] = \int \frac{1}{2} \dot{\phi}^2 d^3 x
$$

and the potential energy

$$
W[\phi] = \int V(\phi(x))d^3 x
$$

where $V(\phi)$ is a function (and it could depend on the gradient of the potential). We have a momentum

$$
\pi(x) = \frac{\delta \mathcal{L}}{\delta \dot{\phi}(x)} = \frac{\delta T}{\delta \dot{\phi}(x)} = \dot{\phi}(x)
$$

Using the momentum $\pi(x)$ we can construct a Hamiltonian

$$
H[\pi, \phi] = \int \left[ \frac{1}{2} \pi(x)^2 + V(\phi(x)) \right] dx
$$

where $\pi(x), \phi(x)$. The potential energy can be something interesting that includes terms like

$$
V(\phi) = \frac{1}{2} |\nabla \phi|^2 + U(\phi)
$$
Using the functional derivative we check
\[
\frac{\delta H}{\delta \pi(y)} = \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int dx [\pi(x) + \epsilon \delta(x - y)]^2 - \pi(x)^2
\]
\[
= \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int dx [\pi(x)^2 + 2\pi(x)\epsilon \delta(x - y) - \pi(x)^2]
\]
\[
= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int dx \pi(x) \epsilon \delta(x - y)
\]
\[
= \pi(y)
\]
\[
\frac{\delta H}{\delta \phi(y)} = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int dx \left[ \frac{1}{2}[\nabla(\phi(x) + \epsilon \delta(x - y))]^2 + U(\phi(x) + \epsilon \delta(x - y)) \right]
\]
\[
- \int dx \left[ \frac{1}{2}[\nabla(\phi(x))]^2 + U(\phi(x)) \right]
\]
\[
= \int dx \left[ \nabla \phi(x) \cdot \nabla \delta(x - y) + U'(\phi(x)) \delta(x - y) \right]
\]
\[
= -\nabla^2 \phi(y) + U'(\phi(y))
\]
where the last step for the left term is done by parts. The equations of motion can be written as
\[
\dot{\phi} = \pi \quad \dot{\pi} = \nabla^2 \phi - U'(\phi)
\]
and
\[
\ddot{\phi} = \nabla^2 \phi - U'(\phi)
\]
Neglecting \(U'(\phi)\) we recover the wave equation.

For this system what is the Poisson bracket? The equation of motion should be consistent with
\[
\dot{\phi}(x) = \{\phi(x), H[\phi, \pi]\} \quad \dot{\pi}(x) = \{\pi(x), H[\phi, \pi]\}
\]
The Poisson bracket is summed over derivatives with respect to all pairs of momenta and coordinates. We expect that in the continuum limit the Poisson bracket will involve an integral and instead of derivatives we will be taking functional derivatives. As \(\pi(x), \phi(x)\) are canonical coordinates the Poisson bracket with \(\pi\) will only involve an integral of a functional derivative of \(H\) with respect to \(\phi\) and the Poisson bracket with \(\phi\) would be the integral of a functional derivative of \(H\) with respect to \(\pi\). In other words we expect something like this
\[
\{\pi(y), H\} = \int dx (-\frac{\delta \pi(y)}{\delta \pi(x)} \frac{\delta H}{\delta \phi(x)})
\]
\[
= \int dx \delta(y - x) \left[ \nabla^2 \phi(x) - U'(\phi(x)) \right]
\]
\[
= \nabla^2 \phi(y) - U'(\phi(y))
\]
and this is consistent with our equations of motion (equation 6). Likewise

\[
\begin{align*}
\{\phi (y), H\} &= \int dx \frac{\delta \phi (y)}{\delta \phi (x)} \frac{\delta H}{\delta \pi (x)} \\
&= \int dx \, \delta (y - x) \pi (x) \\
&= \pi (y)
\end{align*}
\]

This dynamical system is consistent with a Poisson bracket that gives

\[
\begin{align*}
\{\phi (x), \pi (y)\} &= \delta (x - y) \\
\{\pi (x), \pi (y)\} &= \{\phi (x), \phi (y)\} = 0
\end{align*}
\]

This is equivalent in the continuum limit with

\[
\begin{align*}
\{p_i, q_i\} &= \delta_{ij} \\
\{p_i, p_i\} &= \{q_i, q_j\} = 0
\end{align*}
\]

where there are many possible coordinates. Our coordinates and momenta \(\phi (x), \pi (x)\) are canonical because they obey the Poisson bracket that we expect in the continuum limit.

In the finite dimensional setting we can use a single vector \(x = (q, p)\) for both coordinates and momenta. The Poisson bracket can be written

\[
\{x_i, x_j\} = J^{ij}
\]

with \(J\) the symplectic matrix

\[
J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}
\]

where \(I\) is the identity matrix. For functions

\[
\{f, g\} = \partial f / \partial x_i J^{ij} \partial g / \partial x_j
\]

For the wave equation we have momenta and coordinate fields \(\pi\) and \(\phi\). We can describe this in terms of a single field \(\rho_i\) with \(\rho_0 = \phi\) and \(\rho_1 = \pi\). With this notation

\[
\{\rho_i (x), \rho_j (y)\} = J^{ij} \delta (x - y)
\]
Figure 2: Stream lines are level curves of the stream function (in 2 dimensions). In this setting vorticity is low so we might work with potential flow.

Figure 3: Stream lines are level curves of the stream function (in 2 dimensions). In this setting the flow has vorticity. If the flow is incompressible we can use a stream function.
3.2 Two-dimensional incompressible fluid dynamics

In a vorticity free setting we might use potential flow \( \mathbf{v} = \nabla \phi \). However in a two-dimensional incompressible setting we might use a stream function, \( \mathbf{v} = \nabla \times (\psi \hat{z}) \).

Let us use a stream function \( \psi(x,y) \) with velocity \( \mathbf{v} = (u,v) \)

\[
  u = -\partial_y \psi \\
  v = \partial_x \psi
\]

and using short hand \( \partial_x = \partial/\partial x \). Computing

\[
  \nabla \cdot \mathbf{v} = \partial_x u + \partial_y v = -\partial_{xy} \psi + \partial_{yx} \psi = 0
\]

So by describing the system in terms of the stream function we automatically have assumed incompressible flow. The vorticity \( (z \text{ component}) \)

\[
  \omega_z = (\nabla \times \mathbf{v}) \cdot \hat{z} = \partial_x v - \partial_y u = \partial_x \partial_x \psi + \partial_y \partial_y \psi = \nabla^2 \psi
\]

So the function

\[
  q(x,y) = \nabla^2 \psi
\]

is the vorticity.

Using a Poisson bracket

\[
  \{f,g\} = \partial_x f \partial_y g - \partial_y f \partial_x g \tag{8}
\]

we can compute

\[
  \{x,\psi\} = \partial_y \psi \\
  \{y,\psi\} = -\partial_x \psi
\]

Here the coordinate \( x \) is conjugate to the coordinate \( y \) (which is acting like a momentum) and the coordinates are canonical coordinates.

Now let us consider the motions of a tracer particle moving with the flow with a trajectory \( x(t), y(t) \). Because the tracer particle is moving with the flow equations of motion are

\[
  \dot{x}(t) = u(x,y,t) = -\partial_y \psi(x,y,t) = -\{x,\psi\} \\
  \dot{y}(t) = v(x,y,t) = \partial_x \psi(x,y,t) = -\{y,\psi\}
\]

We can associate \(-\psi\), minus the stream function, with the Hamiltonian and the Poisson bracket is canonical.
3.2.1 Dynamics of vortices in 2D

The stream function acts like a potential and satisfies a Poisson equation, $-\nabla^2\psi = \omega$. The Greens’ function for a point vortex at position $x_i$ is

$$\psi(x) = -\frac{\Gamma_i}{2\pi} \log ||x - x_i||$$

corresponding to a vorticity field

$$\omega(x) = \Gamma_i \delta(x - x_i)$$

where $\Gamma_i$ is a constant. We have assumed that we are working on the infinite real plane. The associated velocity field for this vortex is

$$u(x,y) = \nabla \times (\psi \hat{z}) = \frac{\Gamma_i}{2\pi} \frac{1}{||x - x_i||^2} [(y_i - y)(\hat{x}) - (x_i - x)(\hat{y})] \quad (9)$$

For $N$ point vortices the vorticity field

$$\omega(x) = \sum_{i=1}^{N} \Gamma_i \delta(x - x_i)$$

Note: The above Greens function and associated velocity field is good when there is no boundary (or we are working on the plane). Solving Laplace’s equation given non-trivial boundaries (like a circular boundary) (and with point discontinuities) is more difficult!

Since vorticity is advected by the flow, the motion of one vortex depends on the velocity fields caused by all the other vortices. Summing over the velocity fields (equation 9)

$$\dot{x}_i(t) = \sum_{j \neq i} \frac{\Gamma_j}{2\pi} \frac{1}{||x_i - x_j||^2} [(y_i - y_j)(\hat{x}) - (x_i - x_j)(\hat{y})]$$

These equations of motion can also be modeled with Hamiltonian and Poisson bracket

$$H(x_i, y_i, ...) = -\sum_{i \neq j} \frac{\Gamma_i \Gamma_j}{4\pi} \log ||x_i - x_j||$$

$$\{f, g\} = \sum_i \frac{1}{\Gamma_i} \left[ \partial_{x_i} f \partial_{y_i} g - \partial_{y_i} f \partial_{x_i} g \right]$$

The dynamics is that of an $N$-dimensional Hamiltonian system (2N phase space) so the infinite dimensional system has been reduced to a finite dimensional one.
3.2.2 Blinking Vortex model and time dependent models

A time dependent low dimension dynamical system can be constructed by using point vortices that have time dependent vorticity $\Gamma_i(t)$. Even with only 2 vortices this can be a chaotic system. I think one can construct a time dependent Hamiltonian for this system simply by replacing $\Gamma_i$ with $\Gamma_i(t)$ however this requires a time dependent Poisson bracket (and a procedure to handle when $\Gamma_i(t) = 0$). The vortices can be advected with the flow or held fixed. The blinking vortex model by Aref (H. Aref, Journal of Fluid Mechanics 143, 1 (1984)) with 2 fixed vortices that alternate in activity, each taking turns shifting between a fixed vorticity value and zero, became a paradigm for chaotic advection.

4 Inviscid Barotropic Compressible Fluids

4.1 Eulerian and Lagrangian coordinates

We refer to the background or Eulerian coordinates as $x$ and those moving with the fluid or material as $X$ and these are the Lagrangian coordinates. Quantities can be described as functions of $x$ and time $T$ or as functions of $X$ and $T$. We refer to $\partial_t$ as a partial derivative with respect to time with $x$ held fixed (the Eulerian derivative). We refer to $\partial_T$ as a partial derivative with respect to time with $X$ held fixed (the Lagrangian derivative). The deformation can be described in terms of the matrix $\frac{\partial x}{\partial X}$

$$\frac{\partial x^i}{\partial X^j} = F^i_j$$

with inverse

$$\frac{\partial X^i}{\partial x^j} = (F^{-1})^i_j$$

It will be useful later to have as short hand

$$F = Dx$$

and

$$\partial_{X^i} = \frac{\partial}{\partial X^i} = \frac{\partial x^j}{\partial X^i} \partial_{x^j} = F^j_i \partial_{x^j}$$

$$\partial_{x^i} = \frac{\partial}{\partial x^i} = \frac{\partial X^j}{\partial x^i} \partial_{X^j} = (F^{-1})^j_i \partial_{X^j}$$

(11)

The velocity of a material point

$$U(X,T) = \partial_T x(X,T)$$

and that expressed in terms of the background coordinates

$$u(x,T) = u(x(X,T)) = U(X,T)$$
Using the chain rule, the material time derivative

$$\partial_T = \partial_t + \partial \frac{\partial x^i}{\partial t} \frac{\partial}{\partial x^i}$$

$$= \partial_t + u \cdot \nabla_x$$

where $\nabla_x$ refers to derivatives with respect to $x$. We can compute the acceleration of a material point using this time derivative

$$\partial_T U = \frac{\partial u}{\partial t} + u \cdot \nabla_x u$$

We can describe a particle distribution (or density) as a function of either $x$ or $X$. Let us describe the density in material coordinates as $q(X, T)$ and that in the background as $\rho(x, T)$. These two densities are related.

$$\int_V q(X, T) d^3X = \int_V q(X(x, T)) \frac{d^3x}{J}$$

with the Jacobian

$$J = \det F$$

computed from the deformation matrix (equation 10; $\frac{\partial x^i}{\partial X^j} = F_{ij}$). We identify the Eulerian density

$$\rho(x, T) = \frac{q(X(x, T), T)}{J}$$

Usually we consider material distributions that are not time dependent so $q(X)$ and

$$\rho(x, T) = \frac{q(X(x, T))}{J(x)}$$

### 4.2 Inviscid Burger’s equation

Ignoring the density and energy density for the moment, let us look at a Lagrangian containing only a kinetic energy term

$$L[\partial_T x, x] = \int \frac{1}{2} (\partial_T x)^2 d^3X$$

To obtain a momentum

$$\frac{\delta L}{\delta (\partial_T x)} = \partial_T x = \pi = u$$

Using this momentum

$$H[x, \pi] = \int \frac{\pi^2}{2} d^3X$$
The equations of motion (and using a canonical Poisson bracket)
\[
\begin{align*}
\partial_T x &= \pi = u \\
\partial_T \pi &= 0
\end{align*}
\]

Putting these together
\[\partial_T u = 0\]
and in 1 dimension this is equivalent to
\[u_t + uu_x = 0\]
which is known as the inviscid Burgers equation.

### 4.3 Working with the Determinant of the deformation matrix

Below when considering a Hamiltonian or Lagrangian that contains terms dependent on density, we will need to do a functional derivative involving the Jacobian matrix. The determinant in three dimensions for a matrix \(B\) can be written
\[
\det B = \frac{1}{3!} \epsilon_{ijk} \epsilon_{abc} B^{ia} B^{jb} B^{kc}
\]

Previously we defined Jacobian matrix
\[
F^i_j = \frac{\partial x^i}{\partial X^j}
\]
and
\[
J = \det F = \frac{1}{3!} \epsilon_{ijk} \epsilon_{abc} F^i_a F^j_b F^k_c
\]

Using this form for the Jacobian \(J\) (and with some index rearrangement)
\[
\frac{\partial J}{\partial F^m_l} = \partial_{F^m_l} \frac{1}{3!} \epsilon_{ijk} \epsilon_{abc} F^i_a F^j_b F^k_c
\]
\[
= \frac{1}{3!} \left[ \epsilon_{ijk} \epsilon_{abc} F^i_a F^j_b \delta^k c_m + \epsilon_{ijk} \epsilon_{abc} F^i_a F^k_c \delta^j b_m + \epsilon_{ijk} \epsilon_{abc} F^j_b F^k_c \delta^i a_m \right]
\]
\[
= \frac{1}{2} F^i_a F^j_b \epsilon_{lij} \epsilon^{mab}
\]

It may be useful for us to define a matrix \(A\) that looks like this
\[
A^m_l = \frac{\partial J}{\partial F^m_l} = \frac{1}{2} F^i_a F^j_b \epsilon_{lij} \epsilon^{mab} \quad (12)
\]
If we multiply this by the matrix \( F \)

\[
A^m_l F_l^k = \frac{1}{2} F^i_d F^j_b F^l_k \epsilon_{lij} \epsilon^{mab}
\]

\[
= \frac{1}{2} \frac{\partial x^i}{\partial x^a} \frac{\partial x^j}{\partial x^b} \frac{\partial x^l}{\partial x^k} \epsilon_{lij} \epsilon^{mab}
\]

Each term involves an antisymmetric sum of 3 derivatives. If \( m \neq k \) then two of the derivatives are the same and two terms cancel giving zero. If \( m = k \) then two terms are the same and the entire thing is equal to the Jacobian matrix. We can write

\[
A^m_l F_l^k = J \delta^i_k
\]

This relation implies that \( A/J \) is the inverse of \( F \) and that means (see equation 11)

\[
\partial x^k = \frac{1}{J} A^i_k \partial X^i
\]

Using equation 12 we can show that

\[
\frac{\partial}{\partial X^m} A^m_j = \frac{1}{2} \frac{\partial}{\partial X^m} \left[ \frac{\partial x^i}{\partial X^a} \frac{\partial x^j}{\partial X^b} \epsilon_{lij} \epsilon^{mab} \right]
\]

\[
= \left[ \frac{\partial^2 x^i}{\partial X^a \partial X^m} \frac{\partial x^j}{\partial X^b} + \frac{\partial x^i}{\partial X^a} \frac{\partial^2 x^j}{\partial X^b \partial X^m} \right] \epsilon_{lij} \epsilon^{mab}
\]

\[
= \left[ \frac{\partial^2 x^i}{\partial X^a \partial X^m} \frac{\partial x^j}{\partial X^b} - \frac{\partial x^i}{\partial X^a} \frac{\partial^2 x^j}{\partial X^b \partial X^m} \right] \epsilon_{lij} \epsilon^{mab}
\]

\[
= 0
\]

The Jacobian \( J \) depends on the deformation \( F = Dx \) as \( J = \det F \). It might be useful to know that for matrix \( A \)

\[
\det(I + \epsilon A) \sim 1 + \epsilon \ \text{tr} A + O(\epsilon^2)...
\]

This can be shown also using the above index notation.

We take the time derivative of \( J \)

\[
\partial_T J = \frac{\partial J}{\partial F^i_j} \partial_T F^i_j
\]

\[
= A^i_j \partial_T X_j (\partial_T x^i)
\]

\[
= A^i_j \partial_T X_j (u^i)
\]

\[
= A^i_j F^k_j \partial_T x_k u^i
\]

\[
= J \delta^i_k \partial_T x_k u^i
\]

\[
= J \partial_T x_k u^i
\]

\[
= J \nabla_x \cdot u
\]
Using our relation between densities

\[ \rho J = q \]
\[ J \partial_T \rho + \rho \partial_T J = \partial_T q \]
\[ \partial_T \rho + \rho \nabla_x \cdot \mathbf{u} = \frac{\partial_T q}{J} \]  

(16)

We recover the mass continuity equation as long as we set \( \partial_T q = 0 \). As \( q(X, T) \) is mass distribution in material coordinates requiring it to be independent of \( T \) is probably reasonable.

4.3.1 Functional derivatives involving the Jacobian matrix

We can look at a functional

\[ F_1[Dx] = \int d^3 X \ J \]

\[ \frac{\delta F_1}{\delta x^i}(Y) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int d^3 X \ \frac{\partial J}{\partial F_m^i} \frac{\partial}{\partial X^m} \left[ x^i(X) + \epsilon \delta(X - Y) - x^i(X) \right] \]

\[ = \int d^3 X \ \frac{\partial J}{\partial F_m^i} \frac{\partial}{\partial X^m} \delta(X - Y) \]

\[ = -\int d^3 X \ \frac{\partial}{\partial X^m} \left( \frac{\partial J}{\partial F_m^i} \right) \delta(X - Y) \]

\[ = -\frac{\partial}{\partial X^m} A^m_i(Y) \]

A slightly more complicated functional

\[ F_2[Dx] = \int d^3 X \ f(J) \]

(17)

and following the above computation

\[ \frac{\delta F_2}{\delta x^i}(X) = -\frac{\partial}{\partial X^m} \left( f' A^m_i \right) (X) \]  

(18)

Now consider

\[ F_3[Dx] = \int d^3 X \ J^{-1} \]

and we follow the previous computation

\[ \frac{\delta F_3}{\delta x^i}(X) = \frac{\partial}{\partial X^m} \left( A^m_i \frac{1}{J^2} \right) (X) \]  

18
Now consider

\[ F_4(Dx) = \int d^3X \ f(J^{-1}) \]  

and we follow the previous computation again

\[ \frac{\delta F_4}{\delta x^i}(X) = \frac{\partial}{\partial X^m} \left( f'A_m^i \frac{1}{J^2} \right)(X) \]  

4.4 For Barotropic Compressible Fluids

With \( e \) internal energy, \( v = 1/\rho \) the specific volume, \( p \) the pressure, \( T \) the temperature, \( s \) the specific entropy

\[ de = -pdv + Tds \]

The enthalpy \( h = e + pv \) and

\[ dh = vdp + Tds \]

With some arithmetic

\[ d(\rho e) = hdp + \rho Tds \]

Derivatives keeping \( s \) constant are

\[ \partial_\rho e = \frac{p}{\rho^2} \]

\[ \partial_\rho(\rho e) = h \]

\[ \partial_\rho h = \frac{\partial_\rho p}{\rho} \]  

4.5 Lagrangian for Compressible Fluids

We consider a barotropic fluid with internal energy density \( e(\rho) \) a function of Eulerian density alone. With the wave equation the Lagrangian density depended on the time derivative of a field, and the gradient of the field. We expect the Lagrangian density here to depend on the local velocity, giving the kinetic term, and the energy density that depends on the deformation gradient. We integrate the Lagrangian density over the Lagrangian coordinates and our our field is \( x \). The kinetic energy density

\[ \mathcal{T}[x](X) = \frac{1}{2}q(\partial_T x)^2 = \frac{1}{2}qU^2 \]

The internal energy density

\[ \mathcal{U}[x](X) = qe(\rho) = qe(q/J) \]
The Jacobian depends on the derivative $\mathbf{D}x$ which has two indices with components $F^i_j = \frac{\partial x^i}{\partial X^j}$. The Lagrangian density is the difference between the kinetic energy density and the internal energy density

$$\mathcal{L}(\partial_T x, \mathbf{D}x, \mathbf{X}) = \frac{1}{2} q(\partial_T x)^2 - q e(q/J(Dx))$$

depending on the time and spatial derivatives of our field $x$ and our Lagrangian coordinates $\mathbf{X}$. The total Lagrangian is the integral of the Lagrangian density over the material coordinates

$$L[x] = \int \mathcal{L}(\partial_T x, \mathbf{D}x, \mathbf{X}) d^3X = \int \frac{1}{2} q(\partial_T x)^2 - q e(q/J) d^3X$$

(22)

Taking the functional derivative

$$\frac{\delta \mathcal{L}}{\delta (\partial_T x(\mathbf{X}))} = q \partial_T x = q(\mathbf{X}) U(\mathbf{X})$$

(23)

The above equation is a vector equation and is satisfied for each index. The above equation also gives us a momentum

$$\pi(\mathbf{X}) = q U$$

(24)

field that we can use to form a Hamiltonian. We can use Lagrange's equations to find the equations of motion or we can use the Hamiltonian and a canonical Poisson bracket that contains a delta function to generate the equations of motion. If we use Lagrange’s equations, then we need to include a term to them that depends on $\partial_{X^i} \frac{\delta \mathcal{L}}{\delta F^j}$ as the Lagrangian contains gradient terms (through $J$). Lagrange’s equations are

$$\partial_T (\partial_T x \mathcal{L}) + \partial_X (\partial_{\mathbf{D}x} \mathcal{L}) - \partial_X \mathcal{L} = 0$$

or

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial (\partial_T x^i)} + \frac{d}{dX^j} \frac{\partial \mathcal{L}}{\partial (\partial_{X^j} x^i)} - \frac{\partial \mathcal{L}}{\partial X^i} = 0$$

Further manipulation of this can be used to check that we get Euler’s equation. We will transfer this system into a Hamiltonian and check that we get Euler’s equation using that framework instead.

### 4.6 Hamiltonian for Compressible Barotropic Fluids

Our Hamiltonian density corresponding to the Lagrangian in equation 22 is

$$\mathcal{H}[\pi, x](\mathbf{X}) = \frac{\pi^2}{2q} + q e(q/J)$$
and the total Hamiltonian

\[ H[\pi, x] = \int d^3 X \left[ \frac{\pi^2}{2q} + qe(q/J) \right] \]  

(25)

To find the equations of motion we need to understand how to take the variational derivatives. The variational derivative with respect to \( \pi \) is straightforward giving

\[ \partial_x \pi = \frac{\delta H}{\delta \pi} = \frac{\pi}{q} = U \]

and this is our first equation of motion.

For the second equation of motion we need to take the functional derivative of \( H \) with respect to \( x_i \). Writing the potential energy density as

\[ qe(q/J) = U \left( \frac{\partial X_j}{x_i} \right) = U \left( \frac{\partial F_j}{x_i} \right) \]

where these are functions of all possible \( i, j \),

\[ \frac{\delta H}{\delta x_i}(X) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int d^3 Y \left[ U(\partial_{x_j}(x^i + \epsilon \delta(X - Y), ...)) - U(F) \right] \]

\[ = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int d^3 Y \sum_j (\partial_{x_j x^i} U(F)) \epsilon \partial_{x_j} \delta(X - Y) \]

\[ = - \int d^3 Y qe^i(q/J) \frac{q}{J^2} (\partial_{F_j} J) \partial_{x_j} \delta(X - Y) \]

\[ = \partial_{x_j} \left[ e'(\rho) \rho^2 (\partial_{F_j} J) \right] \]

\[ = \partial_{x_j} \left[ p A_i^j \right] \]

and the second to last step we used equation 21 and the last step we have used equation 12 (also see equation 20) for functionals involving \( J^{-1} \). The Lagrangian or material density \( q(X, T) \) does not depend on \( x \) so its spatial derivatives don’t contribute to the variational derivative of \( H \). We want the above expression in terms of \( \partial_x = \nabla_x \) instead of \( \partial_X = \nabla_X \).

\[ \frac{\delta H}{\delta x^i} = p \partial_{x^i} \left( A_j^i \right) + A_j^i \partial_{x^j p} \]

\[ = 0 + A_j^i F_j^k (\partial_{x^k p}) \]

\[ = J \delta^k_i (\partial_{x^k p}) \]

\[ = J \partial_{x^i} p \]

\[ \frac{\delta H}{\delta x} = J \nabla_p \]

(26)

where I have used equation 11 and equation 14 for the gradient and 15 for the zero divergence term. Hamilton’s equations will give

\[ \partial_T \pi = \partial_T (q U) = - \frac{\delta H}{\delta x} \]
Putting this together with equation 26
\[
q \partial_T u + u \partial_T q = -J \nabla_x p \\
\partial_T u = -\nabla_x p - u \partial_T q / \rho
\]

The last term on the right must be zero in order to obey mass conservation, see equation 16. If we ignore it, then we have recovered Euler’s equations.

4.7 Changing coordinates and constructing a new Poisson bracket

In a finite dimensional Hamiltonian system with coordinates \( \mathbf{x} = \mathbf{q}, \mathbf{p} \)

\[
\frac{dx^i}{dt} = J^{ij} \frac{\delta H}{\delta x^j}
\]

(here \( J \) is the symplectic matrix and nothing to do with the det \( \mathbf{F} \)). if we have another set of coordinates that are not necessarily canonical, \( \mathbf{z} \)

\[
\frac{dz^i}{dt} = \frac{\partial z^i}{\partial x^j} J^{jk} \frac{\delta H}{\delta x^k} = \frac{\partial z^i}{\partial x^j} J^{jk} \frac{\partial z^m}{\partial x^k} \frac{\delta H}{\delta x^m} = \{z^i, H\}
\]

So we can change the coordinates and write the Poisson bracket as

\[
\{f, g\} = \frac{\partial f}{\partial z^i} \frac{\partial z^i}{\partial x^j} J^{jk} \frac{\partial z^m}{\partial x^k} \frac{\partial g}{\partial z^m} = \frac{\partial f}{\partial z^i} \{z^i, z^m\} \frac{\partial g}{\partial z^m}
\]

Non-canonical coordinate transformations and the same Hamiltonian (but in new variables) can still give the correct equations of motion if the Poisson bracket is changed. If our original coordinates are canonical fields \( \phi, \pi \) and \( F, G \) are functionals and we want to write the Poisson bracket in terms of a new fields \( q(\phi, \pi), p(\phi, \pi) \) then the Poisson bracket

\[
\{F, G\} = \int dx \left( \frac{\delta F}{\delta \phi} \frac{\delta G}{\delta \pi} - \frac{\delta G}{\delta \phi} \frac{\delta F}{\delta \pi} \right) + \int dx \left[ \left( \frac{\delta F}{\delta q} \frac{\delta q}{\delta \phi} + \frac{\delta F}{\delta p} \frac{\delta q}{\delta \pi} \right) \left( \frac{\delta G}{\delta q} \frac{\delta q}{\delta \phi} + \frac{\delta G}{\delta p} \frac{\delta q}{\delta \pi} \right) + \left( \frac{\delta F}{\delta \pi} \frac{\delta p}{\delta \phi} + \frac{\delta F}{\delta p} \frac{\delta p}{\delta \pi} \right) \left( \frac{\delta G}{\delta \pi} \frac{\delta p}{\delta \phi} + \frac{\delta G}{\delta p} \frac{\delta p}{\delta \pi} \right) \right]
\]
As an example we will work with the barotropic hydrodynamical system and change to new fields, finding a new Hamiltonian with respect to these new fields and a new and more complex Poisson bracket that is computed using these new fields. For this system the original fields $x(X)$ and $\pi(X)$ are functions of Lagrangian variable $X$ and these are canonical variables. Suppose for this compressible fluid system we would rather not work with our deformation field $x$ and associated momentum $\pi = qU$ and instead would prefer to work with the variables $\rho(x, T)$, the Eulerian density, and the Eulerian momentum density $m(x) = \rho(x, T)u(x, T)$. In terms of our canonical variables and Lagrangian quantities

$$\rho = \frac{q}{J} \quad m = \rho u = \frac{q}{J} u = \frac{\pi}{J}$$ (27)

We choose density in Lagrangian coordinates $q = 1$ to simplify the spatial derivatives (they are bad enough to work with even with $q = 1$).

We will try to derive a new Poisson bracket integrated over Eulerian coordinate $x$ instead of integrated over Lagrangian coordinate $X$. We start with any functional

$$F[\rho, m] = \int d^3 x f(\rho(x), m(x))$$ (28)

Transferring the coordinates used to integrate to $X$

$$F[\rho, m] = \int d^3 X Jf(\rho(x(X)), m(\pi(X), x(X)))$$ (29)

Note the appearance of Jacobian $J$ inside the integral.

We will use equation 18 and 20 on our functional $F$ that depends on $J$ and $J^{-1} = \rho$ (setting $q = 1$) to compute $\frac{\delta F}{\delta x}$

$$\frac{\delta F}{\delta x^i} = \partial_{X^m} \left( \frac{\partial f}{\partial \rho} A_i^m \frac{1}{J} \right) + \partial_{X^m} \left( \frac{\partial f}{\partial m^k} A_i^m \pi^k \frac{1}{J} \right) - \partial_{X^m} (f A_i^m)$$

Equation 15 gives $\partial_{X^m} A_i^m = 0$ and we expand the last term

$$\frac{\delta F}{\delta x^i} = A_i^m \partial_{X^m} \left( \frac{\partial f}{\partial \rho} \right) + A_i^m \partial_{X^m} \left( \frac{\partial f}{\partial m^k} m^k \right) - A_i^m \frac{\partial f}{\partial \rho} \frac{\partial m^k}{\partial X^m} - A_i^m \frac{\partial f}{\partial m^k} \frac{\partial m^k}{\partial X^m}$$

$$= A_i^m \rho \partial_{X^m} \left( \frac{\partial f}{\partial \rho} \right) + A_i^m m^k \partial_{X^m} \left( \frac{\partial f}{\partial m^k} \right)$$

We can change from $\partial_{X}$ to $\partial_{x}$ using equation 14

$$\frac{\delta F}{\delta x^i} = J \left[ \rho \partial_{x^i} \left( \frac{\partial f}{\partial \rho} \right) + m^k \partial_{x^i} \left( \frac{\partial f}{\partial m^k} \right) \right]$$

23
Using equation 28 we can write $\frac{\partial f}{\partial \rho}$ as $\frac{\delta F}{\delta \rho}$. Writing the above equation in vector notation

$$\frac{\delta F}{\delta \mathbf{x}} = J \left[ \rho \nabla_{\mathbf{x}} \left( \frac{\delta F}{\delta \rho} \right) + \nabla_{\mathbf{x}} \left( \frac{\delta F}{\delta \mathbf{m}} \right) \cdot \mathbf{m} \right]$$

(30)

It is more straightforward to compute the other functional derivative that we need. Using equation 29

$$\frac{\delta F}{\delta \pi} = \frac{\delta F}{\delta \mathbf{m}}$$

(31)

Because $\pi(X)$ and $x(X)$ are canonical coordinates they satisfy a canonical Poisson bracket. For functionals $F, G$

$$\{F, G\} = \int d^3 X \left( \frac{\delta F}{\delta \mathbf{x}} \frac{\delta G}{\delta \pi} - \frac{\delta G}{\delta \mathbf{x}} \frac{\delta F}{\delta \pi} \right)$$

Inserting equation 30 and 31 into this for both $F$ and $G$ we find

$$\{F, G\} = \int d^3 x \left[ J \left[ \rho \nabla_{\mathbf{x}} \left( \frac{\delta F}{\delta \rho} \right) + \nabla_{\mathbf{x}} \left( \frac{\delta F}{\delta \mathbf{m}} \right) \cdot \mathbf{m} \right] \cdot \frac{\delta G}{\delta \mathbf{m}} 
- J \left[ \rho \nabla_{\mathbf{x}} \left( \frac{\delta G}{\delta \rho} \right) + \nabla_{\mathbf{x}} \left( \frac{\delta G}{\delta \mathbf{m}} \right) \cdot \mathbf{m} \right] \cdot \frac{\delta F}{\delta \mathbf{m}} \right]
= \int \left[ \rho \left( \nabla_{\mathbf{x}} \left( \frac{\delta F}{\delta \rho} \right) \cdot \frac{\delta G}{\delta \mathbf{m}} - \frac{\delta F}{\delta \mathbf{m}} \cdot \nabla_{\mathbf{x}} \left( \frac{\delta G}{\delta \rho} \right) \right) \right] 
+ \left[ \nabla_{\mathbf{x}} \left( \frac{\delta F}{\delta \mathbf{m}} \right) \cdot \mathbf{m} \right] \cdot \frac{\delta G}{\delta \mathbf{m}} - \frac{\delta F}{\delta \mathbf{m}} \cdot \nabla_{\mathbf{x}} \left( \frac{\delta G}{\delta \rho} \right) \right] \right] d^3 x$$

(32)

and the last step we have changed the integral to be in $\mathbf{x}$ rather than $X$. This is our new Poisson bracket!

Remaining is to rewrite our Hamiltonian (equation 25) in terms of our new variables. Setting $q = 1$

$$H[\mathbf{x}, \pi] = \int d^3 X \left[ \frac{\pi^2}{2} + e(1/J) \right]$$

$$= \int d^3 x J \left[ \frac{\pi^2}{2} + e(1/J) \right]$$

$$= \int d^3 x \left[ \frac{m^2}{2\rho} + \rho e(\rho) \right]$$

so our new Hamiltonian is

$$H[\rho, \mathbf{m}] = \int d^3 x \left[ \frac{m^2}{2\rho} + \rho e(\rho) \right]$$
4.8 Another way to write Inviscid irrotational fluid dynamics

Inviscid irrotational fluid dynamics can be described with a Hamiltonian

\[ H[\rho, \phi] = \int d^3x \left[ \frac{1}{2} \rho(x) \left( \nabla \phi(x) \right)^2 + e(\rho(x)) \right] \]

and a Poisson bracket

\[ \{\rho(x), \phi(y)\} = \delta^3(x - y) \]

(we note that these coordinates are canonical). Both fields \( \rho, \phi \) are functions of \( x \). In the Hamiltonian the function \( e(\rho) \) is an internal energy density. For barotropic flow

\[ \frac{\partial^2 e}{\partial \rho^2} = \frac{1}{\rho} \frac{\partial p}{\partial \rho} \quad \frac{\partial e}{\partial \rho} = p \rho^2 \]

Since we have two fields we get two equations of motion

\[ \frac{d\rho}{dt} = -\rho \nabla \cdot \mathbf{u} \quad (35) \]
\[ \frac{d\phi}{dt} = \frac{u^2}{2} + e(\rho) + \rho \frac{\partial e}{\partial \rho} = \frac{u^2}{2} + e(\rho) + \frac{p}{\rho} \quad (36) \]

Let us define a vector field that we call the velocity

\[ \mathbf{u} = \nabla \phi \]

so \( \phi \) is the velocity potential field and there can be no vorticity in this model as

\[ \mathbf{\omega} = \nabla \times \mathbf{u} = \nabla \times \nabla \phi = 0 \]

Inserting the expression for velocity into our equations of motion

\[ \frac{d\rho}{dt} = -\rho \nabla \cdot \mathbf{u} \]
\[ \frac{d\phi}{dt} = \frac{u^2}{2} + e(\rho) + \rho \frac{\partial e}{\partial \rho} = \frac{u^2}{2} + e(\rho) + \frac{p}{\rho} \]

Taking the gradient of the second equation

\[ \frac{d}{dt} \nabla \phi = - (\mathbf{u} \cdot \nabla) \mathbf{u} - \frac{\partial e}{\partial \rho} \nabla \rho + \frac{p}{\rho^2} \nabla \rho - \frac{\nabla p}{\rho} \]
\[ \frac{d\mathbf{u}}{dt} = - (\mathbf{u} \cdot \nabla) \mathbf{u} - \frac{p}{\rho^2} \nabla \rho + \frac{p}{\rho^2} \nabla \rho - \frac{\nabla p}{\rho} \]

\[ = - (\mathbf{u} \cdot \nabla) \mathbf{u} - \frac{\nabla p}{\rho} \]

This is Euler’s equation.

Using the velocity, the first equation of motion (equation 33 or equation 35) is equivalent to the mass continuity equation if we let \( \rho \) be a Lagrangian variable. Why is it that \( \rho \) is a Lagrangian variable and not \( \phi \)?
5 Poisson manifolds

For infinite dimensional systems instead of equations of motions for a single particle that is two differential equations, we would expect partial differential equations, like the Euler equation for fluid mechanics or the KdV equation.

A choice of Hamiltonian and Poisson bracket is not necessarily unique given equations of motion (in this case partial differential equations). When we work with fields, we can relax the requirement that we have a canonical set of coordinates in which to define the Poisson bracket. For finite dimensional systems we often had a canonical basis in which the Poisson brackets of the elements gave $\pm 1$ or 0. Here we can assume instead that we know the Poisson brackets in a particular coordinate basis (but that this coordinate basis is not canonical). The system is symplectic if we ensure that the Poisson brackets satisfy the properties of Poisson brackets.

For example we can write Poisson brackets for two functions that depend on coordinates $x$

$$\{A(x), B(x)\} = \frac{\partial A(x)}{\partial q^i} \frac{\partial B(x)}{\partial p^i} - \frac{\partial A(x)}{\partial p^i} \frac{\partial B(x)}{\partial q^i}$$

$$= \frac{\partial A}{\partial x^j} \frac{\partial x^j}{\partial q^i} \frac{\partial x^k}{\partial p^i} - \frac{\partial A}{\partial x^j} \frac{\partial x^j}{\partial p^i} \frac{\partial x^k}{\partial q^i}$$

$$= \frac{\partial A}{\partial x^j} \left( \frac{\partial x^j}{\partial q^i} \frac{\partial x^k}{\partial p^i} - \frac{\partial x^j}{\partial p^i} \frac{\partial x^k}{\partial q^i} \right)$$

$$= \partial_j A\{x^j, x^k\} \partial_k B$$

where we have assumed in the middle that we know how to compute Poisson brackets for the coordinates $x$ (that are not necessarily canonical) and we have used summation notation and the notation

$$\partial_i \equiv \frac{\partial}{\partial x^i}$$

This we write again

$$\{A(x), B(x)\} = \partial_i A\{x^i, x^j\} \partial_j B$$

(37)

Let us check that the Poisson bracket is consistent for the wave equation which has canonical coordinates (and $\{x^i, x^j\} = J^{ij}$). Using equation 37 but with both functions
inside the Poisson bracket a function of $\phi, \pi$

$$\{\pi(y), H[\pi, \phi]\} = \frac{\partial \pi(y)}{\partial x^i} \{x^i, x^j\} \frac{\delta H}{\delta x^j}$$

$$= \int dx dz \frac{\delta \pi(y)}{\delta \pi(x)} \{\pi(x), \phi(z)\} \frac{\delta H}{\delta \phi(z)}$$

$$= \int dx dz \delta(y - x) \{\pi(x), \phi(z)\} \frac{\delta H}{\delta \phi(z)}$$

$$= \int dz \{\pi(y), \phi(z)\} \frac{\delta H}{\delta \phi(z)}$$

$$= \int dz \delta(y - z) \frac{\delta H}{\delta \phi(z)}$$

For the wave equation

$$\dot{\phi}(y) = \{\phi(y), H[\pi, \phi]\} = \int \delta(y - x) \frac{\delta H}{\delta \pi(x)} dx$$

$$\dot{\pi}(y) = \{\pi(y), H[\pi, \phi]\} = -\int \delta(y - x) \frac{\delta H}{\delta \phi(x)} dx$$

The equations of motion for the wave equation are consistent with the Hamiltonian

$$H[\pi, \phi] = \int \left[ \frac{1}{2} \pi(x)^2 + V(\phi(x)) \right] dx$$

and the Poisson bracket shown in equation 7 that has $\{x^i, x^j\} = J^{ij}$.

In a more general case we might not be able to find coordinates that are everywhere canonical, but we might still have a way of computing something like a Poisson bracket. In this case we consider a Poisson structure on a smooth manifold $M$. This structure is in the algebra of smooth functions on the manifold. For smooth functions $f, g, h$ the Poisson bracket satisfies

1. Skew symmetry

$$\{f, g\} = -\{g, f\}$$

2. Leibnitz rule

$$\{fg, h\} = f\{g, h\} + g\{f, h\}$$

3. Jacobi identity

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$$

For canonical coordinates $\mathbf{x} = (\mathbf{q}, \mathbf{p})$ we can write the Poisson bracket as

$$\{f, g\} = \frac{\partial f}{\partial x_\mu} J^{\mu \nu} \frac{\partial g}{\partial x_\nu} \quad (38)$$

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Now consider the possibility that $J^{ij}$ is replaced by something that depends on the coordinates or
\[ \{f, g\} = \frac{\partial f}{\partial x_\mu} F^{\mu\nu} \frac{\partial g}{\partial x_\nu} \]
We require $F^{\mu\nu} = -F^{\nu\mu}$ so that that is antisymmetric. The Leibnitz rule is satisfied by any Poisson bracket in this form. Inserting the bracket into the Jacobi identity we find that the coefficients must satisfy
\[ F^{ijm} \frac{\partial J^{jk}}{\partial x^m} + F^{km} \frac{\partial J^{ij}}{\partial x^m} + F^{jm} \frac{\partial J^{ki}}{\partial x^m} = 0 \]

5.1 Example: Inviscid Burgers’ equation

Using the inviscid Burgers’ equation to explore an infinite dimensional (continuous) Hamiltonian system with a non-trivial Poisson structure. This system has a Poisson structure that is also used with the KdV equation.

A pressure-less fluid with constant density
\[ \frac{\partial u}{\partial t} + u \cdot \nabla u = 0 \]
In one dimension
\[ u_t + uu_x = 0 \]
also known as the Inviscid Burger’s equation. We can find the equations of motion using a Hamiltonian
\[ H(u) = -\int \frac{1}{3!} u(x)^3 \, dx \]  
(39)
that is only a function of one field and a Poisson bracket that is
\[ \{u(x), u(y)\} = \frac{\partial}{\partial x} \delta(x - y) \]  
(40)

This is antisymmetric, but how do we use it? For two functions of $u$
\[ \{f(u), g(u)\} = \int dx \, dy \, \frac{\delta f}{\delta u(x)} \{u(x), u(y)\} \frac{\delta g}{\delta u(y)} \]
\[ = \int dx \, dy \, \frac{\delta f}{\delta u(x)} \left( \frac{\partial}{\partial x} \delta(x - y) \right) \frac{\delta g}{\delta u(y)} \]
\[ = \int dx \, \frac{\delta f}{\delta u(x)} \frac{\partial}{\partial x} \int dy \delta(x - y) \frac{\delta g}{\delta u(y)} \]
\[ = \int dx \, \frac{\delta f}{\delta u(x)} \left( \frac{\partial}{\partial x} \frac{\delta g}{\delta u(x)} \right) \]
With suitably well behaved boundaries we could have integrated by parts in $x$ giving
\[
\{f(u), g(u)\} = -\int dy \left( \frac{\partial}{\partial y} \frac{\delta f}{\delta u(y)} \right) \frac{\delta g}{\delta u(y)}
\]

Adding these two we can construct an expression that makes the asymmetry of the bracket clearer
\[
\{f(u), g(u)\} = \int \frac{dx}{2} \left[ \frac{\delta f}{\delta u(x)} \left( \frac{\partial}{\partial x} \frac{\delta g}{\delta u(x)} \right) - \left( \frac{\partial}{\partial x} \frac{\delta f}{\delta u(x)} \right) \frac{\delta g}{\delta u(x)} \right]
\]

We can associate $F^{\mu\nu}$ from equation 38 with $\frac{\partial}{\partial x} \delta(x - y)$ from equation 40.

Let’s use the Poisson bracket in equation 40 and the Hamiltonian in equation 39 to compute the equations of motion using
\[
\frac{du(x)}{dt} = \{u(x), H\}
\]

We compute the functional derivative using the Hamiltonian in equation 39
\[
\frac{\delta H[u]}{\delta u(x)} = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int dy \left[ H(u(y) + \epsilon \delta(x - y)) - H(y) \right]
\]
\[
= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int dy \frac{1}{3!} \left[ (u(y) + \epsilon \delta(y - x))^3 - u(y)^3 \right]
\]
\[
= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int dy \frac{1}{3!} 3\epsilon u(y)^2 \delta(y - x)
\]
\[
= -\frac{u(x)^2}{2}
\]

The equations of motion
\[
\{u(x), H\} = \int dy \int dz \frac{\delta u(x)}{\delta u(y)} \left( \frac{\partial}{\partial y} \delta(y - z) \right) \frac{\delta H}{\delta u(z)}
\]
\[
= -\int dy \delta(x - y) \frac{\partial}{\partial y} \frac{u(y)^2}{2}
\]
\[
= -\frac{\partial}{\partial x} \frac{u(x)^2}{2} = -uu_x
\]

And this is consistent with our equation of motion for the inviscid Burger’s equation.
\[
u_{x} = -uu_x
\]

Burger’s equation has one field $u$ but previously we derived it using two fields. I think that the Hamiltonian Poisson manifolds with one field cannot be derived from a Lagrangian. Perhaps there is a way to derive single field Poisson brackets with a projection to a submanifold.
### 5.1.1 The symplectic two form

Our Poisson bracket (equation 40) has a kind of inverse. If you integrate over a delta function you would get a step function so we expect the inverse would involve integrating with a step function.

It is possible to find a symplectic structure consistent with the Poisson bracket of equation 40. The symplectic two form takes two vectors and returns a real number. Here we have infinite dimensional vectors which we can describe with \( u(x) \) with all \( x \) values. So our symplectic structure should take two different \( u \) functions and return a single number. A function that does this is

\[
\Omega(u_1, u_2) = \frac{1}{2} \int_{-\infty}^{\infty} [\hat{u}_1(x)u_2(x) - \hat{u}_2(x)u_1(x)]dx
\]

and where

\[
\hat{u}(x) \equiv \int_{-\infty}^{x} dy \ u(y)
\]

If

\[
u_1(x) = \partial v/\partial x
\]

for some \( v(x) \) then

\[
\hat{u}_1 = \int_{-\infty}^{x} dy \frac{\partial v}{\partial y} = v(x)
\]

and

\[
\Omega(u_1, u_2) = \frac{1}{2} \int_{-\infty}^{\infty} \left[ v(x)u_2(x) - \left( \int_{-\infty}^{x} dy \ u_2(y) \right) \frac{\partial v}{\partial x} \right] dx
\]

\[
= \frac{1}{2} \int_{-\infty}^{\infty} dx \left[ v(x)u_2(x) + u_2(x)v(x) \right] - \left( \int_{-\infty}^{x} u_2(y)dy \right) v(x) \bigg|_{-\infty}^{\infty}
\]

\[
= \int v(x)u_2(x)dx
\]

(where we have integrated by parts) and have assumed that \( u_2, v \) are well behaved (zero) at the boundaries.

Let us use \( v(x) = \frac{\delta H}{\delta u(x)} \) giving \( u_1(x) = \frac{\partial}{\partial x} \frac{\delta H}{\delta u(x)} \) and so equation 42 gives

\[
\Omega \left( \frac{\partial}{\partial x} \frac{\delta H}{\delta u(x)}, u_2 \right) = \int dx \ \frac{\delta H}{\delta u(x)} u_2(x)
\]

The symplectic two-form can be used to generate Hamiltonian flows from a Hamiltonian function \( H \). Previously we did this by finding a vector \( V \) such that the two form \( \omega(V, ?) = \)
\(dH\). The vector \(V\) gives \(\dot{x}\) and generates the flow. Previously \(dH = \partial_x \delta u(x) dx\). Now our variables are \(u(x)\) so by analogy we use the functional derivative

\[
dH = \frac{\delta H[u]}{\delta u(x)} du(x)
\]
and we are implicitly taking into account all \(u(x)\). The vector field \(V = X_H(u)\) giving \(\dot{u}\).

\[
\Omega(X_H(u), u_2) = \int dx \frac{\delta H[u]}{\delta u(x)} u_2(x)
\]
But comparing this we equation 43 implies that \(X_H(u) = \frac{\partial}{\partial x} \frac{\delta H}{\delta u(x)}\) and

\[
\frac{du}{dt} = \frac{\partial}{\partial x} \frac{\delta H}{\delta u(x)}
\]
and this is consistent with what we found using the Poisson bracket of equation 40.

### 5.2 Conserved Quantities and Continuity Equations

An integrated quantity

\[
Q(u, t) = \int f(u(x, t)) dx
\]
that is conserved would satisfy

\[
\frac{dQ}{dt} = \frac{d}{dt} \int f(u(x, t)) dx = 0
\]
Here I have written the local quantity \(f(u)\) depending on a field \(u(x, t)\) that is dependent on \(x, t\).

An example of a continuity equation is mass continuity in one dimension

\[
\partial_t \rho + \partial_x (\rho v) = 0
\]
with \(\rho\) the local mass density. This equation implies that the total mass \(\int dx \rho\) is conserved. Here \(\rho v\) is the mass flux and \(v\) can be any velocity field that moves mass around.

If a function locally (but everywhere) satisfies a continuity equation then its integral is a conserved quantity. For example, suppose that

\[
\partial_t (f(u(x, t))) + \partial_x (g(u(x, t))) = 0
\]
for some functions \(f(u)\) and \(g(u)\). The above relation is called a continuity equation or in conservation law form. We can integrate the continuity equation

\[
\int_{-\infty}^{\infty} dx \partial_t (f(u(x, t))) = - \int_{-\infty}^{\infty} dx \partial_x (g(u(x, t)))
\]
\[
\frac{d}{dt} \int dx f(u(x, t)) = - g(u(x, t))|_{-\infty}^{\infty}
\]
\[
\frac{dQ}{dt} = 0
\]
We have assumed that \( g \) is zero on the boundary. We say that \( g(u(x,t)) \) is the flux or current of \( f(u(x,t)) \).

6 The KdV equation

Another interesting non-linear infinite dimension Hamiltonian system is based on the KdV equation. To put the KdV equation into Hamiltonian form is more challenging because we don’t obviously have conjugate momenta. The KdV differential equation

\[
u_t = uu_x + u_{xxx}
\]

or

\[
\frac{\partial u}{\partial t} = u \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3}
\]

that can also be written in conservation law form

\[
u_t = \partial_x (u^2/2 + u_{xx})
\]

The KdV equation can be derived from a Hamiltonian

\[
H[u] = \int dx \left( \frac{u^3}{3!} - \frac{1}{2}(u_x)^2 \right)
\]

Using our definition of a functional derivative

\[
\frac{\delta H}{\delta u(x)} = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int dy \left[ \frac{(u(y) + \epsilon \delta(y - x))^3}{3!} - \frac{1}{2} [\partial_y (u(y) + \epsilon \delta(y - x))]^2 \right]
\]

\[
= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int dy \left[ \frac{\epsilon u(y)^2 \delta(y - x) - \epsilon \partial_y u(y) \partial_y \delta(y - x)}{2} \right]
\]

\[
= \frac{u^2(x)}{2} + \frac{\partial^2 u(x)}{\partial x^2}
\]

where the last step is done by integrating by parts. We see that this is contained within the derivative inside the equation of motion in conservation law from (equation 44) The equations of motion can be written as

\[
u_t = \partial_x \frac{\delta H}{\delta u(x)}
\]

What type of Poisson bracket gives this equation of motion? We desire

\[
\{ u(x), H[u] \} = \frac{\partial}{\partial x} \frac{\delta H}{\delta u(x)}
\]

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Using equation 37

\[ \{u(x), H[u]\} = \int dydz \frac{\delta u(x)}{\delta u(z)} \{u(z), u(y)\} \frac{\delta H}{\delta u(y)} \]

\[ = \int dy \{u(x), u(y)\} \frac{\delta H}{\delta u(y)} \]

If we use a Poisson bracket

\[ \{u(x), u(y)\} = \frac{\partial}{\partial x} \delta(x - y) \] (46)

then

\[ \{u(x), H[u]\} = \int dy \frac{\partial}{\partial x} \delta(x - y) \frac{\delta H}{\delta u(y)} \]

\[ = \frac{\partial}{\partial x} \int dy \delta(x - y) \frac{\delta H}{\delta u(y)} \]

\[ = \frac{\partial \delta H}{\partial x \delta u(x)} \]

as desired. Thus the KdV equation arises from the Hamiltonian in equation 45 with Poisson bracket in equation 46.

**Remark** It is possible to check that this Poisson bracket obeys the Jacobi identity and so is well behaved. Instead of coordinates and momenta here we only have \(u\). This is why the Poisson bracket has a derivative of a delta function rather than just a delta function for \(p, q\) analogs. How is possible that we can use one function for both canonical coordinates?

### 6.1 Rescaling

We show here that it is not important what coefficients are in front of each term. If there are coefficients in front the terms in the differential equation, it is possible to rescale \(u, x, t\) so that it resembles the form we have used above. For

\[ u_t = au u_x + bu_{xxx} \]

we can rescale with

\[ x \rightarrow b^{1/3} x \quad u \rightarrow a^{-1} b^{1/3} u \]

to get the form used above for the KdV equation.
6.2 Soliton Solutions

Soliton solutions can be found by searching for a solution in the form of a traveling wave \( u(x,t) = f(x + vt) \). This has

\[
\begin{align*}
  u_t &= vf' \\
  u_x &= f' \\
  u_{xxx} &= f'''
\end{align*}
\]

If we insert this into the KdV equation

\[
v f' = ff' + f'''
\]

Treating these as \( x \) derivatives only (like setting time to zero)

\[
v f_x = \frac{d}{dx}(f^2/2 + f_{xx})
\]

Integrating

\[
d^2f + \frac{1}{2}f^2 - vf = \text{constant}
\]

Requiring a zero on the infinite boundaries, implies that the constant is zero. Multiplying by \( f_x \)

\[
f_x \frac{d^2f}{dx^2} + \frac{1}{2}f_xf^2 - vf_xf = 0
\]

\[
\frac{1}{2} \frac{d}{dx}(f_x)^2 + \frac{1}{6}(f^3)_x - \frac{v}{2}(f^2)_x = 0
\]

Integrating and again setting a constant to zero

\[
\frac{1}{2}(f_x)^2 + \frac{1}{6}f^3 - \frac{v}{2}f^2 = 0
\]

Looking for a solution with \( f_x = 0 \) at the origin,

\[
u(x) = 3v \sech^2 \left( \frac{\sqrt{v}x}{2} \right)
\]

and restoring time

\[
u(x,t) = 3v \sech^2 \left( \frac{\sqrt{v}(x + vt)}{2} \right)
\]

This is a traveling wave with amplitude or height directly proportional to its travel speed. The solution is non-dispersive; the wave maintains its shape as a function of time and only travels to the left. Higher amplitude solitons travel faster than (and catch up to) lower amplitude solitons.

Solitons are a characteristic of integrable systems. Two solitons can pass through one another, but delayed in time by the interaction. By considering asymptotic limits and using inverse scattering theory, it might be possible to compute the delay?
6.3 Infinite number of conserved quantities

We start with a transformation

\[ u = v + i\epsilon \partial_x v + \frac{\epsilon^2}{6} v^2 \]  

(47)

This transformation takes a function \( v \), generates a function \( u \), and the transformation depends on the constant \( \epsilon \). We compute some derivatives

\[
\begin{align*}
    u_t &= v_t + i\epsilon v_{xt} + \frac{\epsilon^2}{3} v v_t \\
    u_x &= v_x + i\epsilon v_{xx} + \frac{\epsilon^2}{3} v v_x \\
    u_{xxx} &= v_{xxx} + i\epsilon v_{xxxx} + \epsilon^2 \left( v_x v_{xx} + \frac{v v_{xxx}}{3} \right)
\end{align*}
\]

Putting these expressions into the KdV equation

\[ u_t = uu_x + u_{xxx} \]

and rearranging terms

\[
\left( 1 - i\epsilon \partial_x - \frac{\epsilon^2 v}{3} \right) \left( v_t - vv_x - \frac{\epsilon^2}{6} v^2 v_x - v_{xxx} \right) = 0
\]

If \( v \) satisfies a modified equation

\[ v_t = vv_x + \frac{\epsilon^2}{6} v^2 v_x + v_{xxx} \]  

(48)

then using the transformation 47 we obtain a \( u \) that is a solution to the KdV equation. In the limit of \( \epsilon \to 0 \) the modified equation is the KdV equation. In the limit of \( \epsilon \to \infty \) and after a rescaling \( \epsilon v/\sqrt{6} \to v \) we obtain what is known as the modified KdV or MKdV equation

\[ v_t = v^2 v_x + v_{xxx} \]

A transformation that takes a solution of the MKdV to a solution of the KdV is known as a Miura transformation (and there are various different forms of it depending on the signs and coefficients of the terms in the KdV equation).

We can write our modified equation (equation 48) as a continuity equation

\[ v_t = \partial_x \left( \frac{v^2}{2} + \frac{\epsilon v^3}{18} + v_{xx} \right) \]

If \( v \) satisfies our modified equation then

\[
\int_{-\infty}^{\infty} dx \ v(x)
\]
is a conserved quantity.

Now write

\[ v = \sum_{n=0}^{\infty} \epsilon^n v_n \]

and put this into equation 47

\[ u = \sum_{n=0}^{\infty} (\epsilon^n v_n + i\epsilon^{n+1} \partial_x v_n) + \frac{\epsilon^2}{6} \sum_{n} \sum_{m} \epsilon^{n+m} v_n v_m \]

\[ = \sum_{n=0}^{\infty} \epsilon^n \left( v_n + i\partial_x v_{n-1} + \frac{1}{6} \sum_{m=0}^{n-2} v_m v_{n-2-m} \right) \]

with the convention that \( v_i \) is zero for \( i < 0 \). This is a way of inverting \( v \rightarrow u \). If \( u \) satisfies the KdV equation, then \( v \) satisfies a modified equation and this is true for all possible \( \epsilon \). This means that each \( v_n \) should satisfy the modified equation also. Since each \( v_n \) satisfy an equation in conservation law form, each one should be a conserved quantity.

To the first few orders of \( \epsilon \)

\[
\begin{align*}
    u &= v_0 \\
    0 &= v_1 + i\partial_x v_0 \\
    0 &= v_2 + i\partial_x v_1 + \frac{1}{6} (v_0)^2 \\
    0 &= v_3 + i\partial_x v_2 + \frac{1}{3} v_0 v_1 \\
    0 &= v_4 + i\partial_x v_3 + \frac{1}{6} v_1^2 + \frac{1}{3} v_0 v_2 \\
    0 &= \ldots
\end{align*}
\]

This is a recursive relation! At each step we see \( v_i \) in terms of \( v_j \) with \( j < i \). To each order in \( \epsilon \)

\[ v_n = -i\partial_x v_{n-1} - \frac{1}{6} \sum_{m=0}^{n-2} v_m v_{n-2-m} \]
In the other direction

\[ v_0 = u \]
\[ v_1 = -i \partial_x v_0 = -i \partial_x u \]
\[ v_2 = -i \partial_x v_1 - \frac{1}{6} (v_0)^2 \]
\[ = -u_{xx} - \frac{1}{6} u^2 \]
\[ v_3 = -i \partial_x v_2 - \frac{1}{6} (v_1)^2 - \frac{1}{3} v_0 v_1 \]
\[ = i u_{xxx} + \frac{2i}{3} u u_x = i \partial_x \left( u_{xx} + \frac{u^2}{3} \right) \]
\[ v_4 = -i \partial_x v_3 - \frac{1}{6} v_1^2 - \frac{1}{3} v_0 v_2 \]
\[ = \frac{1}{3} \left( \frac{u^3}{6} - \frac{u_x^2}{2} \right) + \partial_{xx} \left( \frac{u^2}{2} + u_{xx} \right) \]

If we assert that \( u \) is a solution of the KdV equation, then each \( v_n \) must individually must satisfy a continuity equation and each \( v_n \) must be conserved quantity. The iterative nature of the series implies that there are an infinite number of them. The odd ones give rise to imaginary terms that are also derivatives. If they are total derivatives then then integrating by parts with the continuity equation gives a trivial relation. The even quantities are not derivatives so are non-trivial conserved quantities.

Conserved quantities are not necessarily in involution. However if we use

\[ H_n = 3(-1)^n \int_{-\infty}^{\infty} dx \ v_{2n} \]

it is possible to show that

\[ \left( D_x^2 + \frac{1}{3} (D_x u(x) + u(x) D_x) \right) \frac{\delta H_m}{\delta u(x)} = D_x \frac{\delta H_{m+1}}{\delta u(x)} \]  

(49)

and this can be used to show that

\[ \{ H_n, H_m \} = 0 \]

Each of the conserved quantities can be thought of as its own Hamiltonian, each giving rise to its own evolution equation, and these are known as higher order equations of the KdV hierarchy.

As we will see below, the operators on either side of equation 49 are two different Poisson brackets. Thus the Lax equation is related to having a dual Poisson structure.
6.3.1 Integrability

One way to define an infinite dimensional system as integrable is to require that there are an infinite number of conserved quantities that are in involution.

6.4 Lax pair

A Lax pair is a pair of time dependent differential operators $L, B$ such that

$$
\frac{dL}{dt} = [B, L] \tag{50}
$$

Here the commutator $[L, B] = LB - BL$ is not to be confused with the Poisson bracket.

If the Lax relation is satisfied then eigenvalues of $L$ are not time dependent. Suppose $L\phi = \lambda \phi$ with $\lambda$ an eigenvalue and $\phi, t = B\phi$ gives a time derivative of $\phi$. We can show this

$$
\partial_t(L\phi) = \partial_t(\lambda\phi) = \lambda_t\phi + \lambda\phi, t \\
= \lambda_t\phi + \lambda B\phi \\
= \lambda_t\phi + BL\phi
$$

and

$$
\partial_t(L\phi) = (\partial_t L)\phi + L\phi, t \\
= (\partial_t L)\phi + LB\phi
$$

Putting these together gives

$$
(\partial_t L)\phi = [B, L]\phi + \lambda_t\phi
$$

If the Lax equation (equation 54) is satisfied then $\lambda_t = \frac{d\lambda}{dt} = 0$ and the eigenvalues of $\phi$ are time independent. It should be noticed that the time derivative of $L$ refers to the derivative of the operator before it is applied to a function.

The Lax relation implies that $-iB$ acts like a Hamiltonian function for a Shrödinger equation. In analogy

$$
L(0) = U^\dagger(t)L(t)U(t) \tag{51}
$$

with

$$
\frac{dU(t)}{dt} = B(t)U(t) = e^{-iHt}
$$

Taking the time derivative of equation 51

$$
0 = iH^\dagger LU + U^\dagger\partial_t LU - U^\dagger L\partial_t HU \\
0 = U^\dagger (iHL + \partial_t L - iLH)U \\
0 = \partial_t L + [iH, L]
$$
And the eigenvalues (and spectrum) of $L$ are independent of time ($L$ is isospectral as time varies). As a consequence the solution of the equation

$$L\psi = \lambda\psi$$

can be solved at time zero and then propagated to another time with the operator $B$

$$\frac{\partial\psi}{\partial t} = B\psi$$

(again $-iB$ is acting like a Hamiltonian function).

It was noted by Peter Lax that using the differential operators

$$L = D_{xx} + \frac{1}{6}u$$

$$B = 4D_{xxx} + \frac{1}{2}(D_xu + uD_x)$$

the equation (known as the Lax equation) satisfy the following relation

$$\frac{dL}{dt} + [L, B] = 0$$

and that this equation is equivalent to the KdV equation.

Let us check that this is true.

$$L_t = \frac{1}{6}u_t$$

$$LB = 4D_x^5 + \frac{5}{3}uD_x^3 + \frac{5}{2}u_xD_x^2 + \frac{u^2}{6}D_x + 2u_{xx}D_x + \frac{1}{12}uu_x + \frac{1}{2}u_{xxx}$$

$$BL = 4D_x^5 + \frac{5}{3}uD_x^3 + \frac{5}{2}u_xD_x^2 + \frac{u^2}{6}D_x + 2u_{xx}D_x + \frac{1}{4}uu_x + \frac{2}{3}u_{xxx}$$

Hence

$$L_t + [L, B] = \frac{1}{6}(u_t - uu_x - u_{xxx})$$

Terms without $D_x$ or its powers are considered operators using an identity operator.

Here the operators $L, B$ can be written as sums of powers of the operator $D_x = \partial_x$. Lax noted that an equation that can be cast into the framework of a Lax equation with two operators $L, B$ can display many of the characteristics of the KdV equation including an infinite number of conserved quantities.

The operator $L$ to any power, $L^n$, gives conserved quantities (and if $L$ is a matrix take the trace).

A Lax pair is not unique, as new pairs can be generated from a single pair. If $L, B$ are a Lax pair and $g$ an invertible transformation then

$$L' = gLg^{-1} \quad B' = gBg^{-1} + \dot{g}g^{-1}$$

also defines a Lax pair. $L$ to any power gives a Lax pair with $B$. Any operator $\tilde{B}$ with $[L, B - \tilde{B}] = 0$ gives a Lax pair with $L$.

Here $g$ an operator, can it be time dependent? Check this xxxx
6.5 Connection to Schrödinger equation

Consider a time independent Schrödinger equation

\[ D_{xx}\psi + \left( \frac{1}{6} u(x) + \lambda \right) \psi = 0 \]  

(56)

for \( u(x, t) \) is a solution to the KdV equation and the variable \( t \) changes the equation (so we are not solving a time dependent thing, just using \( u \) to set a potential for the equation). This expression should look like it contains the operator \( L \) in it (see equation 53). We can invert the equation for \( u \) giving

\[ u(x, t) = -6 \left( \lambda + \frac{\psi_{xx}}{\psi} \right) \]  

(57)

Taking spatial and time derivatives of this relation we find that \( u \) satisfies the KdV equation if the eigenvalue \( \lambda \) is independent of time.

If \( u \) satisfies the KdV equation then eigenvalues of the associated Schrödinger equation are time independent.

6.6 Connection to Inverse Scattering method

Given \( u(x, t = 0) \), compute the spectrum of \( L(t = 0) \). Choose an eigenvalue and compute the associated wavefunction \( \psi(x, t = 0) \). Then predict the wavefunction at all later times using the evolution operator \( B \) and using equation 52. From this compute \( L(t) \) and \( u(x, t) \) at later times. It may be useful to consider asymptotic limits of \( u \) and the wavefunction. If the asymptotic limits at \( x \to \pm\infty \) are wave-like then the problem can be described in terms of reflection and transmission coefficients. If the asymptotic limits are exponentially dropping then the scattering problem is discussed in terms of bound states. This sounds simple and can be done for any initial \( u \) so is a general technique.

As an example let's look at the soliton solution for the KdV equation.

\[ u(x, t = 0) = 12 \text{ sech}^2 x \]

The associated Schrödinger equation (plugging \( u \) into equation 56)

\[ \psi_{xx} + 2\text{sech}^2 x \psi = -\lambda \psi \]

A solution exists with \( \lambda = -1 \) and

\[ \psi(x) = \frac{1}{2} \text{sech} x \]
Let us make sure that this is a solution to the associated Shrödinger equation.

\[
\begin{align*}
\psi_x &= \frac{1}{2} \sinh x \cosh x \\
\psi_{xx} &= -\frac{1}{2} \sech x \left(2 \sech^2 x - 1\right) \\
\psi_{xxx} &= -\frac{1}{2} \left[\frac{5 \sinh x \cosh x}{\cosh^2 x} - \frac{6 \sinh^3 x}{\cosh^3 x}\right]
\end{align*}
\]

\[
\psi_{xx} + 2 \sech^2 x \psi = -\frac{1}{2} \sech x (2 \sech^2 x - 1) + \sech^3 x = \psi
\]

and so \( \psi \) is an eigenfunction with \( \lambda = -1 \). To take into account time evolution asymptotic behavior must be taken into account. It is not as simple as just applying \( B \) to the wave function. This is the soliton solution so the eigenvalue is essential the wave speed. More generally one discusses bound and free states separately and computes reflection and transmission coefficients.

### 6.7 Zero Curvature Condition

The idea here is that we can look at integrable non-linear problems as a compatibility condition of two linear problems and these can involve a spectral parameter \( \lambda \). We change from linear operators to matrices introducing a vector wavefunction

\[
\psi = \begin{pmatrix} \psi \\ \psi_x \end{pmatrix}
\]

Consider two matrices \( X, T \) that can be a function of a time independent parameter \( \lambda \). Suppose that our wavefunction satisfies

\[
\begin{align*}
\psi_x &= X \psi \\
\psi_t &= T \psi \quad (58)
\end{align*}
\]

Note that these two equations are linear. We need to figure out if they are consistent with each other. Compute

\[
\begin{align*}
\psi_{xt} &= X_t \psi + X \psi_t \\
&= X_t \psi + XT \psi \\
\psi_{tx} &= T_x \psi + T \psi_x \\
&= T_x \psi + TX \psi
\end{align*}
\]
Subtracting these two
\[(X_t - T_x + [X, T]) \psi = 0\]

A pair \(X, T\) satisfies the condition
\[X_t - T_x + [X, T] = 0\] (59)

iff the accompanying differential equation is satisfied. Such a pair of matrix operators can be called a Lax pair.

For the KdV equation we have (equation 53)
\[
L = D_{xx} + \frac{1}{6} u
\]
\[
B = 4D^3 + \frac{1}{2} (uD + Du)
\]

Using \(L \psi = \lambda \psi\)
\[
\psi_{xx} = \left( \lambda - \frac{u}{6} \right) \psi
\]

and
\[
\partial_x \psi = \begin{pmatrix} 0 & 1 \\ \lambda - u/6 & 0 \end{pmatrix} \begin{pmatrix} \psi \\ \psi_x \end{pmatrix} = X \psi
\]

with
\[
X = \begin{pmatrix} 0 & 1 \\ \lambda - u/6 & 0 \end{pmatrix}
\] (60)

Computing \(\psi_t\) using the \(B\) and \(L\) operators (equations 53)
\[
\psi_t = B \psi = 4D \psi_{xx} + \frac{1}{2} u_x \psi + u \psi_x
\]
\[
= \left( 4 \lambda + \frac{u}{3} \right) \psi_x - \frac{1}{6} u_x \psi
\]
\[
\psi_{xt} = \left( 4 \lambda + \frac{u}{3} \right) \psi_{xx} - \frac{1}{6} u_{xx} \psi + \frac{1}{6} u_x \psi_x
\]
\[
= \left( 4 \lambda^2 - \frac{u}{18} - \frac{1}{3} u \lambda - \frac{1}{6} u_{xx} \right) \psi + \frac{1}{6} u_x \psi_x
\]

with
\[
\partial_t \psi = T \psi
\]

we find
\[
T = \begin{pmatrix} 4 \lambda^2 - \frac{u^2}{18} - \frac{u \lambda}{3} - \frac{1}{6} u_{xx} & \frac{1}{6} u_x \\
4 \lambda - \frac{u}{6} & \frac{1}{6} u_x \end{pmatrix}
\] (61)
Using the two matrices we can compute the zero curvature condition and we find
\[ X_t - T_x + [X, T] = \begin{pmatrix} 0 & 0 \\ \frac{1}{6}(u_t - uu_x - u_{xxx}) & 0 \end{pmatrix} \]
and this is zero if \( u \) satisfies the KdV equation. Matrices \( X, T \) can be constructed from a Lax operator pair of operators (if there are higher order derivatives we could use a larger dimension vector for the wavefunction).

The name zero curvature condition comes from the following geometrical interpretation. The equations \((\partial_x - X)\psi = 0\) and \((\partial_t - T)\psi = 0\) define a connection on a two-dimensional vector bundle over the \((x, t)\)-plane. The first equation describes how to parallel-translate a vector \( \psi \) in the \( x \)-direction, and the second equation describes how to parallel-translate a vector \( \psi \) in the \( t \)-direction. The matrices \( X \) and \( T \) are the connection coefficients. A connection is said to have zero curvature if parallel translation of a vector \( \psi \) along a path from a point \((x_0, t_0)\) to another point \((x_1, t_1)\) gives the same result independent of path. This is the same thing as asserting the existence of a full two-dimensional basis of simultaneous solutions of the equations \((\partial_x - X)\psi = 0\) and \((\partial_t - T)\psi = 0\) that must be satisfied by the connection coefficients and the zero curvature condition. Every solution of the KdV equation defines a connection with zero curvature. The zero curvature condition is important when considering evolution or scattering operators and it makes possible to reorder integrals.

Let us see if we start with a zero-curvature condition and derive an operator equation.

\[ X_t - T_x + [X, T] = 0 \]
we can write as
\[ [\partial_t - T, \partial_x - X] = 0 \] (62)
To check this work with the operators
\[ 0 = [\partial_t - T, \partial_x - X] \]
\[ = [\partial_t, \partial_x] - [\partial_t, X] - [T, \partial_x] + [T, X] \]
\[ = 0 - \partial_t(X\phi) + X\partial_t\phi - T\partial_x\phi + \partial_x(T\phi) + [T, X]\phi \]
\[ = -\partial_t X + \partial_x T + [T, X] \]

Now let us define
\[ B = T \] (63)
\[ L = \partial_x - X \] (64)
and rewrite equation 62
\[ 0 = [\partial_t - B, L] \]
\[ = [\partial_t, L] - [B, L] \]
\[ = \partial_t(L\phi) - L\partial_t\phi - [B, L]\phi \]
\[ = \partial_t L - [B, L] \]
and this is the operator form of the Lax equation. It seems that is possible to construct an operator Lax pair from a matrix Lax pair and vice versa. However I am not sure if a one-dimensional operator Lax pair can always be constructed from a matrix Lax pair.

The zero curvature condition looks remarkably similar to the Cartan-Maurer equation. If we associate

\[
X = g^{-1} \partial_x g \\
T = g^{-1} \partial_t g
\]

and have \( g \) be an element of a Lie group. Then the zero curvature equation (or Cartan-Maurer equation) determines the structure constants of the group. \( T, X = A_0, A_1 \) are a gauge potential or a connection. Apparently for \( SL(2, R) \) the Cartan-Maurer equation gives the KdV equation. This implies that every Lie group gives rise to an integrable system (and hierarchy of differential equations). Is this true in the opposite direction? Does every integrable system with a Lax Pair have a Lie group associated with it? Since Lie groups are classified, does this mean all integrable systems are too?

### 6.8 Two different Poisson brackets

Above we found that the KdV equation is the equations of motion for

\[
\{u(x), u(y)\}_1 = \partial_x \delta(x - y) \\
H[u]_1 = \int_{-\infty}^{\infty} dx \left( \frac{u^3(x)}{6} - \left( \frac{\partial u}{\partial x} \right)^2 \right)
\]

The equations of motion

\[
\frac{\partial u(x, t)}{\partial t} = \{u(x), H[u]_1\}_1 \quad \rightarrow \quad u_t = uu_x + u_{xxx}
\]

There is a second pair of Poisson bracket and Hamiltonian that gives the same equations of motion

\[
\{u(x), u(y)\}_2 = \left( \frac{\partial^3}{\partial x^3} + \frac{1}{3} \left( \frac{\partial}{\partial x} u(x) + u(x) \frac{\partial}{\partial x} \right) \right) \delta(x - y) \\
H[u]_2 = \int_{-\infty}^{\infty} dx \left( \frac{u^2(x)}{2} \right)
\]
We can check the equation of motion
\[
\frac{\partial u}{\partial t} = \int_{-\infty}^{\infty} dy \left( \frac{\partial^3}{\partial x^3} + \frac{1}{3} \left( \frac{\partial}{\partial x} u(x) + u(x) \frac{\partial}{\partial x} \right) \right) \delta(x - y)
\]
\[
= u_{xxx} + \frac{2}{3} uu_x + \frac{1}{3} uu_x
\]
\[
= u_{xxx} + uu_x
\]

So
\[
\frac{\partial u(x, t)}{\partial t} = \{u(x), H_2[u]\}_2 \rightarrow u_t = uu_x + u_{xxx}
\]

A conserved quantity can be operated on by either Poisson bracket. A conserved quantity can be used to generate its own flow (a Hamiltonian flow) and using either Poisson bracket. Each conserved quantity can be used to generate a new time evolution equation (a new differential equation).

It might be useful to write each Poisson bracket as an operator
\[
p_1 = D_x
\]
\[
p_2 = D_{xxx} + \frac{1}{3} \left( D_x u + u D_x \right)
\]

Reexamine equation 49 and notice the relation between the different conserved quantities!

The existence of two Poisson brackets is related to the fact that the equation is integrable. Suppose we have Hamiltonians \( H_0 \) and \( H_1 \) and Poisson brackets for each that can be written
\[
\{f(y), g(y)\}_0 = f^{0, \mu} \partial_\mu f \partial_\nu g
\]
\[
\{f(y), g(y)\}_1 = f^{1, \mu} \partial_\mu f \partial_\nu g
\]

Here \( f^{0, \mu} \) can be a function of coordinates \( y \). Time evolution can be computed using \( H_0 \) and the first Poisson bracket or \( H_1 \) and the second Poisson bracket. Inverting \( f_1 \) we define a tensor
\[
S^\mu_\nu = f_{1, \mu \lambda} f^{0, \lambda \nu}
\]
and a tensor
\[
U_\mu^\nu(y) = \partial_\mu y_\nu = \partial_\mu (f^{0, \mu \lambda} \partial_\lambda H_0) = \partial_\mu (f^{1, \mu \lambda} \partial_\lambda H_1)
\]

and after computing some derivatives
\[
\frac{dS^\nu_\mu}{dt} = S^\lambda_\mu U_\lambda^\nu - U_\mu^\lambda S^\nu_\lambda
\]
which can be written as
\[ \frac{dS}{dt} = [S, U] \]
and this is a Lax equation.

Quantities \( K_n = \frac{1}{n} \text{trace} S^n \) (for \( n > 0 \)) and \( K_0 = \log |\det S| \) are conserved quantities. We should go on to prove that conserved quantities are independent and can be involution.

For the KdV equation the two Poisson structures give

\[
\begin{align*}
  f_0(x, y) &= D_x \delta(x - y) \\
  f_1(x, y) &= \left(D^3_x + \frac{1}{3}(uD_x + D_x u)\right) \delta(x - y)
\end{align*}
\]

The inverse of \( f_0 \) is \( D^{-1} \) and from this we get \( L \) and we can check that the Lax equation gives the KdV equation.

The inverse of \( D_x \) is

\[ f_0^{-1}(x, y) = \epsilon(x - y) \]

which is a step function (equal to \(-1/2\) for \( x - y < 0 \) and equal to \(+1/2\) for \( x - y > 0 \)).

\[ \partial_x \epsilon(x - y) = \delta(x - y) \]

\[ S(x - z) = \int dy f_0^{-1}(x - y) f_1(y - z) = \int dy \epsilon(x - y) \left(D^3_x + \frac{1}{3}(uD_x + D_x u)\right) \delta(y - z) \]

In operator language

\[ S = \left(D^3 + \frac{1}{3}(uD + Du)\right) D^{-1} \]

The KdV equation itself we can write

\[ \dot{u} = u_{xxx} + uu_x = D^3 u + u(Du) \]

\[ U = \frac{\delta \dot{u}}{\delta u} = D^3 + Du \]

(though this is not obvious to me).

\[
\begin{align*}
SU &= \left(D^3 + \frac{1}{3}(uD + Du)\right) D^{-1}(D^3 + Du) \\
&= D^5 + D^3 u + \frac{2}{3} uD^3 + DuD^2 + \frac{2}{3} uDu + \frac{1}{3} DuD^2 + \frac{1}{3} u(Du) \\
US &= D^5 + \frac{2}{3} D^3 u + \frac{1}{3} D^3(Du)D^{-1} + DuD^2 + \frac{2}{3} uD^2 + \frac{1}{3} Du(Du)D^{-1}
\end{align*}
\]
and combining we get the KdV equation.

More here to be more careful with computation and definitions!

To summarize, the existence of two Hamiltonians (each with its own Poisson bracket) that are consistent with the equations of motion implies that there must be a Lax pair and infinite number of conserved quantities. Likely the existence of a Lax pair implies the dual Poisson structure.

7 Symplectic Reduction

A goal here is to use a symmetry of the problem to reduce infinite dimensional systems to finite dimensional ones. Or to just simplify the Hamiltonian at the expense of complicating the Poisson bracket. Someday a nice exploration of how to do this here, hopefully with examples! Perhaps look at the recent paper by Podvin and Sergent.

Notes: Much of this is following Ashok Das's book on Integrable Models. The hydrodynamics examples are primarily taken from P. J. Morrison's review Hamiltonian description of the ideal fluid, 1998, Rev Mod Physics, 70, 467 or a nice review entitled Hamiltonian Fluids by John Hunter.